The Martingale Approach to Operational and Financial Hedging

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Motivation

The Martingale Approach to Operational and Financial Hedging
Motivation

Problem:

Select an operational strategy and a financial strategy that optimize the value of the corporation.
Motivation

- Operations and services models are now beginning to consider issues related to risk aversion and hedging.

- There are many good reasons for hedging:
  - costs associated with financial distress
  - taxes
  - costs associated with raising capital
  - principal-agent issues
  - smoothing earnings for the Wall Street analysts!

- And in practice corporations often do hedge:
  - not inconsistent with Modigliani - Miller theory of corporate finance

- Is there a modelling framework that might be useful for operational and financial decision-making?
Motivation

- A Methodology that effectively integrates Operational and Financial activities by
  - Maximizing the Economic Value of the corporation.
  - Modeling the risk preferences of decision makers.
  - Using financial markets for hedging purposes.
  - Recognizing the “incompleteness” of the problem.
  - Taking advantage of all available information.
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A Methodology that effectively integrates Operational and Financial activities by
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\[
\max \{\text{Economic Value}\}
\]
subject to problem dynamics
and subject to risk management constraint(s).
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- A Methodology that effectively integrates Operational and Financial activities by
  - Maximizing the **Economic Value** of the corporation.
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\max \{ \text{Economic Value} \} \\
\text{subject to } \text{problem dynamics} \\
\text{and subject to } \text{risk management constraint(s)}. \\
\]

- **Operational risk** in customers' taste, machine breakdowns, unreliable suppliers, etc...

- **Financial risk** in exchange rates, commodity prices, interest rates, etc...
Outline of Talk

○ **Part I:** The Modelling Framework
  - Motivation for maximizing *economic* value of operating profits.
  - Financial market & informational structure

○ **Part II:** Supply Contracts with Financial Hedging
  - The Wholesale Price Contract with Budget Constraint

○ **Part III:** Models with Tail Constraints
  - A Procurement Example

○ Conclusions & Further Research
Part I

Modelling Framework

Financial Market & Information
Motivating the Modelling Framework

- Consider two series cash-flows arising from two operating policies $I_1$ and $I_2$:

\[
C^{(1)}(I_1) = (c_1^{(1)}(I_1), c_2^{(1)}(I_1), \ldots, c_n^{(1)}(I_1))
\]

\[
C^{(2)}(I_2) = (c_1^{(2)}(I_2), c_2^{(2)}(I_2), \ldots, c_n^{(2)}(I_2))
\]

- If probability distribution of $C^{(1)}$ and $C^{(2)}$ identical then most operations models are indifferent between the two.

- However, if $C^{(1)}$ positively correlated with financial market for example, and $C^{(2)}$ negatively correlated with financial market then economic value of $C^{(2)}$ greater than economic value of $C^{(1)}$.
  - models should take this into account as they get ever more complex and include additional sources of uncertainty

- Do so by using an equivalent martingale measure, $\mathbb{Q}$, or stochastic discount factor
  - consistent with only a hedging motivation for financial trading
The Modelling Framework

- $H(I)$ the time $T$ cumulative payoff from some operating policy, $I$
  - The operating policy may be static or dynamic.
The Modelling Framework

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- $G(\theta)$ the time $T$ gain from a self-financing trading strategy, $\theta$
The Modelling Framework

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  - The operating policy may be static or dynamic.

- $G(\theta)$ the time $T$ gain from a self-financing trading strategy, $\theta$

- Operational and financial hedging problem is

$$\min_{I,G(\theta)} \mathbb{E}^Q [H(I)]$$

subject to problem dynamics

and subject to risk management constraint(s) $\equiv$ \[
\begin{align*}
\mathcal{R}(H(I), G(\theta)) &\leq 0 \quad a.s. \\
\mathbb{E}^p [\mathcal{R}(H(I), G(\theta))] &\leq 0.
\end{align*}
\]
The Modelling Framework

- $H(I)$ the time $T$ cumulative payoff from some operating policy, $I$
  - The operating policy may be static or dynamic.
- $G(\theta)$ the time $T$ gain from a self-financing trading strategy, $\theta$

Operational and financial hedging problem is

$$\min_{I,G(\theta)} E^Q[H(I)]$$

subject to problem dynamics

and subject to risk management constraint(s) \(\equiv\)

$$\mathcal{R}(H(I), G(\theta)) \leq 0 \ a.s.\$$

$$\mathbb{E}^P[\mathcal{R}(H(I), G(\theta))] \leq 0.$$

- consistent with only a hedging motivation for financial trading
  - $G(\theta)$ a martingale under $\mathbb{Q}$ so $E^Q[G(\theta)] = 0$
  - assuming initial capital assigned to hedging strategy is zero
Financial Market Model

- Probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
- \((W_{1t}, W_{2t})\) a standard Brownian motion.
- A single risky stock with price dynamics 
  \[ \, dX_t = \mu X_t \, dt + \sigma X_t \, dW_{1t}. \]
- Cash account available with \(r \equiv 0\).
- \(\mathcal{F}_t\) is the filtration generated by \((W_{1t}, W_{2t})\).
- \(\mathcal{F}^X_t\) is the filtration generated by \(X_t\).
- Operating profits, \(H(I) \in \mathcal{F}_T\), a function of \((W_{1t}, W_{2t})\).
- The gain process, \(G(\theta)_t\) of a self-financing trading strategy \(\theta_t\) is 
  \[ G(\theta)_t = \int_0^t \theta_s \, dX_s. \]
- An equivalent martingale measure, \(\mathbb{Q}\), under which \(X_t\) and \(G(\theta)_t\) are martingales.
Information

In general, operating profits are function of \((X_t, W_{2t})\), that is, \(H(I) \in \mathcal{F}_T\).
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- **Incomplete Information:**
  - Demand, product quality, customer tastes not always observable.
  - Only information related to \(X_t\) is available.

  \[\Rightarrow \text{ Trading strategies must satisfy: } \theta_t \in \mathcal{F}_t^X, \quad t \in [0, T].\]
Information

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  \[\Rightarrow\]  Trading strategies must satisfy: \(\theta_t \in \mathcal{F}_t^{X}, \quad t \in [0, T].\)

- **Complete Information:**
  - The evolution of \(X_t\) and \(B_{2t}\) are both available.

  \[\Rightarrow\]  Trading strategies satisfy: \(\theta_t \in \mathcal{F}_t, \quad t \in [0, T].\)
Part II

Supply Contracts with Financial Hedging

The Wholesale Price Contract with Budget Constraint
The Wholesale Price Contract

Non-Cooperative Operation:

1. At time $t = 0$, Manufacturer offers a wholesale price $w$ to the Retailer.
2. At time $t = 0$, Retailer orders a quantity $q$.
3. At time $t = T$, a random clearance price $P(q)$ is realized: $P(q) = A - q$.

- The random price $A$ is correlated to the financial market $X_t$.
- Retailer operates under a budget constraint: $w q \leq B$. 

Solution Concept: Stackelberg game.
Types of Contracts

(w,q) contract is decided

Production takes place

Clearance Price P(q) is realized
Payoffs are determined

Simple Contract

\( t = 0 \) \hspace{1cm} \( t = T \)
Types of Contracts

Simple Contract

- \((w,q)\) contract is decided
- Production takes place at time \(t = 0\)
- Clearance Price \(P(q)\) is realized and payoffs are determined at time \(t = T\)

Flexible Contract

- \((w_\tau,q_\tau)\) contract is decided
- Market signal \(X_\tau\) is observed at time \(t = \tau\)
- Production takes place at time \(t = \tau\)
- Clearance Price \(P(q)\) is realized and payoffs are determined at time \(t = T\)
Types of Contracts

Simple Contract

- (w,q) contract is decided
- Production takes place at t = 0
- Clearance Price P(q) is realized and Payoffs are determined at t = T

Flexible Contract

- (w, q) contract is decided
- Market Signal X_t is observed at t = τ
- Production takes place at t = τ
- Clearance Price P(q) is realized and Payoffs are determined at t = T

Flexible Contract with Financial Hedging

- (w, q) contract is decided
- Market Signal X_t is observed at t = τ
- Trading Gain G_τ is observed
- Financial hedging takes place at t = 0
- Production takes place at t = τ
- Clearance Price P(q) is realized and Payoffs are determined at t = T
Types of Contracts

(w,q) contract is decided

Production takes place

\[ t = 0 \quad \text{to} \quad t = T \]

Simple Contract

(w,\tau, q_0) contract is decided

Market Signal \( X_\tau \) is observed

Production takes place

\[ t = 0 \quad \text{to} \quad t = \tau \]

Flexible Contract

(w,\tau, q_0) contract is decided

Market Signal \( X_\tau \) is observed

Trading Gain \( G_\tau \) is observed

Financial hedging takes place

Production takes place

\[ t = 0 \quad \text{to} \quad t = \tau \]

Flexible Contract with Financial Hedging

Clearance Price \( P(q) \) is realized

Payoffs are determined

\[ t = \tau \quad \text{to} \quad t = T \]

Remark: The time \( \tau \) can be a decision variable: a deterministic time or a \( \mathcal{F}_\tau^X \)-stopping time.
Some Notation

*) $(\Omega, \mathcal{F}, \mathbb{Q})$ probability space.

*) $X_t$: Time $t$ value of a $\mathbb{Q}$-martingale tradable security ($t \in [0, T]$).

*) $\mathcal{F}_t = \sigma(X_s ; 0 \leq s \leq t)$ filtration generated by $X_t$ with $\mathcal{F}_T \subseteq \mathcal{F}$.

*) $\mathbb{E}_t^Q[E] := \mathbb{E}^Q[E|\mathcal{F}_T]$ for all $E \in \mathcal{F}$.

*) $P(q) = A - q$: retail price at time $T$ with $A \in \mathcal{F}$.

*) $\bar{A}_\tau = \mathbb{E}^Q[A|\mathcal{F}_\tau]$ for any $\mathcal{F}_t$-stopping time $\tau \leq T$.

*) $c_\tau$: per unit manufacturing cost at time $\tau \in [0, T]$.

*) $w_\tau \in \mathcal{F}_\tau$: manufacturer’s wholesale price menu.

*) $q_\tau \in \mathcal{F}_\tau$: retailer’s ordering quantity menu.

*) $G_\tau \in \mathcal{F}_\tau$: retailer’s trading gains (or losses) with $\mathbb{E}^Q[G_\tau] = 0$.

*) Budget constraint: $w_\tau q_\tau \leq B_\tau := B + G_\tau$.

Assumption. *For all $\tau \in [0, T]$, $\bar{A}_\tau \geq c_\tau$. 

Flexible Contract

Consider a fixed $\tau \in [0, T]$.

Step 1: At $t = 0$, the manufacturer offers a wholesale menu $w_\tau \in \mathcal{F}_\tau$.

Step 2: A retailer, with no access to the financial market, decides the ordering level $q_\tau \in \mathcal{F}_\tau$ solving

$$\Pi^F_R(w_\tau) = \mathbb{E}_0^Q \left[ \max_{q_\tau \geq 0} \left\{ \mathbb{E}_\tau^Q [(A - q_\tau - w_\tau) q_\tau] \right\} \right]$$

subject to $w_\tau q_\tau \leq B$, for all $\omega \in \Omega$.

The optimal solution (retailer’s reaction) is

$$q(w_\tau) = \min \left\{ \left( \frac{\bar{A}_\tau - w_\tau}{2} \right)^+, \frac{B}{w_\tau} \right\}.$$

Step 3: The manufacturer problem is (Stackelberg leader)

$$\Pi^F_M = \mathbb{E}_0^Q \left[ \max_{w_\tau \geq c_\tau} \left\{ (w_\tau - c_\tau) q_\tau(w_\tau) \right\} \right].$$
Flexible Contract

**Proposition.** (Flexible Contract Solution)

*Under Assumption*, the equilibrium solution for the flexible contract is

\[ w^F_\tau = \frac{A_\tau + \delta^F_\tau}{2} \quad \text{and} \quad q^F_\tau = \frac{A_\tau - \delta^F_\tau}{4}, \]

where

\[ \delta^F_\tau := \max \left\{ c_\tau, \sqrt{(A^2_\tau - 8B)^+} \right\}. \]

The equilibrium expected payoffs of the players are then given by

\[ \Pi^F_{M|\tau} = \frac{(A_\tau + \delta^F_\tau - 2c_\tau)(A_\tau - \delta^F_\tau)}{8} \quad \text{and} \quad \Pi^F_{R|\tau} = \frac{(A_\tau - \delta^F_\tau)^2}{16}. \]

**Remarks:**

- \( \delta^F_\tau \) is a modified production cost (\( \geq c_\tau \)) that is induced by a limited budget \( B \).

- \( w^F_\tau \) decreases in \( B \) and \( q^F_\tau, \Pi^F_{M|\tau} \) and \( \Pi^F_{R|\tau} \) increase in \( B \).
Flexible Contract vs. Simple Contract

Define
\[ w^F := \mathbb{E}_0^Q[w^F], \quad q^F := \mathbb{E}_0^Q[q^F], \quad \Pi^F_M := \mathbb{E}_0^Q[\Pi^F_M|\tau] \quad \text{and} \quad \Pi^F_R := \mathbb{E}_0^Q[\Pi^F_R|\tau]. \]

**Proposition.** Suppose that \( B \leq \frac{\bar{A}^2 - c^2}{8} \) almost surely. Then
\[ w^F \leq w^S, \quad q^F \geq q^S, \quad \Pi^F_M \leq \Pi^S_M \quad \text{and} \quad \Pi^F_R \geq \Pi^S_R. \]

However, if \( B \geq \max \left\{ \frac{\bar{A}^2 - c^2}{8}, \frac{\bar{A}^2 - c_0^2}{8} \right\} \) almost surely then
\[ w^F = w^S + \frac{c_\tau - c_0}{2} \quad \text{and} \quad q^F = q^S - \frac{c_\tau - c_0}{4} \]

and \( \Pi^F_M \geq \Pi^S_M \) and \( \Pi^F_R \geq \Pi^S_R \) if and only if \( \mathbb{E}_0^Q[(\bar{A}_\tau - c_\tau)^2] \geq (\bar{A} - c_0)^2. \)
Flexible Contract vs. Simple Contract

\[ \bar{A}_\tau \sim \text{Uniform}[1, 3], \quad c_0 = 0.3, \quad c_\tau = 0.35 \text{ (case 1)} \quad \text{and} \quad c_\tau = 0.7 \text{ (case 2)}. \]
Flexible Contract: Efficiency

![Graphs showing Q^F_τ, W^F_τ, and P^F_τ against Budget (B)]

\[ \bar{A}_\tau \geq 3 \ c_\tau \]

\[ \bar{A}_\tau \leq 3 \ c_\tau \]

\[ \bar{A}_\tau = 2, \quad c_\tau = 0.6 \text{ (top)} \quad \text{and} \quad c_\tau = 1.2 \text{ (bottom).} \]
Flexible Contract with Financial Hedging

The Martingale Approach to Operational and Financial Hedging
Flexible Contract with Financial Hedging

**Step 1**: At \( t = 0 \), and for a fixed \( \tau \leq T \), the manufacturer offers a price menu \( w_\tau \in \mathcal{F}^X_\tau \).
Flexible Contract with Financial Hedging

**Step 1:** At $t = 0$, and for a fixed $\tau \leq T$, the manufacturer offers a price menu $w_\tau \in \mathcal{F}_\tau X$.

**Step 2:** In response, at $t = 0$, the retailer selects an optimal ordering menu $q^*_\tau(w_\tau) \in \mathcal{F}_\tau X$ solving

$$\Pi^H_R(w_\tau) = \max_{q_\tau \geq 0, G_\tau} \mathbb{E}^Q [(A - q_\tau) q_\tau - w_\tau q_\tau]$$

subject to

$$w_\tau q_\tau \leq B + G_\tau,$$

for all $\omega \in \Omega$

$$\mathbb{E}^Q[G_\tau] = 0.$$
Flexible Contract with Financial Hedging

- **Step 1:** At $t = 0$, and for a fixed $\tau \leq T$, the manufacturer offers a price menu $w_{\tau} \in F_t^X$.

- **Step 2:** In response, at $t = 0$, the retailer selects an optimal ordering menu $q^*_\tau(w_{\tau}) \in F_t^X$ solving

$$
\Pi^H_R(w_{\tau}) = \max_{q_{\tau} \geq 0, G_\tau} \mathbb{E}^Q[(A - q_{\tau})q_{\tau} - w_{\tau}q_{\tau}]
$$

subject to $w_{\tau}q_{\tau} \leq B + G_\tau$, for all $\omega \in \Omega$

$$
\mathbb{E}^Q[G_\tau] = 0.
$$

- **Step 3:** The manufacturer selects the optimal wholesale price menu $w^*_\tau$ solving

$$
\Pi^H_M(w_{\tau}) = \max_{w_{\tau}} \mathbb{E}^Q[w_{\tau}q^*_\tau(w_{\tau}) - c_{\tau}q^*_\tau(w_{\tau})].
$$
Flexible Contract with Financial Hedging

Proposition. (Retailer’s Optimal Strategy)

Let $Q_\tau$, $\mathcal{X}$ and $\mathcal{X}^c$ be defined as follows

$$Q_\tau \triangleq \left( \frac{\bar{A}_\tau - w_\tau}{2} \right)^+,$$  
$$\mathcal{X} \triangleq \{ \omega \in \Omega : B \geq Q_\tau w_\tau \},$$  
and $\mathcal{X}^c \triangleq \Omega - \mathcal{X}$. 
Flexible Contract with Financial Hedging

Proposition. (Retailer’s Optimal Strategy)

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\[ Q_\tau \triangleq \left( \frac{\bar{A}_\tau - w_\tau}{2} \right)^+ , \quad X \triangleq \{ \omega \in \Omega : B \geq Q_\tau w_\tau \} , \quad \text{and} \quad X^c \triangleq \Omega - X. \]

**Case 1:** Suppose that $\mathbb{E}^Q[Q_\tau w_\tau] \leq B$. Then $q^*_\tau(w_\tau) = Q_\tau$ and there are infinitely many choices of the optimal claim, $G_\tau$. One natural choice is to take

\[ G_\tau = [Q_\tau w_\tau - B] \cdot \left\{ \begin{array}{ll} \delta & \text{if } \omega \in X \\ 1 & \text{if } \omega \in X^c \end{array} \right. \]

\[ \delta \triangleq \frac{\int_{X^c} [Q_\tau w_\tau - B] \, dQ}{\int_X [B - Q_\tau w_\tau] \, dQ}. \]
Flexible Contract with Financial Hedging

Proposition. (Retailer’s Optimal Strategy)

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$$G_\tau = [Q_\tau w_\tau - B] \cdot \begin{cases} \delta & \text{if } \omega \in \mathcal{X} \\ 1 & \text{if } \omega \in \mathcal{X}^c \end{cases}$$

$$\delta \triangleq \int_{\mathcal{X}^c} [Q_\tau w_\tau - B] \, dQ \bigg/ \int_{\mathcal{X}} [B - Q_\tau w_\tau] \, dQ.$$ 

Remark: In this case, it is possible to completely eliminate the budget constraint by trading in the financial market.
Flexible Contract with Financial Hedging

**Proposition.** (Continuation)

**Case 2:** Suppose that $B < \mathbb{E}^Q [Q^* w^*]$. Then

$$q^* (w^*) = \left( \frac{\bar{A}_{\tau} - w^* (1 + \lambda)}{2} \right)^+ \text{ where } \lambda \geq 0 \text{ solves } \mathbb{E}^Q \left[ w^* \left( \frac{\bar{A}_{\tau} - w^* (1 + \lambda)}{2} \right)^+ \right] = B.$$
Flexible Contract with Financial Hedging

Proposition.  (Continuation)

\textbf{Case 2:} Suppose that $B < \mathbb{E}^Q[Q_tw_t]$. Then

$$q_t(w_t) = \left(\frac{\bar{A}_t - w_t(1 + \lambda)}{2}\right)^+$$

where $\lambda \geq 0$ solves $\mathbb{E}^Q\left[w_t \left(\frac{\bar{A}_t - w_t (1 + \lambda)}{2}\right)^+\right] = B$.

Proposition.  (Producer’s Optimal Strategy and the Stackelberg Solution)

Let $\phi^* \triangleq \inf \left\{ \phi \geq 1 \text{ such that } \mathbb{E}^Q\left[\left(\frac{\bar{A}_t^2 - (\phi c_t)^2}{8}\right)^+\right] \leq B \right\}$.

Then, $w^*_t = \frac{\bar{A}_t + \phi^* c_t}{2}$ and $q^*_t = \left(\frac{\bar{A}_t - \phi^* c_t}{4}\right)^+$ and the players’ expected payoffs satisfy

$$\Pi^{H}_{M|\tau} = \frac{(\bar{A}_t + \phi^* c_t - 2c_t)(\bar{A}_t - \phi^* c_t)^+}{8} \quad \text{and} \quad \Pi^{H}_{R|\tau} = \frac{(\bar{A}_t - \phi^* c_t)^+)^2}{16}.$$
Flexible Contract with Financial Hedging

**Proposition.** (Continuation)

**Case 2:** Suppose that $B < \mathbb{E}^Q [Q_\tau w_\tau]$. Then

$$q_\tau(w_\tau) = \left( \frac{\bar{A}_\tau - w_\tau (1 + \lambda)}{2} \right)^+$$

where $\lambda \geq 0$ solves $\mathbb{E}^Q \left[ w_\tau \left( \frac{\bar{A}_\tau - w_\tau (1 + \lambda)}{2} \right)^+ \right] = B$.

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Let $\phi^* \triangleq \inf \left\{ \phi \geq 1 \text{ such that } \mathbb{E}^Q \left[ \left( \frac{\bar{A}_\tau^2 - (\phi c_\tau)^2}{8} \right)^+ \right] \leq B \right\}$.

Then, $w^*_\tau = \frac{\bar{A}_\tau + \phi^* c_\tau}{2}$ and $q^*_\tau = \left( \frac{\bar{A}_\tau - \phi^* c_\tau}{4} \right)^+$ and the players’ expected payoffs satisfy

$$\Pi^H_{M|\tau} = \frac{(\bar{A}_\tau + \phi^* c_\tau - 2c_\tau) (\bar{A}_\tau - \phi^* c_\tau)^+}{8} \quad \text{and} \quad \Pi^H_{R|\tau} = \frac{((\bar{A}_\tau - \phi^* c_\tau)^+)^2}{16}.$$

**Remark:** When $q^*_\tau = 0$, the manufacturer decides to overcharge the retailer making the supply chain non-operative. This is never the case if the retailer does not have not access to the financial market.
Flexible Contract with Financial Hedging

**Proposition.** The manufacturer always prefers the H-contract to the F-Contract. On the other hand, the retailer's preferences are
Flexible Contract with Financial Hedging

**Proposition.** The manufacturer always prefers the H-contract to the F-Contract. On the other hand, the retailer’s preferences are

- **Small Budget**
  - H-Contract
  - Undetermined H-Contract or F-Contract
  - H-Contract
  - H-Contract = F-Contract

- **Large Budget**
  - H-Contract
  - F-Contract

![Graphs showing retailer's preferences](image)
Flexible Contract with Financial Hedging Efficiency

- On path-by-path basis, the Centralized system is not necessarily more efficient than the Decentralized Supply Chain!

\[ \exists \omega \in \Omega \text{ such that } q^H_{C|\tau} = 0 \text{ and } q^H_{\tau} > 0. \]

**Remarks:** This is never the case under a Flexible Contract without Hedging.
Flexible Contract with Financial Hedging
Efficiency

○ On path-by-path basis, the Centralized system is not necessarily more efficient than the Decentralized Supply Chain!

\[ \exists \omega \in \Omega \text{ such that } q_{C|\tau}^H = 0 \text{ and } q_{\tau}^H > 0. \]

Remarks: This is never the case under a Flexible Contract without Hedging.

○ On average, the Centralized solution is more efficient than the Decentralized solution.

\[ \mathbb{E}_0^Q[q_{C|\tau}^H] \geq \mathbb{E}_0^Q[q_{\tau}^H]. \]
Summary

- Simple extension to the traditional wholesale contract.
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- The proposed procurement contracts uses the Financial Market as:
  - A source of public information upon which contracts can be written.
  - A means for financial hedging to mitigate the impact of the budget constraint.
Summary

○ Simple extension to the traditional wholesale contract.

○ The proposed procurement contracts uses the Financial Market as:
  – A source of **public information** upon which contracts can be written.
  – A means for **financial hedging** to mitigate the impact of the budget constraint.

○ Consistent with the notions of production postponement and demand forecast.
Summary

◦ Simple extension to the traditional wholesale contract.
◦ The proposed procurement contracts uses the Financial Market as:
  − A source of **public information** upon which contracts can be written.
  − A means for **financial hedging** to mitigate the impact of the budget constraint.
◦ Consistent with the notions of production postponement and demand forecast.
◦ Managerial Insights:
  − Manufacturer and Retailer incentives are not always aligned as a function of $B$.
  − Manufacturer prefers retailers that have access to the financial market.
  − With hedging, the supply chain might not operate in some states $\omega \in \Omega$.
  − In some cases, financial hedging **eliminates** the budget constraint.
  − Optimal time $\tau$ of the contract balances $\mathbb{V}ar(\bar{A}_\tau)$ and $c_\tau$. 
Summary

- Simple extension to the traditional wholesale contract.
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  - In some cases, financial hedging **eliminates** the budget constraint.
  - Optimal time $\tau$ of the contract balances $\text{Var}(\bar{A}_\tau)$ and $c_\tau$.
- Extensions:
  - Other types of contracts: quantity discount, buy-back, etc.
  - Include other sources of uncertainty: exchange rates, interest rates, credit risk.
Part III

Models with Tail Constraints

Example: A Procurement Model
Problem Formulation

\[
\max_{I, \theta} \mathbb{E}^Q[H(I)] \\
\text{subject to } \mathbb{E}^P[R(H(I) + G(\theta))] \geq \bar{H}.
\]
Problem Formulation

$$\max_{I, \theta} \mathbb{E}^Q[H(I)]$$

subject to $$\mathbb{E}^P[R(H(I) + G(\theta))] \geq \bar{H}.$$
Problem Formulation

\[
\begin{align*}
\max_{I, \theta} \mathbb{E}^Q[H(I)] \\
\text{subject to} \quad \mathbb{E}^P[R(H(I) + G(\theta))] \geq \bar{H}.
\end{align*}
\]

- Value-at-Risk (VaR\textsubscript{\eta}): \( H_\eta \geq \bar{H} \).
Problem Formulation

\[
\max_{I, \theta} E^Q[H(I)] \\
\text{subject to } E^P[R(H(I) + G(\theta))] \geq \bar{H}.
\]

- Value-at-Risk (VaR\(_\eta\)): \( H_{\eta} \geq \bar{H} \).
- Conditional Value-at-Risk (CVaR\(_\eta\)): \( E^P[H(I) \mid H(I) \leq H_{\eta}] \geq \bar{H} \).
Problem Formulation

\[
\max_{I, \theta} \mathbb{E}^Q[H(I)]
\]

subject to \( \mathbb{E}^P[\mathcal{R}(H(I) + G(\theta))] \geq \bar{H}. \)

- Value-at-Risk (VaR\(\eta\)): \( H_\eta \geq \bar{H}. \)
- Conditional Value-at-Risk (CVaR\(\eta\)): \( \mathbb{E}^P[H(I) \mid H(I) \leq H_\eta] \geq \bar{H}. \)
- Mean & Standard-Deviation (MStd\(\eta\)): \( \mathbb{E}^P[H(I)] - \sqrt{\frac{\eta}{1-\eta}} \sqrt{\text{Var}(H(I))} \geq \bar{H}. \)
Tail Constraints: Bounds

- Rockafellar and Ursayev (2002):

\[
\text{CVaR}_\eta(X) = \min_{\zeta} \left\{ \zeta + \frac{1}{1 - \eta} \mathbb{E}^p[(X - \zeta)^+] \right\}.
\]
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\]

- Define \( \mathbb{E}^p[X] = \mu_X \) and \( \text{Var}(X) = \sigma_X^2 \) then Gallego (1992):

\[
\mathbb{E}^p[(X - \xi)^+] \leq \sqrt{\frac{\sigma_X^2 + (\xi - \mu_X)^2 - (\xi - \mu_X)}{2}}.
\]
Tail Constraints: Bounds

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\text{CVaR}_\eta(X) = \min_{\zeta} \left\{ \zeta + \frac{1}{1 - \eta} \mathbb{E}^\mathbb{P}[(X - \zeta)^+] \right\}.
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\]

- Combining these results we get

\[
\text{CVaR}_\eta(X) \leq \mu_X + \sqrt{\frac{\eta}{1 - \eta}} \sigma_X \triangleq \text{MStd}_\eta(X).
\]
Tail Constraints: Bounds

○ Rockafellar and Ursayev (2002):

$$\text{CVaR}_\eta(X) = \min_\zeta \left\{ \zeta + \frac{1}{1 - \eta} \mathbb{E}^p[(X - \zeta)^+] \right\}.$$  

○ Define $\mathbb{E}^p[X] = \mu_X$ and $\text{Var}(X) = \sigma_X^2$ then Gallego (1992):

$$\mathbb{E}^p[(X - \xi)^+] \leq \sqrt{\frac{\sigma_X^2 + (\xi - \mu_X)^2 - (\xi - \mu_X)^2}{2}}.$$ 

○ Combining these results we get

$$\text{CVaR}_\eta(X) \leq \mu_X + \sqrt{\frac{\eta}{1 - \eta}} \sigma_X \triangleq \text{MStd}_\eta(X).$$  

○ Tail Constraint Ordering

$$\text{VaR}_\eta \geq \text{CVaR}_\eta \geq \text{MStd}_\eta.$$
Problem Formulation

○ For some $\eta \in [0, 1]$ and $\bar{H}$, we would like to solve

$$\max_{I,G(\theta)} \mathbb{E}^Q [H(I)]$$

subject to

$$\mathbb{E}^P [H(I) + G(\theta)] - \sqrt{\left( \frac{\eta}{1-\eta} \right) \text{Var} (H(I) + G(\theta))} \geq \bar{H},$$

$$\mathbb{E}^Q [G(\theta)] = 0$$

and $\theta_t \in \mathcal{F}^X_t$ or $\theta_t \in \mathcal{F}_t$.  


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Problem Formulation

- For some $\eta \in [0, 1]$ and $\bar{H}$, we would like to solve

$$\max_{I, G(\theta)} \mathbb{E}^Q[H(I)]$$

subject to

$$\mathbb{E}^P[H(I) + G(\theta)] - \sqrt{(\frac{\eta}{1 - \eta}) \Var(H(I) + G(\theta))} \geq \bar{H},$$

$$\mathbb{E}^Q[G(\theta)] = 0 \quad \text{and} \quad \theta_t \in \mathcal{F}_t^X \quad \text{or} \quad \theta_t \in \mathcal{F}_t^\text{Complete Inf.}.$$  

- Two-step Optimization:

  **Step 1: (Hedging Policy)** For a fixed operating policy, $I$, we solve the hedging problem.

  $$V(I) \triangleq \max_{G(\theta)} \operatorname{MStd}_{\eta}(H(I) + G(\theta)) \quad \text{subject to} \quad \mathbb{E}^Q[G(\theta)] = 0.$$
Problem Formulation

- For some $\eta \in [0, 1]$ and $\bar{H}$, we would like to solve

$$\max_{I,G(\theta)} \mathbb{E}^Q[H(I)]$$

subject to

$$\mathbb{E}^P[H(I) + G(\theta)] - \sqrt{\left(\frac{\eta}{1 - \eta}\right) \text{Var}(H(I) + G(\theta))} \geq \bar{H},$$

$$\mathbb{E}^Q[G(\theta)] = 0 \quad \text{and} \quad \theta_t \in \mathcal{F}_t^X \quad \text{or} \quad \theta_t \in \mathcal{F}_t^C.$$  

- Two-step Optimization:

  **Step 1: (Hedging Policy)** For a fixed operating policy, $I$, we solve the hedging problem.

$$V(I) \triangleq \max_{G(\theta)} \text{MStd}_\eta(H(I) + G(\theta)) \quad \text{subject to} \quad \mathbb{E}^Q[G(\theta)] = 0.$$  

  **Step 2: (Operational Policy)** Then, we compute the optimal operational policy solving

$$\max_I \{\mathbb{E}^Q[H(I)]\} \quad \text{subject to} \quad V(I) \geq \bar{H}.$$
Incomplete Information Model: Step 1

**Theorem.** (Incomplete Information Hedging)

Let us denote by $\pi$ the Radon-Nikodym derivative of $Q$ with respect to $P$. Suppose that the hedging strategy $\theta$ is restricted to be $\mathcal{F}^X_T$-measurable. Then, the following two cases are possible:

**Case 1:** If $\eta \geq \frac{\mathbb{E}^P[\pi^2] - 1}{\mathbb{E}^P[\pi^2]}$ then

$$V(I) = \mathbb{E}^Q[H(I)] - \left[ \left( \frac{\eta - (1 - \eta)(\mathbb{E}^P[\pi^2] - 1)}{1 - \eta} \right) \text{Var}(H(I) - \mathbb{E}^P[H(I)|\mathcal{F}^X_T]) \right]^{\frac{1}{2}}.$$

**Case 2:** If $\eta < \frac{\mathbb{E}^P[\pi^2] - 1}{\mathbb{E}^P[\pi^2]}$ then

$$V(I) = +\infty.$$

**Remark:** In Case 2, financial trading has removed the risk management constraint.
Incomplete Information Model: Step 2

**Case 1:** If $\eta \geq \frac{\mathbb{E}_P[\pi^2] - 1}{\mathbb{E}_P[\pi^2]}$, then the (financially hedged) operational problem is

$$\max_I \mathbb{E}^Q[H(I)]$$

subject to

$$\mathbb{E}^Q[H(I)] - \left[ \left( \frac{\eta - (1 - \eta)(\mathbb{E}_P[\pi^2]-1)}{1 - \eta} \right) \mathbb{V} \mathbb{a} \mathbb{r}(H(I) - \mathbb{E}_P[H(I)|\mathcal{F}_T^X]) \right]^{\frac{1}{2}} \geq \bar{H}.$$

**Case 2:** If $\eta < \frac{\mathbb{E}_P[\pi^2] - 1}{\mathbb{E}_P[\pi^2]}$, then the (financially hedged) operational problem is

$$\max_I \mathbb{E}^Q[H(I)]$$

**Remarks:**

✓ Case 2 is the standard risk neutral operational model but under the “appropriate” EMM.
✓ If $H(I) \in \mathcal{F}_T^X$ then Case 1 reduces to

$$\max_I \mathbb{E}^Q[H(I)] \quad \text{s.t.} \quad \mathbb{E}^Q[H(I)] \geq \bar{H}. $$
The Retailer’s profit is \( H(I) = (P(I) - c) I \).

\( I \): Ordering Level at time 0.

\( P(I) \): Clearance Price at time \( T \) as a function of \( I \).

\( c \): Per unit purchasing cost.

**Problem:**

\[
\max_{I,G} \mathbb{E}^Q[H(I)] \quad \text{subject to} \quad \text{MStd}_\eta(H(I) + G) \geq \bar{H}.
\]

**Assumptions:**

\( X_t \sim (\mu, \sigma)\text{-GBM} : \quad dX_t = \mu X_t dt + \sigma X_t dW_{1t} \).

Linear Demand Model: \( P(I) = A - I, \quad A = \zeta + \gamma \ln(X_T) + \alpha W_{2T} \).
A Procurement Example (contd')

Solution without Financial Hedging

The Martingale Approach to Operational and Financial Hedging
Solution with Financial Hedging if $\eta \geq \frac{\mathbb{E}^P[\pi^2] - 1}{\mathbb{E}^F[\pi^2]}$

The Martingale Approach to Operational and Financial Hedging
A Procurement Example (contd')

\[ \frac{H^\text{opt}_c}{H^\text{opt}_u} \]

No Hedging

Incomplete

Complete

\[ \frac{H^\text{opt}_c}{H^\text{opt}_u} \]

Complete

No Hedging

Incomplete

The Martingale Approach to Operational and Financial Hedging
Conclusions

- **A modelling framework** for selecting operating and financial hedging strategies
  - Consistent with maximizing economic value of the firm.
  - Models risk preferences using risk management constraints.
  - The only motivation for financial trading is hedging in particular, no profit motivation for hedging.
  - Structural properties result in considerable tractability.
  - Considers different information structures.

- **Dynamic trading**
  - Reduces, and in some cases, eliminates the negative effects of risk constraints.
  - Improves the overall performance/efficiency of the operations.

- **Further Research**