HANDBOOK OF INTEGRATED RISK MANAGEMENT IN GLOBAL SUPPLY CHAIN
The Role of Financial Services in Procurement Contracts

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CONTENTS

Acknowledgments v
List of Figures vii
List of Tables ix

1 The Role of Financial Services in Procurement Contracts 1

1.1 Introduction 2
1.2 Model Description 6
1.2.1 Summary of Notation 9
1.3 Wholesale Contract with a Budget Constraint \((w_T, Q_T)\) 10
1.4 Equilibrium Under a Credit Contract \((Q_I, w_I, \alpha_i)\) 11
1.4.1 Retailer’s Optimal Ordering Strategy \((Q_I)\) 11
1.4.2 Supplier’s Optimal Contract \((w_I, \alpha_i)\) 14
1.4.3 Special Case: \(B = 0\) 18
1.5 Equilibrium with External Financing \((Q_E, w_E)\) 20
1.6 Computational Experiments 23
1.7 Concluding Remarks and Extensions 30
1.8 Appendix: Main Proofs 35

References 41
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# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Effect of limited liability on retailer’s payoff</td>
<td>13</td>
</tr>
<tr>
<td>1.2</td>
<td>Supplier’s payoff function</td>
<td>17</td>
</tr>
<tr>
<td>1.3</td>
<td>Equilibrium wholesale price and ordering level</td>
<td>24</td>
</tr>
<tr>
<td>1.4</td>
<td>Retailer’s and supplier’s equilibrium payoffs</td>
<td>25</td>
</tr>
<tr>
<td>1.5</td>
<td>Supply chain equilibrium payoff</td>
<td>26</td>
</tr>
<tr>
<td>1.6</td>
<td>Wholesale price and supplier’s payoff</td>
<td>27</td>
</tr>
<tr>
<td>1.7</td>
<td>Retailer’s true and reported budget</td>
<td>34</td>
</tr>
<tr>
<td>1.8</td>
<td>Retailer and supplier expected payoffs as function of the retailer’s true budget</td>
<td>35</td>
</tr>
</tbody>
</table>
LIST OF TABLES

1.1 Retailer’s optimal ordering level 14
1.2 Retailer and supplier payoffs for the external financing and traditional wholesale contracts 28
1.3 Competition penalty 30
In this chapter we investigate the interplay between operational and financial decisions within the context of a two-echelon supply chain. A retailer purchases a single product from a supplier and afterwards sells it in the retail market to a random demand. The retailer, however, is budget-constrained and is therefore limited in the number of units that he may purchase from the supplier. We study two alternative forms of financing that the retailer can use to overcome the limitations imposed by the budget constraint. First, we consider the case of internal financing in which the supplier offers financial services to the retailer in the form of a procurement contract. The type of contracts that we consider allows the retailer to pay in arrears a fraction of the procurement cost after demand is realized. Second, we consider the case of external financing in which a third party financial institution (e.g., a bank) offers a commercial loan to the retailer. Our results show that the performance of the entire supply chain can be severely affected by the lack of financing and that it is in the interest of both players to find ways to finance the retailer’s operations. We also show that, for the most part, both the supplier and the retailer (and the entire supply chain) are better off by using internal financing rather than by relying on external financing. From the supplier’s point of view, we show that the value of offering internal financing
decreases with the size of the retailer’s initial budget. This is despite of the fact that the risk of the retailer defaulting on his contractual obligations decreases with his initial budget. Interestingly, from the retailer’s standpoint the value of internal financing is non-monotonic in his budget and there is an intermediate value at which his expected payoff is maximized.

1.1 INTRODUCTION

A key function of supply chain management is the effective coordination of material flows, information flows and financial flows across different organizations. For the most part, the research in operations management has focused on optimizing the interplay between material flows and information flows ignoring the issue of coordinating material and financial flows. In practice, however, companies have limited working capital to operate and financial constraints play a pivotal role in determining firms’ production and procurement decisions. In the global supply chain, and specially during financial crisis and economic downturns, these financial constraints are strengthened by common cash-management practices that promote collecting account receivables as quick as possible while postponing payments to providers and suppliers. This is not only true for small start-up companies with limited access to credit but also for large corporations. However, this ‘war for cash’ (see [1]) puts extra pressure on small companies that usually find themselves making suboptimal procurement decisions to balance their cash reserves increasing their credit risk and their chances of getting out of business.

In this chapter we investigate the connection between operational and financial decisions within the context of a two-echelon supply chain. We consider a stylized model in which a retailer purchases a single product from a supplier and afterwards sells it in the retail market to a random demand. The retailer has a fixed budget that limits his procurement decisions. In this setting, we propose and analyze two alternative forms of financing that the retailer can use to overcome the limitations imposed by the budget constraint. First, we consider the case of what we call internal financing in which the supplier offers financial services to the retailer in the form of a procurement contract. The type of credit contracts that we consider allow the retailer to pay in arrears a fraction of the procurement cost after demand is realized. By offering this option to delay payments, the supplier is effectively allowing the retailer to order more so as to take advantage of a possible upside on market demand. On the downside, the supplier is also internalizing some of the retailer’s demand risk. The second form of financing that we consider is what we call external financing in which a third party financial institution (e.g., a bank) offers a commercial loan to the retailer. In this case, the supplier offers a traditional wholesale contract in which the retailer must pay in advance (possibly using the bank’s loan) the totality of the procurements
costs. In contrast to the credit contract, in this case it is the bank, and not the supplier, who is getting exposed to the retailer’s credit risk.

The main goal of this chapter is to shed some light on how financial services impact agents’ operational and financial decisions and how they should be designed and used to create value in a supply chain environment. As their names suggest, our choice of these two contrasting forms of financing, internal and external, is intended to highlight how this process of value creation is affected by the level of involvement that the agent providing the financing has within the supply chain. For instance, in the case of internal financing, it is in the supplier best interest to support the retailer’s business as this will have a direct impact on the supplier’s own business. On the other hand, a bank providing external financing has no formal stakes in the supply chain and so should be less concern with its operations as long as the loan and interests are repaid.

Our particular interest in the internal financing model is motivated by some successful applications in real business operations. As an example, consider the case of Chinese Material Shortage Transportation Group (CMST), one of the largest logistics enterprise in China. It is common for many small and medium size paper manufacturers in mainland China to purchase materials from international suppliers. While most of these companies have limited working capital, it is still the case that the financial system is unable to provide adequate services to support their businesses. As a result, most of these small manufacturers find themselves making suboptimal procurement decisions. CMST viewed this gap as a business opportunity and, since 2002, started financing these small paper manufacturers to buy paper materials from international suppliers while simultaneously providing the logistics services required in these transactions. The financial service offered by CMST is essentially a credit contract under which the paper manufacturer pays a fraction of the wholesale price charged by the supplier as a deposit and CMST covers the difference. Then, the manufacturer repays CMST the remaining fraction of the wholesale price when the sale season is over. With this supply chain financing service, CMST has become one of the leading logistics-financing service providers in China. In 2006, CMST financing supply chain business was about 1.1 billion dollars, up from 750 million dollars in 2005. Currently, this operations is one of most popular modes of financing for small-and-medium companies in China.

As the CMST example unveils, bringing financial services into supply chain management has the potential to improve the operational efficiency and the profits of the entire supply chain. It is because of this value-creation capacity that we expect financial services to play an even more predominant role in the future growth of global supply chain management and it is precisely this fact that motivates this research.

We conclude the introduction by attempting to position our chapter within—but not reviewing—the vast literature on supply chain management. We refer the reader to the books by [2] and [3] for a general overview of supply chain
management issues and to the survey article by [4] for a review of supply chain management contracts.

One of the distinguishing features of our model with respect to most of the existing literature in supply chain management is the consideration of a budget constraint that limits the retailer’s procurement capacity. A few recent exceptions are the papers by [5], [6], [7], [8], and [9].

[9] analyze a single-period newsvendor model which is used to illustrate how a firm’s inventory decisions are affected by the existence of a budget constraint and the firm’s capital structure (debt/equity ratio). [10] use a multi-period production model to examine the interdependence of a firm’s capital structure and its short-term operating decisions concerning inventory, dividends, and liquidity. In a similar setting, [7] consider a budget-constrained newsvendor that can borrow from a bank that acts strategically when choosing the terms (interest rate) of the loan. The paper characterizes the Stackelberg equilibrium and investigates conditions under which channel coordination (i.e., the budget-constrained newsvendor orders the same amount than the unconstrained newsvendor) can be achieved.

[5] consider a deterministic multi-period production/inventory control model and investigate the interplay between inventory decisions and asset-based financing. In their model, a retailer finances its operations by borrowing from a commercial bank. The terms of the loans are structured contingent upon the retailer’s balance sheets and income statements (in particular, inventories and receivables). The authors conclude that asset-based financing allows retailers to enhance their cash return over what it would be if they were only able to use their own capital.

The work by [6] and [8] are the most closely related to this chapter. They both consider a two-echelon supply chain system in which the retailer is budget constrained and investigate different types of procurement contracts between the agents using a Stackelberg equilibrium concept. In [6], the supplier offers a menu of wholesale contracts (with different execution times and wholesale prices) and the retailer choose the optimal timing at which to execute the contract. The reason why the timing of the contract is important is because in their model the retailer’s demand is partially correlated to a financial index that the retailer (and the supplier) can track. As a result, the retailer can dynamically trade in the financial market to adjust his budget to make it contingent upon the evolution of the index and choose the appropriate time at which to execute the contract with the supplier. Their model shows how financial markets can be used as a source of public information upon which procurement contracts can be written and as a means for financial hedging to mitigate the effects of the budget constraint.

Similar to our work, in [8] the supplier takes a pro-active role in offering different type of contracts designed to provide financial services to the budget-constrained retailer. The authors analyze a set of alternative financing scheme including supplier early payment discount, open account financing, joint supplier financing with bank, and bank financing. They conclude that
in an optimally designed scheme it is in the supplier’s best interest to offer financing to the retailer and the retailer will always prefer financing from the supplier rather than the bank. We reach similar conclusions in our setting. A noticeable difference between their formulation and ours is that in our model we impose a budget constraint at the time the contract is signed that explicitly limits the retailer’s ordering decision. With the possibility of the retailer declaring bankruptcy, this constraint limits the supplier’s default risk exposure.

Finally, it is worth mentioning that there exits a somehow related stream of research that investigates the use of financial markets and financial instruments (such as forwards and futures contracts, options, swaps, etc) to hedge operational risk exposure (see [11] for a detailed review). For example, [12] consider the general problem of dynamically hedging the profits of a risk-averse corporation when these profits are partially correlated with returns in the financial markets. [13] consider a risk-averse newsvendor model in which demand is perfectly and partially correlated with a marketable risky security. In both cases, they show how the retailer can hedge demand uncertainty by trading on the risky security. [14] examine the joint impact of operational flexibility and financial hedging on a firm’s performance and their complementary/substitutability with the firm’s overall risk management strategy. [15] and [16] examine the interaction of operational and financial decisions from an integrated risk management standpoint. [17] analyze the effect of capital market imperfections on a firm’s operational and financial decisions in a capacity investment setting. The authors consider that the firm can use tradeable asset’s forward contracts and commercial loans to relax its budget constraint. [18] propose an infinite-horizon discounted Markov decision process in which an IPO event is treated as a stopping time. The value of the IPO is modeled as a random variable whose distribution depends on the firm’s current assets, its most recent sales revenue, and its most recent profits. Every period the firm must decide on capacity expansion, production, and loan size. They characterize an optimal capacity-expansion and financing policy so as to maximize the expected present value of the firm’s IPO. [19] study how trade credit financing affect the relationship among firms in supply chain, supplier selection, and supply chain performance.

The rest of this chapter is organized as follows. In the next section we present the mathematical formulation including the retailer and supplier payoff functions, the main features of the credit contract and some fundamental assumptions about the market demand. We also characterize the Stackelberg equilibrium under a traditional wholesale contract which we use as a benchmark for comparison throughout the chapter. In Section 1.4 we investigate the optimal design of the credit contract using a non-cooperative game theoretical framework. First, we compute the retailer best response ordering strategy as a function of the parameters of the credit contract. Then, we solve the supplier’s optimization problem to determine the optimal parameters of this contract. We conclude this section analyzing in detail the extreme case.
in which the retailer has no initial budget. Section 1.5 is devoted to the study of the external financing model. We first define the notion of a feasible loan in a competitive financial market environment. We then use this concept to solve for the optimal loan. We show that the outcome of the game in this case is equivalent to a model in which the retailer has an unlimited budget (a reminiscence of the Modigliani and Miller’s irrelevance principle). Section 1.6 presents a set of numerical experiments that we use to highlight the main features and insights of our model. In these experiments, we compare the outcome of the internal and external financing models from the point of view of the agents’ payoffs as well as the entire supply chain efficiency. We also investigate the impact of demand variability on the outcome of the game and the choice of the best contract. Concluding remarks and possible extensions of our model are discussed in Section 1.7. In particular, we emphasize the extension that considers the case in which the retailer’s initial budget is private knowledge and propose some simple variations of the credit contract that could be used to partially solve the supplier’s adverse selection problem.

1.2 MODEL DESCRIPTION

We model an isolated portion of a competitive supply chain with one supplier that produces a single product and one retailer that faces a random market demand $D$ that is realized at a future time $T$. We assume that $D$ is a nonnegative random variable with distribution function $F(D)$. We will make the following assumptions about $F$ throughout the chapter.

Assumption 1 The demand distribution function $F$ satisfies the following properties:

(i) It has a smooth density $f(D) > 0$ in $(a, b)$, for $0 \leq a \leq b \leq \infty$, 

(ii) It has a finite mean, and 

(iii) Its hazard function $h(D) := f(D)/\bar{F}(D)$ is increasing in $D \geq 0$, where $\bar{F}(D)$ is the tail distribution $\bar{F}(D) := 1 - F(D)$.

These are not particularly restrictive requirements on $F(D)$ that we impose to guarantee the existence and uniqueness of an equilibrium (see Lemma 1 below).

A distinctive feature of our model is that the retailer is restricted by a budget constraint that limits his ordering decisions. In particular, we assume that the retailer has an initial budget $B$ that may be used to purchase product units from the supplier. On the other hand, we assume that the supplier has

$^{1}$Similar models are discussed in detail in Section 2 of [4], See also [20].

$^{2}$They are satisfied by most popular distributions such as the Uniform, Log-Normal, Exponential and Weibull, among many others (see [21]).
deep pockets, that is, enough working capital to pay for manufacturing costs independent of the size of the production batch.

At time \( t = 0 \), the retailer and supplier negotiate the terms of a procurement contract that specifies the following quantities:

- **Credit Contract** \((w, \alpha)\): where \( w \) is the wholesale price per unit and \( \alpha \in [0, 1] \) is the fraction of this wholesale price that the retailer must pay in advance at time \( t = 0 \). We will refer to \( \alpha \) as the credit parameter.

- **Order Quantity** \( Q \): The number of units purchased by the retailer.

We assume that in the negotiation of this contract the supplier acts as a Stackelberg leader. That is, at \( t = 0 \) the supplier moves first and proposes the contract \((w, \alpha)\), to which the retailer then reacts by selecting the ordering level \( Q \) and paying the supplier the amount \( \alpha w Q \). The remaining portion \((1 - \alpha) w Q \) is paid in arrears at time \( t = T \), after the value of \( D \) is realized and the retailer’s revenues are collected. These revenues are equal to \( p \min\{D, Q\} \), where \( p \) is a fixed retail price. Neither a salvage value nor a return policy for unsold units are considered in this model. For notational convenience, we will normalized all prices in this economy so that \( p = 1 \). As a result, \( w, c \) and \( B \) are measured relative to the retail price.

The implication of the retailer’s limited budget on the execution of the contract is twofold. First, the order quantity placed at \( t = 0 \) must satisfy the budget constraint \( \alpha w Q \leq B \). Second, if the realized demand \( D \) is sufficiently low then the retailer will be unable to pay the supplier the full amount \((1 - \alpha) w Q \) due at time \( T \). In this case (which occurs if \( \min\{D, Q\} + B < w Q \)) the retailer declares bankruptcy and the supplier collects \( \min\{D, Q\} + B - \alpha w Q \) instead of \((1 - \alpha) w Q \). Hence, it follows that the supplier faces a trade-off when selecting the optimal credit contract \((w, \alpha)\) to offer. On one hand, the supplier would like to choose a small \( \alpha \) to minimize the impact of the budget constraint so as to boost the retailer’s ordering level. Indeed, by choosing the contract \((w, \alpha)\) the supplier is effectively offering the retailer a loan of \((1 - \alpha) w Q \) to procure more units. On the other hand, the supplier would like to choose a large \( \alpha \) to minimize the credit risk associated with the retailer defaulting at time \( T \). We will discuss in full detail this trade-off in the following section.

In terms of who knows what, we consider for most part of this chapter the symmetric information case where all information is common knowledge. In particular, we assume that the supplier knows the retailer budget \( B \), the demand probability distribution \( F(D) \), and the retail price. We will discuss this symmetric information assumption in Section 1.7 where we discuss the case in which the retailer’s budget \( B \) is private information.

For a given contract \((w, \alpha)\), we define the retailer’s net expected payoff as a function of \( Q \) to be equal to \( \pi^r_\alpha(Q) := E\left[\left(\min\{D, Q\} + B - w Q\right)^+ - B\right] \), where \( E[\cdot] \) denotes expectation with respect to \( F \). (We use the subscript/superscript ‘\( r \)’—which stands for Internal financing— to denote quantities related to the credit contract.) Note that in our definition of \( \pi^r_\alpha \) we have subtracted the
The role of financial services in procurement contracts

Initial budget $B$ from the retailer's profits. Hence, $\pi^n$ measures the net contribution to earnings that the retailer gets by operating in this supply chain. For example, with this definition if the retailer chooses $Q = 0$ then his net payoff would be 0 reflecting the fact that he has gained nothing from his retail business. The retailer's optimal net expected payoff is obtained solving

$$\Pi^R = \max_{Q \geq 0} \pi^n(Q) = \max_{Q \geq 0} \mathbb{E} \left[ \left( \min\{D, Q\} + B - wQ \right)^+ \right] - B \quad (1.1)$$

subject to $wQ \leq B \quad (1.2)$

The positive part in the definition of $\Pi^R$ captures the retailer's limited liability in case of bankruptcy.

Let us denote by $Q^I$ the optimal solution to (1.1)-(1.2), which represents the retailer's best response to the supplier contract $(w, \alpha)$. Naturally, $Q^I$ depends on $w, \alpha$ and $B$. When we wish to emphasize this dependence we will include these quantities as part of its argument. For example, we will write $Q^I(w, \alpha)$ when discussing the dependence of the optimal ordering level on the terms of the contract $(w, \alpha)$. A similar convention applies to $\Pi^R$ and other quantities that we introduce below.

Under the common knowledge assumption, the supplier is able to anticipate the retailer's best response $Q^I$. As a result, the supplier chooses an optimal credit contract by solving

$$\Pi^S = \max_{w, \alpha} \mathbb{E} \left[ (w - c)Q^I(w, \alpha) - \left( \min\{D, Q^I(w, \alpha)\} + B - wQ^I(w, \alpha) \right)^- \right], \quad (1.3)$$

where $c$ is the per unit manufacturing cost incurred by the supplier. Let us denote by $w^I(B)$ and $\alpha^I(B)$ the optimal solution to (1.3). Since we have assumed that the supplier is not budget constrained, $\Pi^S$ represents the supplier's operating profits which can be negative.

To ensure the operability of the supply chain the market price must exceed the manufacturing cost, that is, $c \leq 1$. The difference $1 - c$ represents the net margin per unit sold made by the entire supply chain. This margin is split into $1 - w$ that goes to the retailer and $w - c$ that goes to the supplier. Naturally, we expect in equilibrium the supplier to set $w$ so that $c < w \leq p$ creating the so-called double marginalization inefficiency (e.g., [22]). We will measure this inefficiency by computing the competition penalty, which is defined as one minus the decentralized supply chain payoff divided by the centralized supply chain payoff (see [23]). That is,

$$\mathcal{P} := 1 - \frac{\Pi^R + \Pi^S}{\Pi^C}, \quad (1.4)$$

where $\Pi^C$ is the optimal centralized payoff obtained by a central planner that owns both the supply and retail operations. (We will use a subscript/superscript ‘C’ to denote quantities related to this centralized solution.) In order to have a fair comparison between the centralized and decentralized operations, we
need to make certain assumptions about $\Pi^C$. In particular, we will assume
that the central planner, like the supplier, has enough working capital to pay
for the manufacturing costs. Hence, using this centralized solution as a bench-
mark provides a measure to compute the benefits of vertical integration. It
follows that

\[
\Pi^C = \max_{Q \geq 0} \mathbb{E} \left[ \min\{D, Q\} - cQ \right]
\]

and the optimal centralized inventory level is $Q_C = \bar{F}^{-1}(c)$.

The following are some additional remarks about the model.

1. The credit parameter, $\alpha$, allows to transfer part of the demand risk
from the retailer to the supplier. Indeed, if $\alpha = 0$ then the supplier is
effectively offering the retailer the option to buy as many units as he
wants and to pay for them in arrears after demand is realized. This can
be a risky strategy for the supplier, specially if the retailer overestimates
market demand. On the other hand, when $\alpha = 1$ the supplier faces no
risk since all payments are made at time $t = 0$.

2. In our formulation of $\Pi^R$ we have implicitly assumed that the retailer
has no other investment opportunity besides his retail operations. As a
result, his excess wealth $B - \alpha wQ$ at time $t = 0$ generates no interest at
time $t = T$. Alternatively, we can think that the retailer has access to
a risk-free cash account with zero interest rate $r_f = 0$. The reason why
we restrict the retailer’s investment opportunities is to isolate the role
that financial supply chain management (represented here by our credit
contract) plays as a value generating activity. We refer the reader to [6]
for a model that explicitly considers the existence of financial markets
as an alternative investment opportunity for the retailer.

3. It is worth noticing that the traditional wholesale contract (e.g. see [4]),
in which the retailer must pay the full procurement cost $wQ$ at time $t = 0$, is a special case of our credit contract with $\alpha = 1$. Because of the
popularity of the wholesale contract in both research and practice, we
will use it throughout this chapter as a benchmark for comparisons.

### 1.2.1 Summary of Notation

For future references, and to help the reader keep track of the different com-
ponents of our model, let us summarize here some of the notation that we will
use throughout the chapter. Additional notation will be introduced later on
as needed.

- $H(Q) := Q h(Q)$, the generalized failure rate function of the demand $D$.
- $\bar{w} := \arg\max\{w \ \bar{F}^{-1}(w) : w \in [0, 1]\}$.
- $w_{\mu} := \arg\max\{(w - c) \ \bar{F}^{-1}(w) : w \in [c, 1]\}$. 
THE ROLE OF FINANCIAL SERVICES IN PROCUREMENT CONTRACTS

- \( \hat{w}(B) := \max\{ w \in [0, 1] : w \bar{F}^{-1}(w) \geq B \} \) if \( B \leq \hat{w} \bar{F}^{-1}(\hat{w}) \).
- \( \underline{w}(B) := \min\{ w \in [0, 1] : w \bar{F}^{-1}(w) \geq B \} \) if \( B \leq \underline{w} \bar{F}^{-1}(\underline{w}) \).
- \( \bar{Q} := \bar{F}^{-1}(\bar{w}) \).
- \( \bar{Q}(w, B) \) solves \( \bar{F}(\bar{Q}) = w \bar{F}(\bar{Q} - B) \).
- \( Q_c := \bar{F}^{-1}(c) \).
- \( Q_e := \bar{F}^{-1}(w_e) \).
- \( \hat{B} := \hat{w} \bar{Q} \).
- \( B_e := w_e Q_e \).

It follows from Assumption 1 that function \( w \bar{F}^{-1}(w) \) and \( (w - c) \bar{F}^{-1}(w) \) are unimodal. As a result, \( w \bar{F}^{-1}(w) \geq B \) for all \( w \in [w(B), \bar{w}(B)] \). Also, one can show that \( \bar{Q} \) solves \( H(\bar{Q}) = 1 \). Such a solution exists and is unique since, by Assumption 1, \( H(\bar{Q}) \) is an increasing function such that \( H(0) = 0 \) and \( \lim_{Q \to \infty} H(\bar{Q}) > 1 \) (see Theorem 2 in [21]). The existence and uniqueness of \( \bar{Q}(w, B) \) are discussed below in Lemma 1.

1.3 WHOLESALE CONTRACT WITH A BUDGET CONSTRAINT \((w_T, Q_T)\)

Let us conclude this section reviewing the Stackelberg-Nash equilibrium for the traditional wholesale for the case in which the retailer is budget constrained. We will use a subscript/superscript ‘T’ to denote quantities related to the market equilibrium under this contract such as the wholesale price \( w_T(B) \), the retailer’s ordering level \( Q_T(B) \), and the agents’ expected payoff \( \Pi_R^T(B) \) and \( \Pi_S^T(B) \), as a function of the retailer’s budget \( B \).

To compute this equilibrium, we first solve the retailer’s optimization problem which is given by

\[
\max_{0 \leq Q \leq B/w} \mathbb{E}[\min\{D, Q\} - w Q].
\]

This is a newsboy problem and the optimal ordering level is given by \( Q^*(w, B) = \min\{B/w, \bar{F}^{-1}(w)\} \). It follows from the definition of \( \bar{w}(B) \) and \( \underline{w}(B) \) in the previous section that \( Q^*(w, B) = B/w \) if and only if \( w \in [\underline{w}(B), \bar{w}(B)] \).

Based on the retailer’s optimal ordering level, the supplier chooses the wholesale price that maximizes his payoff, that is,

\[
w_T(B) = \arg\max_{c \leq w \leq 1} \{ (w - c) Q^*(w, B) \}.
\]

Using the results in [20] we can show that the first-order optimality condition is also sufficient when \( F \) has increasing generalized failure rate (IGFR). In our
case, Assumption 1 ensures that $F$ has IGFR. An alternative characterization of the wholesale price $w_r(B)$ is given by $w_r(B) = \max \{w_e, \bar{w}(B)\}$. The ordering quantity is given by

$$Q_r(B) = \min \left\{ \frac{B}{w_r(B)}, \bar{F}^{-1}(w_r(B)) \right\}$$

and the retailer and supplier equilibrium payoffs are

$$\Pi^R_r(B) = \mathbb{E} \left[ \min \{D, Q_r(B)\} - w_r(B) Q_r(B) \right]$$

and

$$\Pi^S_r(B) = (w_r(B) - c) Q_r(B).$$

The following result summarizes some useful properties of the equilibrium under the traditional wholesale contract as a function of $B$. The proof is straightforward and it is omitted.

**Proposition 1** The wholesale price $w_r(B)$ is decreasing in $B$ while $Q_r(B)$, $\Pi^R_r(B)$ and $\Pi^S_r(B)$ are all increasing in $B$. In particular, the wholesale price and ordering level satisfy

$$(w_r(B), Q_r(B)) = \begin{cases} (1, 0) & \text{if } B = 0 \\ (w_e, Q_e) & \text{if } B \geq B_e. \end{cases}$$

According to this result, the budget $B_e$ is the minimum budget that the retailer needs to purchase the optimal unconstrained quantity $Q_e$. The budget $B_e$ will be useful to formalize the notion of a large budget (see Proposition 5). The subscript/superscript ‘E’ stands for **E**xternal financing, and the reason for this choice is that (as we will see in Section 1.5) $w_e$ and $Q_e$ are the equilibrium wholesale price and production level when the retailer uses a third party financial institution to finance his operations instead of the credit contract.

### 1.4 EQUILIBRIUM UNDER A CREDIT CONTRACT ($Q_i, w_i, \alpha_i$)

In this section, we characterize the equilibrium for a supply chain that operates under the credit contract described in the previous section. As it is customary when determining a Stackelberg-Nash equilibrium, we first compute the follower’s best response as a function of an arbitrary strategy selected by the leader. That is, we start solving the retailer’s optimization problem in (1.1)-(1.2) to find $Q_i(w, \alpha)$ for a fixed contract ($w, \alpha$). Then, we will plug this solution into the supplier’s optimization in (1.3) to compute the optimal contract ($w_i, \alpha_i$).

#### 1.4.1 Retailer’s Optimal Ordering Strategy ($Q_i$)

In the process of computing the equilibrium we would like to be able to identify the main differences between our solution and existing results on procurement
contracts. In particular, we would like to understand the impact that both the retailer’s budget constraint and limited liability have on \( Q \). For this, we find convenient to isolate these two effects by considering the following optimization problem

\[
\max_{Q \geq 0} \mathbb{E} \left[ \left( \min(D, Q) + B - wQ \right)^+ - B \right], \tag{1.6}
\]

which corresponds to the retailer’s original problem without the budget constraint \( wQ \leq B \). Let \( \tilde{Q} \) be the solution to this unrestricted problem. Note that we can interpret (1.6) as the retailer’s problem if the supplier offers the contract \((w, 0)\) with full financing at time 0, and so \( \tilde{Q}(w, B) = Q_r(w, 0, B) \).

The following is a useful intermediate step in our characterization of \( Q \).

**Lemma 1** Suppose \( F \) satisfies the conditions in Assumption 1. Then, if \( w < 1 \) there exists a unique nonnegative solution \( \tilde{Q}(w, B) \) to (1.6) that solves

\[
\tilde{F}(\tilde{Q}) = w \tilde{F}(w \tilde{Q} - B).
\]

\( \tilde{Q}(w, B) \) is a decreasing function of both \( w \) and \( B \), and satisfies

\[
\tilde{Q}(w, B) = \tilde{F}^{-1}(w), \quad \text{for all } B \geq w \tilde{F}^{-1}(w).
\]

The function \( w \tilde{Q}(w, B) \) is unimodal in \( w \in [c, 1] \) and attains its maximum at \( w_0 \) such that \( \tilde{Q}(w_0, B) = \hat{Q} \).

**Proof:** See the Appendix at the end. \( \square \)

The reason to introduce \( \tilde{Q} \) is twofold. On one hand, it will be useful in our characterization of \( Q \) in Proposition 2. Second, it represents the optimal inventory level for a retailer that has exclusively limited liability and no budget constraint when ordering at \( t = 0 \).

To get some intuition about the value of \( \tilde{Q}(w, B) \), recall that in the traditional newsvendor model (e.g., [24]), with full liability, the retailer’s optimal solution solves the familiar fractile equation \( \tilde{F}(Q) = w \) (or equivalently, \( Q = \tilde{F}^{-1}(w) \)). This is a first-order optimality condition that requires the marginal revenue of an extra unit, \( \tilde{F}(Q) \), to be equal to the marginal cost of this extra unit, \( w \). In our case, the value of \( \tilde{Q} \) solves a modified fractile equation \( \tilde{F}(\tilde{Q}) = w \tilde{F}(w \tilde{Q} - B) \). Since \( w \geq w \tilde{F}(w \tilde{Q} - B) \), it follows that a retailer with limited liability faces lower marginal costs and so \( \tilde{Q} \geq \tilde{F}^{-1}(w) \).

This is not particularly surprising if we interpret this limited liability as an option that reduces the retailer’s downside risk associated with low demand outcomes. Interestingly, this marginal cost \( w \tilde{F}(w \tilde{Q} - B) \) increases with the retailer’s budget \( B \). As a result, \( \tilde{Q} \) decreases with \( B \). Hence, for \( B \) sufficiently large \( Q_r = \hat{Q} \) and so the retailer optimal ordering level decreases with his initial budget. Intuitively, this happens because retailers with smaller budgets have less stakes at risk when choosing their inventory levels (as their potential loses are bounded by \( B \)) and so they order more aggressively to take full advantage of high demand outcomes.
Figure 1.1 Retailer’s net payoff $\pi^h(D, Q) = (\min\{D, Q\} + B - wQ)^+ - B$ as a function of $D$ for a fixed order $Q$ and three different budgets $0 = B_0 < B_1 < B_2$.

Figure 1.1 depicts the retailer net payoff as a function of $D$ for a fixed order $Q$ and three different budgets $0 = B_0 < B_1 < B_2$. As we can see, the retailer’s downside loses increase with $B$ while his upside gains remain constant, all else being equal. This asymmetry in the retailer’s payoff explains why small-budget retailers have the incentives to order more than their high-budget counterparts so as to take advantage of high demand scenarios.

Based on Lemma 1 we derive in the following Proposition the optimal ordering level for the retailer.

**Proposition 2** Suppose the retailer has a budget $B$ and the supplier chooses the contract $(w; \alpha)$. Then, under the assumptions on Lemma 1 the retailer’s optimal solution $Q_I$ satisfies:

$$Q_I(w; \alpha, B) = \min \left\{ \tilde{Q}(w, B), \frac{B}{\alpha w} \right\}.$$  

Proof: See the Appendix at the end. □

Table 1.1 provides a summary of the possible values of $Q_I$ as well as a comparison with $\bar{F}^{-1}(w)$ as a function of $B$. (Recall that $w \bar{F}^{-1}(w)$ represents the minimum budget required to finance the optimal non-cooperative solution $\bar{F}^{-1}(w)$ under a wholesale contract with no budget constraint.)

It follows from Table 1.1 that a budget constrained retailer orders less than his unconstrained counterpart only if $B \leq \alpha w \tilde{F}^{-1}(w)$. However, we will show below (in Corollary 1) that this is never the case under an optimal contract $(w_i, \alpha_i)$. Also, according to Table 1.1, $w \tilde{F}^{-1}(w)$ is an upper bound on how much a retailer is willing to pay the supplier at $t = 0$, that is, $\alpha w Q_I(w, \alpha) \leq w \tilde{F}^{-1}(w)$ for all contract $(w, \alpha)$. However, this condition does not rule out
Table 1.1 Retailer’s optimal ordering level as a function of $B$.

<table>
<thead>
<tr>
<th>Budget Range</th>
<th>$Q_i$</th>
<th>$Q_i$ v.s. $F^{-1}(w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>$B \leq \alpha w , \tilde{F}^{-1}(w)$</td>
<td>$B/(\alpha w)$</td>
</tr>
<tr>
<td>Small to Medium</td>
<td>$\alpha w , \tilde{F}^{-1}(w) \leq B \leq \alpha w , \check{Q}(w)$</td>
<td>$B/(\alpha w)$</td>
</tr>
<tr>
<td>Medium to Large</td>
<td>$\alpha w , \check{Q}(w) \leq B \leq w , \tilde{F}^{-1}(w)$</td>
<td>$\check{Q}(w)$</td>
</tr>
<tr>
<td>Large</td>
<td>$w , \tilde{F}^{-1}(w) \leq B$</td>
<td>$\check{F}^{-1}(w)$</td>
</tr>
</tbody>
</table>

that $w \, Q_i(w, \alpha) > w \, \tilde{F}^{-1}(w)$. Indeed, we will show that in equilibrium this strict inequality does hold when $B$ is relatively small.

Table 1.1 was built for a fixed contract $(w, \alpha)$. In the following section we characterize the supplier’s optimal credit contract $(w_i, \alpha_i)$ and show that at optimality $\alpha_i \, w_i \, \check{F}^{-1}(w_i) \geq B$. Hence, it is in the supplier best interest to select a contract that induces the retailer to take advantage of the credit line offered by the supplier.

### 1.4.2 Supplier’s Optimal Contract $(w_i, \alpha_i)$

It follows from equation (1.3) that the supplier’s optimal contract solves

$$
\Pi^*_S = \max_{w \in [c, 1], \alpha \in [0, 1]} \mathbb{E} \left[ (w - c) \, Q_i(w, \alpha) - \left( D + B - w \, Q_i(w, \alpha) \right)^+ \right]
= \max_{w \in [c, 1], \alpha \in [0, 1]} \left\{ (w - c) \, Q_i(w, \alpha) - \int_0^{\left( w \, Q_i(w, \alpha) - B \right)^+} \check{F}(D) \, dD \right\}.
$$

We compute $(w_i, \alpha_i)$ in two steps. First, we determine the optimal value $\alpha_i(w)$ for a fixed value of $w$. Then, we compute the optimal wholesale price $w_i$.

According to Proposition 2, the retailer’s optimal ordering decision $Q_i(w, \alpha)$ is given by

$$
Q_i(w, \alpha) = \min \left\{ \check{Q}(w), \frac{B}{\alpha w} \right\}, \text{ where } \check{Q}(w) \text{ solves } \check{F}(\check{Q}) = w \, \check{F}(w \, \check{Q} - B).
$$

For a fixed $w$, let us define $\alpha(w, B) := B/(w \, \check{Q}(w))$. It follows that if $\alpha \leq \alpha(w, B)$ then $Q_i(w, \alpha) = \check{Q}(w)$ which is independent of $\alpha$. Hence, without a significant loss in generality, we can assume that at optimality $\alpha_i(w) \geq$
min\{1, \tilde{\alpha}\}$. In particular, when $B$ is large (i.e., $B \geq w \tilde{F}^{-1}(w)$), $\tilde{\alpha} \geq 1$ and so $\alpha_{i}(w) = 1$. That is, in this case the supplier does not need to offer financial support to the retailer. On the other hand, for $B \leq w \tilde{F}^{-1}(w)$ we have $\tilde{\alpha} \leq 1$ and we can restrict the search for an optimal $\alpha_{i}(w)$ to the domain $[\tilde{\alpha}(w), 1]$.

In this range $Q_{i}(w, \alpha) = B/(\alpha w)$ and the supplier’s problem reduces to

$$
\Pi^s = \max_{\alpha \in [\alpha(w), 1]} \left(1 - \frac{c}{w}\right) \frac{B}{\alpha} - \mathbb{E}\left[D - \left(1 - \alpha\right) \frac{B}{\alpha}\right], \quad (B \leq w \tilde{F}^{-1}(w)).
$$

**Proposition 3** Suppose $F$ satisfies the conditions in Assumption 1. Then, for a fixed $w$ such that $c < w < 1$ the optimal $\alpha_{i}(w, B)$ is given by

$$
\alpha_{i}(w, B) = \max\left\{\frac{B}{B + \tilde{F}^{-1}\left(\frac{c}{w}\right)}, \frac{B}{B + \tilde{F}^{-1}\left(\frac{1}{w} \tilde{Q}(w)\right)}\right\} \in [0, 1].
$$

In particular, $\alpha_{i}(w, B) = 1$ if and only if $B \geq w \tilde{F}^{-1}(w)$.

**Proof:** See the Appendix at the end. $\square$

It follows from Proposition 3 –and the fact that $\tilde{Q}(w)$ decreases with $B$—that $\alpha_{i}(w, B)$ is monotonically increasing in the retailer’s budget. When $B = 0$, the supplier offers the retailer 100% financing by setting $\alpha_{i}(w, 0) = 0$. (Of course, this is the only choice of $\alpha$ that allows the retailer and the entire supply chain to operate.) On the other hand, when the budget is large (i.e., $B \geq w \tilde{F}^{-1}(w)$), the retailer does not need any financial support as he is able to pay the entire procurement cost at $t = 0$. As a result, the supplier sets $\alpha_{i}(w, B) = 1$ for $B \geq w \tilde{F}^{-1}(w)$.

As a byproduct of Proposition 3, we can obtain a simple lower bound for $\alpha_{i}(w, B)$ (uniform on $w$). From the first term inside the maximum it follows that

$$
\alpha_{i}(w, B) \geq \frac{B}{B + \tilde{F}^{-1}(c)}, \quad \text{for all } w \in [c, 1] \text{ and } B \geq 0.
$$

Interestingly, this lower bound suggests that retailers selling low-margin products (those with large $c$) should receive less financial support from the supplier than retailers selling more profitable products.

\footnote{We assume that if the supplier is indifferent between $\alpha_1$ and $\alpha_2$ then he always selects $\max\{\alpha_1, \alpha_2\}$ so as to maximize the payment he receives at time 0.}
THE ROLE OF FINANCIAL SERVICES IN PROCUREMENT CONTRACTS

**Corollary 1** Under the assumptions in Proposition 3, the retailer’s initial payment to the supplier is equal to

\[
\alpha_i(w) w Q_i(w, \alpha_i(w)) = \begin{cases} 
B & \text{if } B \leq w \tilde{F}^{-1}(w), \\
\tilde{w} \tilde{F}^{-1}(w) & \text{otherwise}.
\end{cases}
\]

**Proof:** Follows directly from Proposition 3 and is omitted. \(\square\)

According to Corollary 1, except when the retailer’s budget is large \((B \geq w \tilde{F}^{-1}(w))\) the supplier is willing to offer financial support (by setting \(\tilde{I} < 1\)) to induce the retailer to expend all his budget at time 0. Of course, by doing so the supplier is also increasing his credit risk exposure since \(w Q_i \geq B\) in these cases.

Let us now discuss the optimal choice of \(w_i(B)\) for the case \(B > 0\). This condition is imposed to ensure that there exists a unique solution \(\tilde{Q}(w, B)\) to \(\tilde{F}(\tilde{Q}) = w \tilde{F}(w \tilde{Q} - B)\) for any \(w \in [c, 1]\). The special case \(B = 0\) is discussed in detail in Section 1.4.3.

**Proposition 4** Let \(B > 0\) and suppose \(F\) satisfies the conditions in Assumption 1. Let \(\tilde{Q}(w, B)\) be the unique solution to \(\tilde{F}(\tilde{Q}) = w \tilde{F}(w \tilde{Q} - B)\) and define \(\tilde{w} \in [c, 1]\) such that \(\tilde{F}(\tilde{Q}(\tilde{w})) = c\). Then, the optimal wholesale price \(w_i(B)\) is bounded below by \(\tilde{w}\) and solves

\[
w_i(B) = \arg \max_{0 \leq w \leq 1} \Pi_s^i(w, B) = \left\{ (w - c) \tilde{Q}(w, B) - \mathbb{E} \left[ (D + B - w \tilde{Q}(w, B))^{-} \right] \right\}.
\]

The retailer’s optimal ordering level is \(Q_i(B) = \tilde{Q}(w_i(B), B)\) and the optimal credit parameter is \(\alpha_i(B) = \min\{1, B/(w_i(B) Q_i(B))\}\). The supplier’s optimal expected payoff \(\Pi_s^i(w_i(B), B)\) is monotonically decreasing in \(B\).

**Proof:** See the Appendix at the end. \(\square\)

In general, the optimal wholesale price \(w_i(B)\) cannot be computed in closed-form, however, it can be easily found numerically. Nevertheless, Proposition 4 leads to a number of useful properties of the resulting equilibrium. For instance, it shows that the supplier prefers to do business with small budget retailers. The intuition behind this result is again driven by the retailer’s limited liability that induces small budget retailers to take more risks by ordering more units compared to large budget retailers.

Figure 1.2 depicts the supplier’s expected payoff \(\Pi_s^i(w)\) as well as two auxiliary functions \(w \tilde{F}^{-1}(w)\) and \((w - c) \tilde{F}^{-1}(w)\) that will be useful for the discussion of the following properties of an optimal solution (we refer the reader to Section 1.2.1 for the definitions of the notation used in the figure).
Figure 1.2  Supplier’s expected payoff $\Pi^s(w)$ as function of the wholesale price $w$ given a fixed retailer’s budget $B$.

- Suppose $B \geq \hat{B}$ or $c \geq \bar{w}(B)$. It follows from Lemma 1 that $\hat{Q}(w) = F^{-1}(w)$ for all $w \in [c, 1]$. In this case, the retailer has a sufficiently large budget and the deposit contract coincides with the traditional wholesale contract with no budget constraint. That is, $\alpha_i(B) = 1$, $w_i(B) = w_i$ and $Q_i(B) = Q_i = F^{-1}(w_i)$.

- Suppose $B < \hat{B}$ and $c < \bar{w}(B)$. In this case, we can get upper and lower bounds on $w_i$. First, after some straightforward manipulations we can rewrite the supplier’s expected payoff as follows

$$\Pi^s_i(w, B) = (w \hat{Q} - B) \hat{F}(w \hat{Q} - B) + \int_{0}^{(w \hat{Q} - B)^+} x dF(x) - c \hat{Q} + B. \quad (1.7)$$

Combining the following facts: (i) the function $xF(x) + \int_{0}^{x} y f(y) \, dy$ is increasing in $x$, (ii) $\hat{Q}$ is a decreasing function of $w$ and (iii) $w \hat{Q}$ is a unimodal function of $w$¹, we conclude that $\Pi^s_i(w, B)$ is increasing in the range $[c, w_0(B)]$ where $w_0(B) = \arg\max\{w \hat{Q}(w, B)\}$. It follows from the proof of Lemma 1 that $\hat{Q}(w_0(B)) = \hat{Q} = F^{-1}(\hat{w})$. As a result, $w_0(B)$ solves the equation $\hat{w} = w_0 \hat{F}(w_0 \hat{F}^{-1}(\hat{w}) - B)$ and so $w_0(B) \geq \hat{w}$. One can also show from the previous equation and the

¹See Lemma 1 for a proof of (ii) and (iii).
definition of \( w(B) \) that \( w_0(B) \geq w(B) \). We conclude that \( w_1(B) \geq \max\{c, w_0(B)\} \geq \max\{c, \bar{w}, w(B)\} \).

On the other hand, a simple upper bound on the optimal wholesale price is given by \( w_1 \leq \max\{\bar{w}, w_E\} = w_T \), the wholesale price under a traditional wholesale contract with the budget constraint (see Proposition 1). It follows that the retailer is better off using if the supplier offers a credit contract instead of the traditional wholesale contract.

- Condition \( \alpha_t = B/(w_t Q_t) \) together with Proposition 3 imply that at optimality \( Q_t \leq Q_c \), where \( Q_c = F^{-1}(c) \) is the optimal centralized ordering quantity. Hence, the double marginalization inefficiency persists under a credit contract for all values of \( B > 0 \).

We conclude this section with the following result that provides a partial characterization of the notion of a ‘large budget’, i.e., a budget \( \bar{B} \) such that the credit contract equilibrium is invariant for \( B \geq \bar{B} \).

**Proposition 5** Suppose the density function of \( D \) satisfies \( f(0) = 0 \). Then, \( \bar{B} = B_U \) and \( (w_1(B), Q_1(B)) = (w_U, Q_U) \) for all \( B \geq B_U \). In addition, the optimal wholesale price \( w_1(B) \) is continuous in \( B \). On the other hand, if \( f(0) > 0 \) then \( B_U \leq \bar{B} \leq \hat{B} \).

**Proof:** See the Appendix at the end. □

The following corollary follows directly from Corollary 1 and Proposition 5.

**Corollary 2** Suppose \( f(0) = 0 \). Then, for \( B \leq B_U \) the retailer utilizes all his budget at time 0, that is, \( \alpha_t(B) w_1(B) Q_1(B) = B \).

### 1.4.3 Special Case: \( B = 0 \)

Let us discuss in more detail the case in which the retailer has no initial budget, i.e., \( B = 0 \). This is an important extreme case as it allows us to isolate the benefits of a credit contract in a situation in which the use of a traditional wholesale contract would lead to a non-operative supply chain.

From Proposition 4, and the fact for \( B = 0 \) the ordering level \( \tilde{Q} \) satisfies \( F(\tilde{Q}) = w \tilde{F}(w \tilde{Q}) \), it follows that the optimal wholesale price solves

\[
    w_1 = \arg \max_{c \leq w \leq 1} \left\{ \tilde{Q} (F(\tilde{Q}) - c) + \int_0^{w \tilde{Q}} D dF(D) \right\}.
\]
Note that for $w = 1$, the equation $\bar{F}(\tilde{Q}) = w \bar{F}(w \tilde{Q})$ is satisfied for all $\tilde{Q} \geq 0$. This multiplicity of solutions follows from the fact that for $w = 1$ and $B = 0$ the retailer’s payoff is identically 0 for any nonnegative $\tilde{Q}$. We consider two alternative solutions to address this problem.

1. **Cooperative Retailer:** In this case, we assume that the retailer is indifferent among all possible values of $\tilde{Q} \geq 0$ when the supplier sets a wholesale price $w_1 = 1$. It follows that the supplier can achieve the same outcome (and profit) as the centralized system. Indeed, it is not hard to see that it is in the supplier’s best interest to set $w = 1$ and to induce the retailer to order a quantity $\tilde{Q}$ that maximizes the supplier’s payoff

$$\max_{\tilde{Q} \geq 0} \left\{ \tilde{Q} \bar{F}(\tilde{Q}) + \int_0^{\tilde{Q}} D \, dF(D) - c \tilde{Q} \right\} = \max_{\tilde{Q} \geq 0} \left\{ E[\min\{D, \tilde{Q}\}] - c \tilde{Q} \right\},$$

which is exactly the payoff of a centralized system. The corresponding optimal credit contract satisfies $w_1 = 1$, $\alpha_1 = 0$, $Q_1 = Q_c = \bar{F}^{-1}(c)$, $\Pi^P_1 = 0$ and $\Pi^C_1 = \int_0^{\bar{F}^{-1}(c)} D \, dF(D)$.

2. **Non-Cooperative Retailer:** In this case, rather than assuming that the retailer is indifferent and willing to order any quantity if $w = 1$, we assume that the retailer ordering quantity, $\tilde{Q}(w)$, is continuous on $w$ so that $\tilde{Q}(1) = \lim_{w \uparrow 1} \tilde{Q}(w)$. It follows then that $\tilde{Q}(1) = \hat{Q}$ (recall that $\hat{Q}$ is the unique solution to $Q h(Q) = 1$).

The continuity of $\tilde{Q}(w)$ (together with its monotonicity, see Lemma 1) allows us to invert this function and write the supplier expected payoff as a function of $\hat{Q}$ instead of $w$. For this, we must express $w$ as a function of $\hat{Q}$ solving the equation $\hat{Q} \hat{F}(\hat{Q}) = w \hat{Q} \hat{F}(w \hat{Q})$. By Assumption 1, the function $Q \hat{F}(Q)$ is unimodal and achieves its maximum at $\hat{Q}$. This together with the fact that $w \leq 1$ implies that the function $w(\hat{Q})$ is defined for $\hat{Q} \geq \hat{Q}$. Hence, as a function of $\hat{Q}$, the supplier maximizes

$\uparrow$An alternative way of modeling the behavior of a non-cooperative retailer would be to assume that the retailer does not operate (i.e., selects $\tilde{Q} = 0$) if the supplier charges the wholesale price $w = 1$. However, from the discussion that follow, we can show that under this assumption the supplier optimization problem is ill-posed in the sense that he would like to select a wholesale price that is strictly less but as close as possible to 1.
his payoff solving

$$\max_{\tilde{Q} \geq 0} \Pi^T_\mathcal{S}(\tilde{Q}) = \left\{ \tilde{Q} \left( \tilde{F}(\tilde{Q}) - c \right) + \int_0^{w(\tilde{Q})} \tilde{Q} D dF(D) \right\}. $$

The unimodality of $Q \tilde{F}(Q)$ together with the condition $\tilde{Q} \tilde{F}(\tilde{Q}) = w(\tilde{Q}) \tilde{Q} \tilde{F}(w(\tilde{Q}) \tilde{Q})$ imply that the function $w(\tilde{Q}) \tilde{Q}$ is also unimodal on $\tilde{Q}$ attaining its maximum at $\hat{Q}$. As a result, $\Pi^T_\mathcal{S}$ is maximized at $\tilde{Q} = \hat{Q}$ which implies $w(\hat{Q}) = 1$. In summary, the outcome of the game in this case is given by $w = 1$, $\Pi^T_\mathcal{S} = 0$, $Q^T_\mathcal{S} = \hat{Q}$, $\Pi^T_R = \hat{Q} \left[ \hat{F}(\hat{Q}) - c \right] + \int_0^{\hat{Q}} D dF(D)$.

If we compare the equilibrium outcomes of these two cases, we see that when $B = 0$ the supplier is always better off charging a full wholesale price $(w = 1)$ independently of whether the retailer is cooperative or not and, as a result, the retailer ends up making no profit.

The main difference between these two cases is the ordering level and corresponding payoff of the supplier. On one hand, the cooperative retailer is willing to order, $Q^T_\mathcal{S} = Q_C$, a quantity that gives the supplier the same payoff as the centralized system. In other words, the cooperative retailer is essentially transferring his retail business to the supplier at no cost. On the other hand, the non-cooperative retailer chooses a quantity $Q^T_\mathcal{S} = \hat{Q}$, which is in general suboptimal from the supplier’s point of view, $\Pi^T_\mathcal{S} \leq \Pi^T_\mathcal{S}(Q_C)$. As for the ordering quantity, whether $Q_C \leq \hat{Q}$ or $Q_C \geq \hat{Q}$ depends on the production cost $c$ and the demand distribution $F$. Indeed, $Q_C \leq \hat{Q}$ if and only if $F^{-1}(c) h(F^{-1}(c)) \leq 1$. This is an example of a situation in which decentralization increases the market output with the corresponding benefits for the end consumers.

### 1.5 EQUILIBRIUM WITH EXTERNAL FINANCING ($Q_e$, $w_e$)

In this section we consider an alternative mode of financing for the retailer. Instead of negotiating the term of a credit contract with the supplier, as in the previous section, we assume that the retailer gets a loan from a financial institution (e.g., a commercial bank) to fund his operation. The retailer uses this loan together with his initial budget to pay the supplier. To contrast the effects of this type of *external financing* with the one discussed in the previous section, we assume that the supplier offers a traditional wholesale
contract that forces the retailer to pay the supplier in full at time 0 (i.e., \( \alpha = 1 \)).

We represent the terms of the loan in the form of a triplet \((L, \mathcal{L}, r)\) where \(L\) is the amount that the retailer borrows at time 0, \(r\) is the nominal interest rate charged by the financial institution at time \(T\) and \(\mathcal{L}\) are the earnings (possibly random) that the retailer has available at time \(T\) to repay the loan plus the interests. That is, the retailer is required to pay \(\min\{\mathcal{L}, L (1 + r)\}\) at the end of the selling season when demand uncertainty is resolved and revenues are collected. Of course, \(r\) should depend on \(L\) and \(\mathcal{L}\) to capture the underlying market price of risk in the economy. To model this dependency we assume that the financial market is competitive in the following sense.

**Definition 1** (Competitive Financial Market) We say that the financial market is competitive if the terms of a loan \((L, \mathcal{L}, r)\) satisfy

\[
L (1 + r_f) = \mathbb{E}[\min\{\mathcal{L}, L (1 + r)\}],
\]

where \(r_f\) is the risk-free rate.

Intuitively, this condition says that on average (taking into account the risk of default) the financial market lends money at the risk-free rate \(r_f\). Without loss of generality, in what follows we will normalize interest rates in this economy so that \(r_f = 0\).

According to this definition, it should be clear that some loans \((L, \mathcal{L}, r)\) are infeasible in a competitive financial market. The following Lemma provides a necessary and sufficient condition that ensures that a loan \((L, \mathcal{L}, r)\) can be effectively negotiated.

**Lemma 2** In a competitive financial market, a loan \((L, \mathcal{L}, r)\) is feasible if and only if \(\mathbb{E}[\mathcal{L}] \geq L\).

**Proof:** See the Appendix at the end. \(\square\)

Let us now discuss the retailer’s problem under this alternative source of (external) financing. After observing the wholesale price \(w\) offered by the supplier, the retailer must determine the quantity \(Q\) to order and the size \(L\) of the loan needed to fund this ordering level. For a given order quantity \(Q\), and given the retailer’s initial budget \(B\), the retailer will always choose to minimize the size of the loan, that is, \(L = (wQ - B)^+\). This is a consequence of two facts: (i) the interest rate increases with the size of the loan and (ii) our assumption that the retailer has no other investment opportunity besides
his retail business (and possibly a cash account that pays the risk-free interest rate). As a result, we can write the retailer’s optimization problem exclusively in terms of $Q$.

According to Lemma 2, the retailer has limited access to the credit market. Indeed, to get a loan $L = (wQ - B)^+$ the retailer’s revenues at time $T$ must satisfy $E[\min\{D, Q\}] \geq (wQ - B)^+$. This condition defines an upper bound $\bar{Q}$ on the quantity that the retailer can order. For all $Q \leq \bar{Q}$, the interest rate $r(Q)$ that the retailer pays to get the loan $L = (wQ - B)^+$ is the unique solution to the equation

$$E[\min\{\min\{D, Q\} + B - wQ, (wQ - B)^+ r(Q)\}] = 0.$$ 

Combining these conditions we get that the retailer’s optimization problem is given by

$$\Pi^R = \max_{0 \leq Q \leq \bar{Q}} E[\min\{\min\{D, Q\} + B - wQ\}] - \mathbb{E}[\min\{\min\{D, Q\} + B - wQ, (wQ - B)^+ r(Q)\}] - B \quad (1.8)$$

subject to $E[\min\{\min\{D, Q\} + B - wQ, (wQ - B)^+ r(Q)\}] = 0$. \quad (1.9)

As before, $\Pi^R$ denotes the retailer’s expected payoff net of his initial budget $B$.

We can solve the retailer’s problem in a rather simple way. Indeed, if we use the identity $(x - y)^+ = x - \min\{x, y\}$ the retailer’s objective can be rewritten as follows

$$\Pi^R = \max_{0 \leq Q \leq \bar{Q}} E[\min\{D, Q\} + B - wQ] - \mathbb{E}[\min\{\min\{D, Q\} + B - wQ, (wQ - B)^+ r(Q)\}] - B.$$

And so by virtue of constraint (1.9) the retailer’s optimization reduces to

$$\Pi^R = \max_{0 \leq Q \leq \bar{Q}} E[\min\{D, Q\} - wQ]$$

subject to $E[\min\{\min\{D, Q\} + B - wQ, (wQ - B)^+ r(Q)\}] = 0$.

Note that the objective function corresponds to the standard payoff of the retailer under a wholesale contract (see the discussion at the end of Section 1.2). In particular, this objective is independent of the interest rate $r(Q)$ and the retailer’s initial budget $B$. The optimal ordering level is $\bar{F}^{-1}(w)$ that trivially satisfies the feasibility constraint $\bar{F}^{-1}(w) \leq \bar{Q}$.

It is worth noticing that the retailer optimal solution is independent of $B$ and coincides with the one that is obtained in the traditional wholesale
contract with no budget constraint. In order words, access to a competitive financial market has enabled the retailer to completely separate the operation of his retail business (in particular his procurement decisions) from its financing. The following proposition summarizes this solution.

**Proposition 6** Suppose the retailer has an initial budget $B$ and has access to a competitive financial market. Then, under a wholesale contract with wholesale price $w$, the retailer’s optimal strategy is to order a quantity $\bar{F} - 1(w)$ and to get a loan (if necessary) for an amount equal to $L = (w \bar{F} - 1(w) - B)^+$. The interest rate, $r$, that the retailer pays on this loan is the unique solution to

$$
\mathbb{E} \left[ \min \{ \min \{ D, \bar{F} - 1(w) \} + B - w \bar{F} - 1(w), (w \bar{F} - 1(w) - B)^+ r \} \right] = 0.
$$

Given the retailer’s best strategy $\bar{F} - 1(w)$, the supplier problem reduces to

$$
\Pi^w_s = \max_{c \leq w \leq 1} \{(w - c) \bar{F} - 1(w)\}
$$

with optimal solution $w^*_E$. As a result, the optimal ordering level is $Q^*_E = \bar{F} - 1(w^*_E)$.

Based on our discussion about the traditional wholesale contract at the end of Section 1.2, it follows that $w^*_E = \lim_{B \to \infty} w^*_T(B)$ and $Q^*_E = \lim_{B \to \infty} Q^*_T(B)$. In other words, the existence of a competitive financial market allows the entire supply chain to operate in exactly the same way as it would operate if the retailer had no budget constraint.

### 1.6 COMPUTATIONAL EXPERIMENTS

In this section we conduct a set of numerical computations to compare the equilibrium under internal financing of Section 1.4 and the equilibrium under external financing of the previous section. We do not attempt to have an exhaustive set of experiments that describe all possible outcomes of the game. Instead, we have chosen a subset of instances that highlight some of the most important features of the equilibrium and the main differences between the alternative forms of financing.

To perform these experiments we use a $(\lambda, k)$-Weibull distribution\(^1\) to model demand uncertainty. The reasons for this choice are (i) the Weibull

\(^1\)The tail distribution of a $(\lambda, k)$-Weibull is given by $\bar{F}(x) = \exp \left( - (x/\lambda)^k \right)$. Its mean and variance are given by $\mu = \lambda \Gamma \left( 1 + \frac{1}{k} \right)$ and $\sigma^2 = \lambda^2 \left[ \Gamma \left( 1 + \frac{2}{k} \right) - \Gamma^2 \left( 1 + \frac{1}{k} \right) \right]$, respectively.
satisfies Assumption 1, (ii) it simplifies the computation of the equilibria of the game and (iii), as a two-parameter distribution, it offers enough flexibility to fit at least the first two moments of the demand. For the case $B = 0$, we have chosen the option of a cooperative retailer (see Section 1.4.3 for details).

Our first set of experiments compares the market equilibrium for different types of contracts as a function of the retailer’s budget $B$. Figure 1.3 plots the wholesale price (left panel) and ordering level (right panel) for the credit contract $(w_{I}(B), q_{I}(B))$, for the case of external financing $(w_{E}(B), q_{E}(B))$, and for the traditional wholesale contract $(w_{T}(B), q_{T}(B))$. The right panel also includes the centralized production quantity $Q_{C}$.

![Figure 1.3](image)

**Figure 1.3** Equilibrium wholesale price (left panel) and ordering level (right panel) as a function of the retailer’s initial budget (B) under three alternative contracts: credit contract $(w_{I}, q_{I})$, external financing $(w_{E}, q_{E})$ and traditional wholesale contract $(w_{T}, q_{T})$. Data: $\epsilon = 0.7$ and $\bar{F}(x) = \exp(-0.01x^2)$.

It follows from Figure 1.3 that all three modes of operations (credit contract, external financing, or traditional wholesale contract) are equivalent if the retailer’s budget is sufficiently large, in this case $B \geq B_{E}$.\(^{1}\) It is also worth noticing that under a traditional wholesale contract (i.e., when the retailer does not receive any type of financial support), the supplier charges a higher wholesale price and the retailer orders less compared to the cases with internal or external financing. Therefore, the supply chain (as a whole) is worst off if

\(^{1}\)This follows from Propositions 1 and 5 and the fact that $f(0) = 0$ in this example.
the retailer is unable to get financial support (either from the supplier or a third party financial institution).

If we compare the outcome under internal versus external financing, we see that under internal financing the retailer orders more units than with external financing. As a result, final consumers are better off under a credit contract. In terms of the wholesale price, there is no absolute comparison between the two types of financing. If $B$ is relatively small, the credit contract leads to higher wholesale prices but if $B$ gets large then external financing leads to higher wholesale prices. Hence, from the point of view of the retailer neither mode of financing dominates the other, that is, his preferences depend on his initial budget.

Figure 1.4 shows the retailer (left panel) and supplier (right panel) expected payoff for the three contracts. It follows that the supplier is always better off under the credit contract independently of the retailer’s initial budget. One of the remarkable features of the credit contract is that the retailer’s expected payoff $\Pi^R_B(B)$ is non-monotonic on his initial budget $B$. There is a finite budget, $B_m$ in Figure 1.4, at which the retailer’s payoff is maximized (for all three types of contracts considered in our discussion). This non-monotonic feature of $\Pi^R_B(B)$ suggests that implementing a credit contract can be difficult in practice since in those cases in which $B \geq B_m$ the retailer has incentives to underreport his true budget at the time the credit contract is negotiated. In Section 1.7, we extend our model to the case in which there is asymmetric information between the retailer and supplier about the true value of $B$. Interestingly, since the supplier’s payoff under a credit contract,
\( \Pi^r(B) \), is monotonically decreasing in \( B \), then for \( B \geq B_m \) both the retailer and the supplier would be better off if the retailer had a smaller initial budget. It also follows from Figure 1.4 that both agents are worst off if they operate using a traditional wholesale contract. Hence, it is in the retailer and supplier best interests to get some type of financing for the retailer.

As Figure 1.5 reveals, the overall payoff of the entire supply chain is monotonically decreasing in \( B \) under a credit contract. Despite this fact, the credit contract is the best mode of operations for the entire supply chain independent of the value of \( B \). On the contrary, the traditional wholesale contract produces the worst possible outcome for the aggregate system (including consumers surplus).

\[ \bar{F}(x) = \exp\left(-0.01x^2\right). \]

Our next computational experiment highlights a curiosity of the equilibrium under a credit contract. The left panel in Figure 1.6 plots the wholesale price for the three type of contracts as a function of the retailer’s budget. As we can see, in this example the wholesale price for the credit contract \( w_i(B) \) has a discontinuity at \( B_d \approx 36 \). In particular, \( w_i(B) \) is monotonically decreasing for \( B \in [0, B_d) \) jumps upward at \( B_d \) and remains constant for \( B \geq B_d \).

To understand the nature of this discontinuity, the right panel in Figure 1.6 plots the supplier’s payoff \( \Pi^s(w, B) \) as a function of \( w \) for four values of \( B \) (with \( B_a < B_b < B_c < B_d \)). For \( B = B_d \), the budget is sufficiently large
Figure 1.6  Wholesale price (left panel) and Supplier’s Payoff (right panel). Data: $c = 0.1$ and $F(x) = \exp(-0.01x)$.

and it is not a constraint, as a result $\Pi_S(w, B_d) = (w - c) F^{-1}(w)$ (this curve is represented by the dashed line in the figure). As we can see, the optimal wholesale price decreases from $a$ to $b$ and from $b$ to $c$ and then jumps from $c$ to $d$. This discontinuity on the optimal wholesale price implies that the retailer’s expected payoff (as well as the payoff of the entire supply chain) has a discontinuity at $B_d$, specifically a downward jump. It is worth mentioning that this discontinuity does not contradict the result in Proposition 5 since in the example in Figure 1.6 demand has an exponential distribution and so $f(0) > 0$. This is in contrast to the example in Figure 1.3 for which demand has a Weibull distribution (with $f(0) = 0$) and the wholesale price $w_i(B)$ is continuous.

We conclude this section measuring the impact of demand variability on the performance of the three contracts. Table 1.2 shows the retailer and supplier payoffs for the External Financing contract and Traditional Wholesale contract relative to their payoffs under a Credit contract as a function of the retailer’s budget ($B$) and the coefficient of variation (StDev/Mean) of the market demand. To isolate the effect of demand variability, we keep the mean demand constant and vary the standard deviation.

From the point of view of the retailer, the option of External financing is generally the best for all levels of demand variability. The Credit contract comes next and the Traditional Wholesale contract is the worst. On the other hand, for the supplier the Credit contract generally is the best option.
Retailer and Supplier Normalized Payoffs

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Table 1.2 Retailer and supplier payoffs for the External Financing and Traditional Wholesale contracts relative to their payoffs under a Credit contract as a function of the retailer’s budget (B) and the coefficient of variation (StDev/Mean) of the market demand. Demand has a Weibull distribution with fixed mean $\mu = 80$ and the manufacturing cost is $c = 0.5$. 

except when the budget is small and the variability is medium to high ($B \leq 5$ and CV=200%) in which case the External financing gives the supplier a higher payoff. In other words, suppliers serving small retailers facing high demand variability are better off if a bank takes the risk of providing financial services to the retailer. Similarly to the retailer, the Traditional Wholesale contract is the worst option for the supplier. We can also see from Table 1.2
that all three contracts produce the same outcome when demand variability is high (CV=500%) independent of the budget. The reason for this is that (for a Weibull distribution) the retailer’s optimal ordering quantity converges to 0 as the demand variability increases.

In Table 1.3 we compare the efficiency of the three contracts from the point of view of the entire supply chain. For each type of contract we compute the percentage competition penalty

\[ \mathcal{P} := 1 - \frac{\Pi^R + \Pi^S}{\Pi^C} \times 100\% \]

as a function of the retailer’s budget \( B \) and the coefficient of variation of the market demand.

As we can see from Table 1.2, the credit contract is generally the most efficient of the three contracts while the traditional wholesale contract is the less efficient. Again, for the case when the budget is small and the variability is medium to high \( (B \leq 5 \text{ and } CV=200\%) \) the option of External financing is the most efficient. We also note that the efficiency of the credit contract decreases with the retailer’s budget \( B \) and the opposite is true for the traditional wholesale contract. Interestingly, for small values of \( B \), the efficiency of the credit contract is U-shape as a function of the level of variability of the demand. In these cases, the competition penalty is maximized at intermediate values of the coefficient of variation. However, as \( B \) increases the efficiency of the credit contract becomes monotonically decreasing in the level of demand uncertainty.
### Table 1.3

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Table 1.3 Competition penalty as a function of the retailer’s budget (B) and the coefficient of variation (StDev/Mean) of the market demand. Demand has a Weibull distribution with fixed mean \( \mu = 80 \) and the manufacturing cost is \( c = 0.5 \).

### 1.7 CONCLUDING REMARKS AND EXTENSIONS

In this chapter we investigate how financial constraints affect the operations of a two-echelon supply chain and discuss two alternative modes of financing that could be used to limit the effects of these constraints. Our model focuses on the case in which the retailer has a limited amount of working capital (his
We set our mathematical formulation within the framework of the newsven-
dor model, that is, we consider a single-period model in which the retailer
procures products from the supplier to satisfy a future stochastic demand.
We also restrict attention to simple (yet popular) linear transfer payments
(or wholesale contracts) between the agents. In these contracts, the supplier
first specifies a per unit wholesale price \( w \) and the retailer then selects the
order size \( q \). Typically, in the absence of a budget constraint, the retailer
pays the full amount \( wq \) at the time the contract is signed. In our model, this
traditional wholesale contract needs to be modified to include the additional
budget constraint \( wq \leq B \) which could seriously affect the profitability of the
entire supply chain if \( B \) is small. To overcome this limitation, we discuss
two possible types of financing for the retailer. First, we consider the case of
*internal financing* in which the supplier charges only a fraction \( \alpha \in [0, 1] \)
of the wholesale price at the time the contract is signed. The remaining fraction
\((1 - \alpha)\) is paid after market demand is realized. With this contract, the effec-
tive budget constraint at the moment the order is placed is \( \alpha wq \leq B \) which,
depending on the value of \( \alpha \), allows the retailer to increasing his order size.
From the supplier standpoint, this *credit contract* increases the overall pro-
duction of the supply chain helping his own business. On the downside, the
supplier faces a new source of risk under a credit contract when the retailer
defaults on his obligation to paid the remaining portion \((1 - \alpha)wq\) after de-
mand is realized. Nevertheless, it is obvious that the supplier is always better
off (on average) using the credit contract instead of a traditional wholesale
contract since he can always set \( \alpha = 1 \).

The second form of financing that we consider is the retailer applying for
a commercial loan from a financial institution (*e.g.*, a bank). For the sake of
simplicity, we assume that the interests charged on this loan are set based on a
competitive financial market assumption (see Definition 1 for details). In this
*external financing* case, the supplier offers a traditional wholesale contract
requiring full payment at the time the contract is signed. In Section 1.5
we study this contract and show that its equilibrium is equivalent to the
traditional wholesale contract in which the retailer has an unlimited budget.
This conclusion is a reminiscence of the well-known Modigliani and Miller’s
irrelevance principle. An interesting extension regarding this part of our model

budget \( B \) to procure products from the supplier (alternatively, we could have
considered the reversed situation in which the supplier is the small player; a
scenario that could be more appropriate in certain industries).
would be to consider a less competitive financial market in which the interest rate are set using a different equilibrium concept (e.g., the CAPM model).

Our analysis and results for the credit contract are somehow less palpable. First of all, we show that for any given value of $w$ and $\alpha$ the retailer’s optimal ordering quantity is decreasing in $B$. That is, small retailer’s are willing to order more (i.e., take more risk) since they have less stakes at risk in case of default. The implication of this behavior is that the optimal ordering quantity and the supplier’s optimal expected payoff are also decreasing in the retailer’s budget $B$. In other words, the supplier prefers to do business with a small retailer willing to take more risk than with a larger retailer (with more collateral) that is more cautious in his procurement decisions. In the extreme case in which $B = 0$ we show that, depending on the retailer’s preferences, it is possible for the supplier to achieve the same expected payoff as in the centralized solution. Neither the traditional wholesale contract nor the external financing option are able to achieve this first best (actually the traditional wholesale contract leads to a non-operative supply chain if $B = 0$). Another interesting feature of the credit contract is that the optimal wholesale price, $w_i(B)$, is a non-monotonic function of the retailer initial budget. Starting at $B = 0$, the wholesale price first decreases as $B$ increases reaching a minimum at some value $w_m$ when $B = B_m$ and then increasing (either smoothly or with an upward jump) to a value $w_E$. This value $w_E$ is equal to the optimal wholesale price under external financing (or equivalently, under a traditional wholesale contract if retailer has sufficiently large budget).

Probably, one of the main assumptions that we make in our model is that the retailer’s demand information and budget are public knowledge. Under this complete information assumption, the supplier can offer a credit contract $(\alpha(B), w(B))$ specially designed and targeted to the retailer with initial budget $B$ that maximizes the supplier’s expected payoff. In practice, however, we do expect some degree of asymmetric information that limits the supplier’s ability to tailor the credit contract using the retailer’s initial wealth. In this setting, the supplier must design a contract that takes into account the adverse selection problem (e.g., [25]) under which the retailer (potentially) misrepresents his initial budget to improve the terms of the credit contract. We conclude this section measuring the impact of this behavior. However, we do not attempt a rigorous treatment of this adverse selection problem here as it goes beyond the scope of this chapter. We do show, however, a simple
extension of the credit contract that can take care of this problem in certain cases.

To understand the nature of this adverse selection problem and get more insights about its implications on the actions of the retailer, let us consider the case in which the supplier offers naïvely the optimal credit contract menu \{\( (\alpha_t(B), w_t(B)) \); \( B \geq 0 \)} that we derived in Proposition 4 under full information. The sequence of events is as follows. First, the supplier offers the menu of credit contracts \{\( (\alpha_t(B), w_t(B)) ; B \geq 0 \}\}. The retailer then reports his initial budget \( \tilde{B} \) and the specific contract \( (\alpha_t(\tilde{B}), w_t(\tilde{B})) \) is selected from the menu. Finally, the retailer orders a quantity \( Q \) and pays the supplier the amount \( \alpha_t(\tilde{B}) w_t(\tilde{B}) Q \) at time \( t = 0 \). The remaining portion \( (1 - \alpha_t(\tilde{B})) w_t(\tilde{B}) Q \) is paid in arrears at time \( T \) after market demand is realized. Because of the information asymmetry, the supplier has no concrete mechanism to verify that the retailer is effectively reporting his true initial wealth. Furthermore, to be consistent with our previous discussion, we assume that the retailer, after reporting his budget \( \tilde{B} \), is free to choose any \( Q \) that satisfies the budget constraint. Naturally, at this point, the supplier could realize that the retailer is misreporting his budget if \( Q \neq Q_t(\tilde{B}) \), the optimal ordering level under full information. To avoid this problem, the supplier can modify the terms of the contract and offer a menu of enlarged credit contracts that also specifies the ordering quantity, that is, a menu \{\( (\alpha_t(B), w_t(B), Q_t(B)) ; B \geq 0 \}\}. We will discuss this variation of the credit contract below. We also assume that if the retailer declares bankruptcy at time \( T \) then the supplier is able to audit the retailer’s assets so that \( B \) becomes public knowledge at this point.

Figure 1.7 plots the retailer’s true budget \( B \) and reported budget \( \tilde{B} \) as a function of \( B \) for the case in which demand is uniformly distributed in \([0,1]\). As we can see from the figure, when the retailer’s true budget is below a threshold \( B_m \) the retailer has incentives to report a larger budget while when his budget exceeds \( B_m \) he has incentives to report a smaller budget. The threshold \( B_m \) is the one at which the wholesale price offered by the retail is minimized, that is, \( B_m = \text{argmin}\{w_t(B) ; B \geq 0\} \). Note that for small values of \( B \) the retailer reports a budget \( \tilde{B} \) that is strictly less than \( B_m \). The reason is that the credit parameter \( \alpha_t(B) \) is in \( B \). Hence, even if reporting \( B_m \) minimizes the wholesale price the corresponding budget constraint \( \alpha_t(B_m) w_t(B_m) Q \leq B \) imposes a rather restrictive upper bound on the feasible ordering quantity \( Q \).
Figure 1.7 Retailer’s true and reported budget. Demand follows a Uniform distribution in [0, 1] and \( c = 0.6 \)

The effect on the agents’ payoffs of this misreporting can be significant. Figure 1.8 depicts the retailer (left panel) and supplier (right panel) expected payoffs as function of the retailer’s true budget (\( B \)) for the cases in which the retailer reports his true budget (Full Information) and the case in which he misreports his budget (Incomplete Information). As expected, the retailer can take advantage of this asymmetry of information to increase his payoff. For the supplier, this misreporting can lead to a sharp decrease in his payoff, specially when \( B \) is small. It is interesting to note that from the point of view of the entire supply chain, the system is better off under Incomplete Information for \( B \geq B_m \) while the system expected payoff is larger under Full Information for \( B \leq B_m \).

As we mentioned before, the supplier can try to solve this adverse selection problem by including the quantity \( Q_1(B) \) in the contract offering the menu \( \{ (w_1(B), \alpha_1(N), Q_1(B)) : B \geq 0 \} \). Based on our previous discussion, it is not hard to see that this modified contract takes care of the adverse selection problem only if \( B \leq B_m \). In this range, the retailer will truthfully declare his real budget. This follows from the fact that \( \alpha_1(B) w_1(B) Q_1(B) = B \) in this range and so the retailer is unable to report a higher budget. However, the retailer will still underreport his budget if \( B > B_m \) to get a lower wholesale
Figure 1.8  Retailer (left panel) and supplier (right panel) expected payoffs as function of the retailer’s true budget for the cases in which the retailer reports his true budget (Full Information) and the case in which he misreports his budget (Incomplete Information). Demand follows a Uniform distribution in $[0, 1]$ and $c = 0.6$

price. To solve this problem, the supplier would have to offer some sort of compensation (information rents) to the retailer if he wants him to report his true budget. This compensation could come in the form of a transfer payment after demand is realized and payoffs are collected. An interesting extension to our model would be to formalize the supplier’s mechanism design problem that could be used to identify such an optimal compensation scheme.

1.8 APPENDIX: MAIN PROOFS

Proof of Lemma 1: We will prove that there is a unique solution to the first-order optimality condition

$$\pi^h(Q) \triangleq \frac{d\pi^h(Q)}{dQ} = \bar{F}(Q) - w \bar{F}(wQ - B) = 0$$

where $\pi^h(Q) \triangleq \mathbb{E}[(\min(Q, D) + B - wQ)^+ - B]$. Indeed, based on the inequality

$$w \bar{F}(wQ - B) \geq w \bar{F}(wQ), \quad \text{for all } Q \geq 0$$

it follows that a sufficient condition for the existence of $\bar{Q}$ is that the equation $\bar{F}(Q) = w \bar{F}(wQ)$ admits a nonnegative solution. Suppose, by contradiction, that $\bar{F}(Q) > w \bar{F}(wQ)$, for all $Q \geq 0$. Integrating over $Q \in [0, \infty)$, we get

$$\mathbb{E}[D] = \int_0^\infty \bar{F}(Q) dQ > w \int_D \bar{F}(wQ) dQ = \mathbb{E}[D].$$
This contradictions prove the existence of $\tilde{Q}$. To prove the uniqueness of $\tilde{Q}$ note that
\[
\frac{d^2 \pi^n}{d \bar{Q}^2} \bigg|_{\bar{Q}} = -f(\bar{Q}) + w^2 f(w \bar{Q} - B) = -F(\bar{Q}) \left[ h(\bar{Q}) - \frac{w^2 F(w \bar{Q} - B)}{F(\bar{Q})} h(w \bar{Q} - B) \right] = -F(\bar{Q}) \left[ h(\bar{Q}) - w h(w \bar{Q} - B) \right] < 0,
\]
where the last equality uses the fact that $\bar{Q}$ satisfies $\pi^n(\bar{Q}) = 0$ and the inequality uses the fact that $w < 1$ and the hazard function $h(D)$ is strictly decreasing in $D$. As a result every local extrema $\bar{Q}$ is a local maxima. By the continuity of $\pi^n$ we conclude that there exists a unique $\tilde{Q}$ that solves the first-order optimality condition for $\pi^n$.

Let us now prove that $\tilde{Q}$ is decreasing in $w$. Recall the definitions of $\tilde{w}$, $\tilde{w}(B)$ and $w(B)$ in Section 1.2.1. The following cases are possible.

- $c \geq \tilde{w}$. In this case $\tilde{Q}(w) = \tilde{F}^{-1}(w)$ for all $w \in [c, 1]$, which is decreasing in $w$.

- $c < \tilde{w}$. For all $w \in [c, w] \cup [\tilde{w}, 1]$ we have that $\tilde{Q}(w) = \tilde{F}^{-1}(w)$, again a decreasing function of $w$. So let us focus on the case $w \in (w, \tilde{w})$. In this case, $w \tilde{Q}(w) > B$ and we can use the identity $\tilde{F}(\tilde{Q}(w)) = w \tilde{F}(w \tilde{Q}(w) - B)$ to compute the derivative
\[
\frac{d\bar{Q}(w)}{dw} = \frac{\tilde{F}(w \tilde{Q} - B) - w \tilde{Q} f(w \tilde{Q} - B)}{w^2 f(w \tilde{Q} - B) - f(\tilde{Q})} = \frac{1 - w \tilde{Q} h(w \tilde{Q} - B)}{w^2 h(w \tilde{Q} - B) - w h(\tilde{Q})}, \quad w \in (w, \tilde{w}),
\]
where the second equality uses the fact that $w \tilde{F}(w \tilde{Q}(w) - B) = \tilde{F}(\tilde{Q}(w))$. By Assumption 1 the hazard function $h(\tilde{Q})$ is increasing, this together with the fact $w < 1$ imply that the denominator is negative. Hence, we would like to show that the nominator is nonnegative. For this, we will prove the sufficient condition $1 - w \tilde{Q} h(w \tilde{Q}) \geq 0$. Let us define $w^0 = \arg\max\{w \tilde{Q}(w) : w \in [0, 1]\}$ which satisfies the first-order optimality condition
\[
\frac{dw \tilde{Q}(w)}{dw} \bigg|_{w = w^0} = 0 \quad \text{which is equivalent to} \quad \tilde{Q}(w^0) h(\tilde{Q}(w^0)) = 1.
\]
As a result, (by the monotonicity of $h(\tilde{Q})$) we get that
\[
1 - w \tilde{Q}(w) h(w \tilde{Q}(w)) \geq 1 - w^0 \tilde{Q}(w^0) h(w^0 \tilde{Q}(w^0)) \geq 1 - \tilde{Q}(w^0) h(\tilde{Q}(w^0)) = 0, \quad \text{for all} \ w \in [w, \tilde{w}],
\]
as required.
The monotonicity of $\tilde{Q}$ on $B$ follows from noticing that $w \tilde{F}(\frac{w + B}{p})$ increases with $B$ and remains constant at $w$ for $B \geq wQ$.

Finally, the unimodality of $w \tilde{Q}(w, B)$ follows from the fact that $\tilde{Q}(w, B)$ solves the equation $\tilde{F}(\tilde{Q}) = w \tilde{F}(w \tilde{Q} - B)$. Taking derivative with respect to $w$ (and after some straightforward manipulations) we get that
\[
\frac{d}{dw}(w \tilde{Q}) = \tilde{Q} h(\tilde{Q}) - 1 = \tilde{Q} h(\tilde{Q}) h(w \tilde{Q} - B).
\]
But $\tilde{Q}$ is decreasing in $w$ and by Assumption 1 $h(\tilde{Q})$ is increasing. It follows that $w \tilde{Q}$ is unimodal as a function of $w$ and attains its maximum at a $w_0$ such that $\tilde{Q}(w_0, B) h(\tilde{Q}(w_0, B)) = 1$.

**Proof of Proposition 2:** Let us define $R(Q) = \mathbb{E}[(\min f Q; D + B w Q)] + B$ and note that $\Pi_R = \max_Q R(Q)$. We will show that $R(Q)$ is unimodal in $Q$ and that it admits the maximizer $Q^*$ described in the statement of the Proposition. Indeed,
\[
\pi^R_Q(Q) = \begin{cases} F(Q) - w & \text{if } w Q \leq B \\
F(Q) - w F(w Q - B) & \text{if } w Q \geq B \end{cases}
\]
In the region $w Q \leq B$, the derivative $\pi^R_Q(Q)$ is decreasing in $Q$ and so $\pi^R(Q)$ is concave in this region. In the region $w Q \geq B$, it follows from the proof of Lemma 1 that $\pi^R(Q)$ is unimodal in this region. Since $\pi^R(Q)$ is differentiable at $Q = B/w$, we conclude that $\pi^R(Q)$ is unimodal in its entire domain $Q \geq 0$. The value of $Q^*$ then follows from solving the first-order condition $\pi^R_Q(Q) = 0$ and imposing the budget constraint $\alpha w Q^* \leq B$.

**Proof of Proposition 3:** Let us write the supplier’s problem as
\[
\max_{\alpha \in [0(w), 1]} \pi^S(\alpha),
\]
for the auxiliary function
\[
\pi^S(\alpha) = \left(1 - \frac{c}{w}\right) \frac{B}{\alpha} - \mathbb{E} \left[\left(D - (1 - \alpha) \frac{B}{\alpha}\right)^+\right].
\]
Let us prove that $\pi^S(\alpha)$ is a unimodal function for $\alpha \in [0, 1]$. Indeed, it is not hard to show that
\[
\pi^S_\alpha(\alpha) = \frac{d}{d\alpha}(\pi^S(\alpha)) = \frac{B}{\alpha^2} \left[F\left(\left(1 - \alpha\right) \frac{B}{\alpha}\right) - \frac{w - c}{w}\right]
\]
and so $\pi^S(\alpha)$ has a unique extreme value $\hat{\alpha}(w) \in (0, 1)$ (solution of $\pi^S_\alpha(\alpha) = 0$) given by
\[
\hat{\alpha}(w) = \frac{B}{B + F\left(\frac{w}{c}\right)}.
\]
In addition,
\[
\left. \frac{d^2 \pi^*(\alpha)}{d \alpha^2} \right|_{\alpha = \hat{\alpha}} = - \frac{B^2}{\alpha} f \left( \frac{(1 - \alpha) B}{\alpha} \right) < 0.
\]
This proves that \( \hat{\alpha} \) is a maximum and so \( \pi^*(\alpha) \) is unimodal as claimed. As a result, the solution to supplier’s problem above is given by
\[
\alpha^*(w) = \max \{ \hat{\alpha}(w), \tilde{\alpha}(w) \}.
\]
Finally, using the definition of \( \tilde{\alpha}(w) \) and \( \tilde{Q}(w) \) is a matter of straightforward manipulations to show that
\[
\tilde{\alpha}(w) = \frac{B}{B + \tilde{Q}(w)} \left( \frac{1}{w} \tilde{F}(\tilde{Q}(w)) \right). \quad \square
\]

**Proof of Proposition 4:** In the following proof, we use some of the notation introduced in the proof of Proposition 3.

By Lemma 1, the function \( \tilde{Q}(w) \) is decreasing in \( w \). Furthermore, we have \( \tilde{F}(\tilde{Q}(c)) = c \tilde{F}(c \tilde{Q}(c) - B) \leq c \) and \( \tilde{F}(\tilde{Q}(1)) = 1 \geq c \) (since \( \tilde{Q}(1) = 0 \) when \( B > 0 \)), and so by continuity we must have that there exists a \( \hat{w} \in [c, 1] \) such that \( \tilde{F}(\tilde{Q}(\hat{w})) = c \) and \( \tilde{F}(\tilde{Q}(w)) \leq c \) for \( w \in [c, \hat{w}] \) and \( \tilde{F}(\tilde{Q}(w)) \geq c \) for \( w \in [\hat{w}, 1] \). It follows from Proposition 3 and Corollary 1 that for all \( w \in [c, \hat{w}] \)
\[
(\alpha_1(w), Q_1(w)) = \left( \tilde{\alpha}(w), \frac{B}{w \tilde{\alpha}(w)} \right),
\]
and
\[
\pi^*(w) = \left( 1 - \frac{c}{w} \right) \frac{B}{\tilde{\alpha}(w)} - \mathbb{E} \left[ D - \left( 1 - \tilde{\alpha}(w) \right) \frac{B}{\tilde{\alpha}(w)} \right].
\]
Furthermore, from the definition of \( \tilde{\alpha}(w) \), it follows that \( \partial \pi^*(w) / \partial \tilde{\alpha}(w) = 0 \) and so
\[
\frac{d \pi^*(w)}{d w} = \frac{\partial \pi^*(w)}{\partial \tilde{\alpha}(w)} \frac{d \tilde{\alpha}(w)}{d w} + \frac{\partial \pi^*(w)}{\partial w} = \frac{\partial \pi^*(w)}{\partial w} > 0, \quad \text{for all } w \in [c, \hat{w}].
\]
We conclude that the optimal \( w^* \) belongs to \([\hat{w}, 1]\). In this region, \( \alpha_1(w) = \tilde{\alpha}(w) \) and \( Q_1(w) = \tilde{Q}(w) \) as a result the supplier’s payoff is given by
\[
(\hat{\pi}^*(w), Q_1(w)) = \left( \tilde{Q}(w) \right) \mathbb{E}[(D + B - w \tilde{Q}(w))^{-}], \quad \text{for all } w \in [\hat{w}, 1].
\]
To prove the monotonicity of the supplier’s payoff on \( B \), let us define the function \( w(\tilde{Q}), B \) as the unique solution of \( \tilde{F}(\tilde{Q}) = w \tilde{F}(\tilde{Q} - B) \). In other words, \( w(\tilde{Q}, B) \) is the inverse function of \( \tilde{Q}(w, B) \), which is guaranteed to exist by Lemma 1. Then, for a fixed \( \tilde{Q} \), we can show that the supplier’s payoff satisfies
\[
\Pi^*_S(\tilde{Q}, B) = \tilde{Q}(\tilde{F}(\tilde{Q}) - c) + \int_0^{w(\tilde{Q}, B) \tilde{Q} - B} [D - (w(\tilde{Q}, B) \tilde{Q} - B)] d\tilde{F}(D).
For notational convenience, in what follows we drop the dependence of \( w(\tilde{Q}, B) \) on both argument. Taking partial derivative with respect to \( B \), it follows that

\[
\frac{\partial \Pi_{SI}(\tilde{Q}, B)}{\partial B} = F(w \tilde{Q} - B) + \tilde{F}(w \tilde{Q} - B) \tilde{Q} \frac{\partial w}{\partial B}
\]

\[
= F(w \tilde{Q} - B) - F(w \tilde{Q} - B) \left[ \frac{w \tilde{Q} f(w \tilde{Q} - B)}{\tilde{F}(w \tilde{Q} - B) - w f(w \tilde{Q} - B)} \right]
\]

\[
= 1 - \frac{\tilde{F}(w \tilde{Q} - B)}{1 - w \tilde{Q} h(w \tilde{Q} - B)} \leq 1 - \frac{\tilde{F}(w \tilde{Q} - B)}{1 - (w \tilde{Q} - B) h(w \tilde{Q} - B)} := g(\tilde{Q} - B).
\]

The second equality follows from the definition of \( w(\tilde{Q}, B) \). By Assumption 1, the function \( h(x) \) is increasing and so the function

\[
g(Q) := \frac{\tilde{F}(Q)}{1 - Q h(Q)}
\]

satisfies \( g(Q) \geq 1 \) for all \( Q < \hat{Q} \), where \( \hat{Q} \) is the unique root of \( Q h(Q) = 1 \). We conclude that

\[
\frac{\partial \Pi_{SI}(\hat{Q}, B)}{\partial B} \leq 0,
\]

for all \( w \) such that \( w \tilde{Q} - B < \hat{Q} \). It is not hard to show that \( \hat{Q} \) maximizes \( Q \hat{F}(Q) \).

This property together with the fact that \( w(\hat{Q}, B) \) is the unique solution of \( \hat{Q} \hat{F}(\hat{Q}) = w \hat{Q} \hat{F}(w \tilde{Q} - B) \) imply that for any \( \hat{Q} \), we must have \( w \tilde{Q} - B < \hat{Q} \). We conclude that

\[
\frac{\partial \Pi_{SI}(\hat{Q}, B)}{\partial B} \leq 0
\]

for all \( \hat{Q} \), which completes the proof. \( \square \)

**Proof of Proposition 5:** Consider \( B \leq \hat{B} \) and recall that \( \Pi_{S}(w, B) = (w - c) \tilde{F}^{-1}(w) \) for \( B \geq \tilde{w}(B) \). Let us compare the left and right derivatives of \( \Pi_{S}(w, B) \) at \( w = \tilde{w}(B) \). For the left derivative of \( \Pi_{S}(w, B) \) we use equation (1.7) at \( \tilde{w}(B) \) and the fact that \( \tilde{w}(B) \hat{Q}(\tilde{w}(B)) - B = 0 \) to get

\[
\left. \frac{d \Pi_{S}(w, B)}{d w} \right|_{w = \tilde{w}(B)} = \tilde{Q}(\tilde{w}(B)) + (w - c) \left. \frac{d \tilde{Q}(w)}{d w} \right|_{w = \tilde{w}(B)}.
\]

Using the fact that \( \tilde{Q}(\tilde{w}(B)) = \tilde{F}^{-1}(\tilde{w}(B)) \), and the expression for the derivative of \( \tilde{Q}(w) \) with respect to \( w \) obtained in the proof of Lemma 1, we get that
The (weak) inequality follows from the fact that the term in the square bracket is negative, a fact that can be proved using a similar argument to the one used in the proof of Lemma 1.

Now, if \( f(0) = 0 \) then \( h(0) = 0 \) and the inequality is an equality. Hence, the supplier’s payoff is smooth at \( w = \bar{w} \). The result follows from this observation and the following facts: (i) \( \Pi^S_I(w; B) \) is unimodal in \( w \in [\bar{w}(B), \bar{w}(B)] \), (ii) \( F^{-1}(w) \) is unimodal in \( w \in [\bar{w}(B), 1] \), and (iii) \( F^{-1}(w) \) is maximized at \( w = w_E \). Therefore, if \( B \geq B_R \) then \( \bar{w}(B) \leq w_E \) and we must have \( w_I(B) = w_E \). □

**Proof of Lemma 2:** Let us first suppose (by contradiction) that \( \mathbb{E}[L] < L \) then by Jensen’s inequality we get that

\[
\mathbb{E}[\min\{L, L(1+r)\}] \leq \min\{\mathbb{E}[L], L(1+r)\} < L
\]

for all \( r \). Therefore, for a loan \((L, L, r)\) to be feasible in a competitive financial market we must have \( \mathbb{E}[L] \geq L \).

To prove the sufficient condition, let us define \( G(r) = \mathbb{E}[\min\{L, L(1+r)\}] - L \) which is a continuous function of \( r \). Next note that if \( \mathbb{E}[L] \geq L \) then \( G(0) \leq 0 \) and \( G(\infty) \geq 0 \). Therefore, by continuity there exists and \( r \geq 0 \) such that \( G(r) = 0 \). □
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