1. Rice, problem 52, page 170. (Omit part e.)
   a. 1 is obviously better – higher mean and lower standard deviation
   b. $0.5\mu_1 + 0.5\mu_2 = 0.9$. The variance will be $0.5^2(0.1^2) + 0.5^2(0.12^2) + 2(0.5)(0.5)(-0.8)(0.1)(0.12) = 0.0013$. The standard deviation is 0.036
   c. $0.8(1) + 0.2(0.8) = 0.96$. The variance will be $0.8^2(0.1^2) + 0.2^2(0.12^2) + 2(0.8)(0.2)(-0.8)(0.1)(0.12) = 0.003904$. The standard deviation will be 0.062482.
   d. return = $w*1 + (1-w)*0.8$
   risk = $\text{sqr}(w^2*0.01 + (1-w)^2*0.0144 + 2*w*(1-w)*0.1*0.12*(-0.8))$
2. Consider a variable with exponential distribution, \( f(x) = \frac{1}{\alpha} \exp\left(-\frac{x}{\alpha}\right), x \geq 0, \alpha > 0. \) We have found that \( E[X] = \alpha \) and \( \text{Var}[X] = \alpha^2. \) Consider the random variable \( Y = X \) if \( X \leq T, \) and \( Y = T \) if \( X > T. \) \( Y \) is a censored version of \( X. \) Find \( E[Y] \) and \( \text{Var}[Y]. \) A different random variable is \( Z = X \) if \( X \leq T. \) (\( Z \) is a truncated version of \( X. \)) Find the density, mean, and variance of \( Z. \)

Solution of this problem will require repeated use of the following two integrals (search online for ‘table of integrals’ if you are interested in the sources):

\[
\int xe^{cx} \, dx = \frac{e^{cx}}{c^2}(cx-1) \quad \text{and} \quad \int x^2 e^{cx} \, dx = e^{cx} \left( \frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3} \right)
\]

For the censored version, \( E[Y] = \text{Prob}[y < T] \cdot E[y|y < T] + \text{Prob}[y \geq T] \times T \)

\[
\text{Prob}[y < T] = 1 - \exp\left(-\frac{T}{\alpha}\right). \quad \text{Prob}[y \geq T] = \exp\left(-\frac{T}{\alpha}\right)
\]

The density of \( y \) has the two parts. When \( x < T, y = x. \) So that part of the density is \( f(y|y < T) = f(x \mid x < T) = f(x)/\text{Prob}(x < T). \) To get the first term in \( E[y], \) the two probabilities will cancel, leaving:

\[
E[y] = \left[ \int_0^T \frac{x}{\alpha} e^{-x/\alpha} \, dx \right] + \left\{ T \times e^{-T/\alpha} \right\}
\]

The integral is one of the two listed above. Plugging in the terms, we have \( c = \frac{-1}{\alpha} \)

\[
E[y] = \frac{1}{\alpha} \left\{ \frac{1}{(-1/\alpha)^2} \left[ \left( -\frac{1}{\alpha} \right) x - 1 \right] \right\}^T_0 + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \frac{1}{\alpha} \left\{ \frac{e^{-T/\alpha}}{\left(-\frac{T}{\alpha} - 1\right)} \right\} - \frac{1}{\alpha} \left\{ \frac{e^{-0/\alpha}}{\left(-\frac{0}{\alpha} - 1\right)} \right\} + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \frac{1}{\alpha} \left\{ \frac{e^{-T/\alpha}}{\left(-\frac{T}{\alpha} - 1\right)} \right\} + \left\{ \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \right) \right\} + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \alpha \left[ 1 - e^{-T/\alpha} \left( 1 + T / \alpha \right) \right] + Te^{-T/\alpha}
\]

The variance of \( y \) would be obtained by using the preceding to compute \( E[y^2] \) in the same fashion.

\[
E[y^2] = \left[ \int_0^T x^2 \frac{1}{\alpha} e^{-x/\alpha} \, dx \right] + \left\{ T^2 \times e^{-T/\alpha} \right\}
\]

Use the integral forms given above again,

\[
E[y^2] = \frac{1}{\alpha} e^{-x/\alpha} \left[ x^2 (-\alpha) - 2x(-\alpha)^2 + 2(-\alpha)^3 \right]_0^T + \left\{ T^2 \times e^{-T/\alpha} \right\}
\]

Collecting terms, \( E[y^2] = 2\alpha^2 - \frac{1}{\alpha} e^{-T/\alpha} \left[ \alpha T^2 + 2T\alpha^2 + 2\alpha^3 \right] + e^{-T/\alpha} T^2 \)

\[
= 2\alpha^2 + e^{-T/\alpha} \left[ T^2 - T^2 - 2T\alpha - 2\alpha^2 \right] = 2\alpha^2 - 2\alpha e^{-T/\alpha} \left[ T + \alpha \right]
\]

\[
= 2\alpha^2 \left\{ 1 - e^{-T/\alpha} \left[ T + \alpha \right] \right\} = 2\alpha^2 \left\{ 1 - e^{-T/\alpha} \left[ T + \alpha \right] \right\}
\]

Finally, the variance is \( \text{Var}[y] = E[y^2] - \{E[y]\}^2, \) which has a lot of terms, and no intuition.
The truncated distribution is manipulated in the same way. The integrals are a little simpler, as the density has only one part:

$$f(x | x < T) = \frac{f(x)}{\text{Prob}(x < T)} = \frac{(1/\alpha)e^{-\alpha x}}{1 - e^{-T/\alpha}}$$

The mean is

$$\frac{1}{1 - e^{-T/\alpha}} \int_0^T (1/\alpha)xe^{-\alpha x} \, dx = \frac{(1/\alpha)}{1 - e^{-T/\alpha}} \left[ \frac{e^{(-1/\alpha)x}x}{(-1/\alpha)} \right]_0^T$$

$$= \frac{1}{1 - e^{-T/\alpha}} \left[ \frac{e^{(-1/\alpha)x}T}{(-1/\alpha)} - 1 \right]$$

$$= \frac{1}{1 - e^{-T/\alpha}} \left[ e^{(-1/\alpha)x}T \{(-1/\alpha)T - 1\} + 1 \right]$$

The variance is found similarly, by obtaining $E[y^2]$ as before.

3. Suppose $x$ is distributed uniformly from 0 to 1. What is the density of $z = \exp(x^2)$?

$$f(x) = 1, 0 < x < 1.$$  
$$z = \exp(x^2), 1 < z < e.$$  
$$x^2 = \log z and x = (\log z)^{1/2}.$$  
$$dx = \frac{1}{2} \cdot (\log z)^{-1/2} \cdot 1/z \, dz$$  
$$f(z) = 1 \cdot \text{Jacobian} = \frac{1}{2} \cdot (\log z)^{-1/2} \cdot 1/z$$

4. If the random variable $x$ has gamma density with $\lambda, P = \frac{1}{2}, \frac{1}{2}$, what is the density of $x^2$. What is the density of $\log(x)$?

$$z = x^2 so x = z^{1/2}.$$  
$$dx = \frac{1}{2} \cdot z^{1/2} \, dz$$ is the Jacobian  
$$f(z) = (1/2)^{(1/2)} \Gamma(1/2) \exp(-1/2 \cdot 2^{1/2}) (z^{1/2})^{1/2 - 1} \left( \frac{1}{2} \cdot z^{1/2} \right) \left( \frac{1}{2} \cdot z^{1/2} \right)$$  
$$= (1/2)^{(1/2)} \cdot \Gamma(1/2) \cdot \exp(-1/2 \cdot 2^{1/2}) \cdot \exp(z)$$

$$f(z) = \log(x) so x = \exp(z) and dx = \exp(z) \, dz$$ is the Jacobian.

$$f(z) = (1/2)^{(1/2)} \cdot \Gamma(1/2) \cdot \exp(-1/2 \cdot \exp(z)) \cdot \exp(z)^{1/2 - 1} \cdot \exp(z) \cdot \exp(z)$$

$$f(z) = (1/2)^{(1/2)} \cdot \Gamma(1/2) \cdot \exp(-1/2 \cdot \exp(z)) \cdot \exp(z)^{1/2}$$

5. Suppose $x$ has standard normal distribution, $\mu = 0, \sigma = 1$. Derive the covariance between $x$ and $x^2$.

Covariance is $E[x \times x^2] - E[x]E[x^2] = E[x^3] - \mu(1)$. But, the normal distribution is symmetric, so the third moment about the mean of zero is zero. They are uncorrelated. The covariance is zero.

6. Rice, problem 12, page 189. The central limit theorem can be used to analyze round-off error. Suppose that the round-off error is represented as a uniform random variable on $[-1/2,+1/2]$. If 100 numbers are added, approximate the probability that the round-off error exceeds (a) 1, (b) 2, (c) 5.

The mean of the sum of 100 variables is 100*0 since all have mean 0. The variance is the sum of the variances, which is 100 * 1/12 = 8.333. The standard deviation is the square root, 2.887.

a. The probability that the sum is greater than 1 is $\text{Prob}[x > 1] = \text{Prob}[(x - 0)/2.887 > (1 - 0)/2.887] = 0.3645$

b. $\text{Prob}[x > 2] = \text{Prob}(z > 2/2.887) = 0.245$ (or .490).

c. $\text{Prob}[x > 5] = \text{Prob}(z > 5/2.887) = 0.0416$ (or .0892).

7. Rice, Problem 10, page 189. Use the continuity correction in your computation.

The exact probability for any $N$ is $\binom{N}{1} / (1/6)^N$ (5/6)^{100-N}.

For 100 tosses, the expected number of 6s is $N \pi = 100(1/6) = 16.67$. The variance is $100(1/6)(5/6) = 13.89$, so the standard deviation is 3.73.

a. $\text{Prob}(15 < x < 20) = \text{Prob}(16 \leq x \leq 19)$. Using the continuity correction, we use 15.5 and 19.5.

Using the central limit theorem, we then use $\text{Prob}(15.5 \leq x^* \leq 19.5)$ from the normal distribution.
Standardizing, this is 
$$\text{Prob}\left[ \frac{(15.5 - 16.67)}{3.73} \leq z \leq \frac{(19.5 - 16.67)}{3.73} \right] = 0.7760 - 0.3769 = 0.3991$$

b. The sum of the face values has mean 100*3.5 = 350 and variance 100*2.917 or standard deviation 17.078. 
$$\text{Prob}[\text{sum} < 300] = \text{Prob}\left[ \frac{(\text{sum} - 350)}{17.078} < \frac{(300 - 350)}{17.078} \right] = \text{Prob}[z < -2.928] = 0.00171$$

Expected loss on any game is zero (fair game) = $-5*(.5) + (+5)(.5)$
Variance of loss on any .5*(-5-0)$^2$ + .5*(5-0)$^2$ = 25.
Expected winning on 50 games is zero. Variance of winning is 50*25 = 1250. Standard deviation is 35.35.
$$\text{Prob}(\text{sum} < -75) = \text{Prob}(z < \frac{(-75 - 0)}{35.35}) = \text{Prob}(z < -2.12) = 0.01695.$$