Dynamic Pricing for Non-Perishable Products with Demand Learning

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Motivation

![Inventory](image1)

![Price](image2)

Dynamic Pricing with Demand Learning
Motivation

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Product 1

New Product

Regular Season

Clearance

$N_0$

$N'_0$

$\tau_1$

$\tau_1 + \tau_2$

Dynamic Pricing with Demand Learning
Motivation

- For many retail operations, “capacity” is measured by store/shelf space.

- A key performance measure in the industry is Average Sales per Square Foot per Unit Time.

- Trade-off between short-term benefits and the opportunity cost of assets. Margin vs. Rotation.

- As opposed to the airline or hospitality industries, selling horizons are flexible.

- In general, most profitable/unprofitable products are new items for which there is little demand information.
Outline

✓ Model Formulation.

✓ Perfect Demand Information.

✓ Incomplete Demand Information.
  - Inventory Clearance
  - Optimal Stopping ("outlet option")

✓ Conclusion.
Model Formulation

I) Stochastic Setting:
- A probability space \((\Omega, \mathcal{F}, \mathbb{P})\).
- A standard Poisson process \(D(t)\) under \(\mathbb{P}\) and its filtration \(\mathcal{F}_t = \sigma(D(s) : 0 \leq s \leq t)\).
- A collection \(\{\mathbb{P}_\alpha : \alpha > 0\}\) such that \(D(t)\) is a Poisson process with intensity \(\alpha\) under \(\mathbb{P}_\alpha\).
- For a process \(f_t\), we define \(I_f(t) := \int_0^t f_s \, ds\).

II) Demand Process:
- Pricing strategy, a nonnegative (adapted) process \(p_t\).
- A price-sensitive unscaled demand intensity
  \[\lambda_t := \lambda(p_t) \iff p_t = p(\lambda_t).\]
- A (possibly unknown) demand scale factor \(\theta > 0\).
- Cumulative demand process \(D(I_{\lambda}(t))\) under \(\mathbb{P}_\theta\).
- Select \(\lambda \in \mathcal{A}\) the set of admissible (adapted) policies
  \[\lambda_t : \mathbb{R}_+ \rightarrow [0, \Lambda].\]
Model Formulation

III) Revenues:

- Unscaled revenue rate \( c(\lambda) := \lambda p(\lambda) \), \( \lambda^* := \arg\max_{\lambda \in [0, \Lambda]} \{ c(\lambda) \} \), \( c^* := c(\lambda^*) \).

- Terminal value (opportunity cost): \( R \)

- Normalization: \( c^* = r R \).

IV) Selling Horizon:

- Inventory position: \( N_t = N_0 - D(I_\lambda(t)) \).

- \( \tau_0 = \inf\{ t \geq 0 : N_t = 0 \} \), \( T := \{ F_t - \text{stopping times } \tau \text{ such that } \tau \leq \tau_0 \} \)

V) Retailer’s Problem:

\[
\max_{\lambda \in \mathcal{A}, \tau \in T} \mathbb{E}_{\theta} \left[ \int_0^\tau e^{-r t} p(\lambda_t) \, dD(I_\lambda(t)) + e^{-r \tau} R \right]
\]

subject to \( N_t = N_0 - D(I_\lambda(t)) \).
Suppose \( \theta > 0 \) is known at \( t = 0 \) and an inventory clearance strategy is used, i.e., \( \tau = \tau_0 \).

Define the value function
\[
W(n; \theta) = \max_{\lambda \in A} \mathbb{E}_\theta \left[ \int_0^{\tau_0} e^{-rt} p(\lambda_t) dD(I_\lambda(t)) + e^{-r \tau} R \right]
\]
subject to \( N_t = n - D(I_\lambda(t)) \) and \( \tau_0 = \inf\{t \geq 0 : N_t = 0\} \).

The solution satisfies the recursion
\[
\frac{r W(n; \theta)}{\theta} = \Psi(W(n-1; \theta) - W(n; \theta)) \quad \text{and} \quad W(0; \theta) = R,
\]
where
\[
\Psi(z) \triangleq \max_{0 \leq \lambda \leq \Lambda} \{\lambda z + c(\lambda)\}.
\]

**Proposition.** For every \( \theta > 0 \) and \( R \geq 0 \) there is a unique solution \( \{W(n) : n \in \mathbb{N}\} \).

- If \( \theta \geq 1 \) then the value function \( W \) is increasing and concave as a function of \( n \).
- If \( \theta \leq 1 \) then the value function \( W \) is decreasing and convex as a function of \( n \).
- \( \lim_{n \to \infty} W(n) = \theta R \).
Value function for two values of $\theta$ and an exponential demand rate $\lambda(p) = \Lambda \exp(-\alpha p)$.

The data used is $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = \Lambda \exp(-1)/(\alpha r) \approx 3.68$. 
Corollary. Suppose $c(\lambda)$ is strictly concave.

The optimal sales intensity satisfies:

$$\lambda^*(n; \theta) = \arg\max_{0 \leq \lambda \leq \Lambda} \{ \lambda (W(n-1; \theta) - W(n; \theta)) + c(\lambda) \}.$$  

- If $\theta \geq 1$ then $\lambda^*(n; \theta) \uparrow n$.
- If $\theta \leq 1$ then $\lambda^*(n; \theta) \downarrow n$.
- $\lambda^*(n; \theta) \downarrow \theta$.
- $\lim_{n \to \infty} \lambda^*(n, \theta) = \lambda^*$.

What about inventory turns (rotation)?

**Proposition.** Let $s(n, \theta) \triangleq \theta \lambda^*(n, \theta)$ be the optimal sales rate for a given $\theta$ and $n$.

If $$\frac{d}{d\lambda} (\lambda p'(\lambda)) \leq 0,$$ then $s(n, \theta) \uparrow \theta$. 

Exponential Demand $\lambda(p) = \Lambda \exp(-\alpha p)$.
$\Lambda = 10$, $\alpha = r = 1$, $\theta_1 = 1.2$, $\theta_2 = 0.8$, $R = 3.68$. 

Dynamic Pricing with Demand Learning 

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Full Information

**Summary:**

- A tractable dynamic pricing formulation for the inventory clearance model.
- $W(n; \theta)$ satisfies a simple recursion based on the Fenchel-Legendre transform of $c(\lambda)$.
- With full information products are divided in two groups:
  - High Demand Products with $\theta \geq 1$: $W(n, \theta)$ and $\lambda^*(n)$ increase with $n$.
  - Low Demand Products with $\theta \leq 1$: $W(n, \theta)$ and $\lambda^*(n)$ decrease with $n$.
- High Demand products are sold at a higher price and have a higher selling rate.
- If the retailer can stop selling the product at any time at no cost then:
  - If $\theta < 1$ stop immediately ($\tau = 0$).
  - If $\theta > 1$ never stop ($\tau = \tau_0$).
- In practice, a retailer rarely knows the value of $\theta$ at $t = 0$!
Incomplete Information: Inventory Clearance

Setting:
- The retailer does not know $\theta$ at $t = 0$ but knows $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq 1 \leq \theta_H$.
- The retailer has a prior belief $q \in (0, 1)$ that $\theta = \theta_L$.
- We introduce the probability measure $\mathbb{P}_q = q\mathbb{P}_{\theta_L} + (1 - q)\mathbb{P}_{\theta_H}$.
- We assume an inventory clearance model, i.e., $\tau = \tau_0$.

Retailer's Beliefs:
Define the belief process $q_t := \mathbb{P}_q[\theta | \mathcal{F}_t]$.

Proposition. $q_t$ is a $\mathbb{P}_q$-martingale that satisfies the SDE
\[
dq_t = -\eta(q_t) [dD_t - \lambda_t \bar{\theta}(q_t)dt],
\]
where $\bar{\theta}(q) := \theta_L q + \theta_H (1 - q)$
and $\eta(q) := \frac{q (1 - q) (\theta_H - \theta_L)}{\theta_L q + \theta_H (1 - q)}$. 
Retailer’s Optimization:

\[ V(N_0, q) = \sup_{\lambda \in \mathcal{A}} \mathbb{E}_q \left[ \int_0^{\tau_0} e^{-rt} p(\lambda_t) \, dD(I(\lambda(s))) + e^{-r\tau_0} R \right] \]

subject to

\[ N_t = N_0 - \int_0^t dD(I(\lambda(s))), \]

\[ dq_t = -\eta(q_t-) [dD_t - \lambda_t \bar{\theta}(q_t-) \, dt], \quad q_0 = q, \]

\[ \tau_0 = \inf \{ t \geq 0 : N_t = 0 \}. \]

HJB Equation:

\[ rV(n, q) = \max_{0 \leq \lambda \leq \Lambda} \left[ \lambda \bar{\theta}(q)[V(n - 1, q - \eta(q)) - V(n, q) + \eta(q)V_q(n, q)] + \bar{\theta}(q) c(\lambda) \right], \]

with boundary condition \( V(0, q) = R, V(n, 0) = W(n; \theta_H), \) and \( V(n, 1) = W(n; \theta_L). \)

Recursive Solution:

\[ V(0, q) = R, \quad V(n, q) + \Phi \left( \frac{r V(n, q)}{\bar{\theta}(q)} \right) - \eta(q) V_q(n, q) = V(n - 1, q - \eta(q)). \]
**Proposition.**

- The value function $V(n, q)$ is
  
  a) monotonically decreasing and convex in $q$,
  
  b) bounded by
  
  $$W(n; \theta_L) \leq V(n, q) \leq W(n; \theta_H),$$

  and

  c) uniformly convergent as $n \uparrow \infty$,

  $$V(n, q) \xrightarrow{n \to \infty} R \bar{\theta}(q),$$

  uniformly in $q$.

- The optimal demand intensity satisfies

  $$\lim_{n \to \infty} \lambda^*(n, q) = \lambda^*.$$ 

**Conjecture:**

The optimal sales rate $\bar{\theta}(q) \lambda^*(n, q) \downarrow q$. 

\[\theta(q)R = [\theta_L q + \theta_H (1-q)]R\]
Incomplete Information: Inventory Clearance

**Asymptotic Approximation:** Since

\[
\lim_{{n \to \infty}} V(n, q) = R \bar{\theta}(q) = \lim_{{n \to \infty}} \{q W(n, \theta_L) + (1 - q) W(n, \theta_H)\},
\]

we propose the following approximation for \( V(n, q) \)

\[
\tilde{V}(n, q) := q W(n, \theta_L) + (1 - q) W(n, \theta_H).
\]

**Some Properties of \( \tilde{V}(n, q) \):**

- Linear approximation easy to compute.
- Asymptotically optimal as \( n \to \infty \).
- Asymptotically optimal as \( q \to 0^+ \) or \( q \to 1^- \).
- \( \tilde{V}(n, q) = \mathbb{E}_q[W(n, \theta)] \neq W(n, \mathbb{E}_q[\theta]) =: V_{CE}^n(n, q) = \text{Certainty Equivalent.} \)
Relative Error (%) := $\frac{V^{\text{approx}}(n, q) - V(n, q)}{V(n, q)} \times 100\%$.

Exponential Demand $\lambda(p) = \Lambda \exp(-\alpha p)$:

Inventory = 5, $\Lambda = 10$, $\alpha = r = 1$, $\theta_H = 5.0$, $\theta_L = 0.5$. 
Incomplete Information: Inventory Clearance

For any approximation $V^{\text{approx}}(n, q)$, define the corresponding demand intensity using the HJB

$$\lambda^{\text{approx}}(n, q) := \arg \max_{0 \leq \lambda \leq \Lambda} [\lambda \bar{\theta}(q)[V^{\text{approx}}(n-1, q-\eta(q)) - V^{\text{approx}}(n, q)] + \lambda \kappa(q)V^{\text{approx}}_q(n, q) + \bar{\theta}(q)c(\lambda)].$$

Relative Price Error (%) := $$\frac{p(\lambda^{\text{approx}}) - p(\lambda^*)}{p(\lambda^*)} \times 100\%.$$
When should the retailer engage in selling a given product?

When \( V(n, q) \geq R \).

Using the asymptotic approximation \( \tilde{V}(n, q) \), this is equivalent to

\[
q \leq \tilde{q}(n) := \frac{W(n; \theta_H) - R}{[W(n; \theta_H) - R] + [R - W(n; \theta_L)]}.
\]

Exponential demand rate \( \lambda(p) = \Lambda \exp(-\alpha p) \).
Data: \( \Lambda = 10, \alpha = 1, r = 1, \theta_H = 1.2, \theta_L = 0.8 \).
**Incomplete Information: Inventory Clearance**

**Summary:**

- Uncertainty in market size ($\theta$) is captured by a new state variable $q_t$ (a jump process).

- $V(n, q)$ can be computed using a recursive sequence of ODEs.

- Asymptotic approximation $\tilde{V}(n, q) := \mathbb{E}_q[W(n, \theta)]$ performs quite well.
  - Linear approximation easy to compute.
  - Value function: $V(n, q) \approx \tilde{V}(n, q)$.
  - Pricing strategy: $p^*(n, q) \approx \tilde{p}(n, q)$.

- Products are divided in two groups as a function of $(n, q)$:
  - Profitable Products with $q < \tilde{q}(n)$ and
  - Non-profitable Products with $q > \tilde{q}(n)$.

- The threshold $\tilde{q}(n)$ increases with $n$, that is, the retailer is willing to take more risk for larger orders.
Incomplete Information: Optimal Stopping

**Setting:**
- Retailer does not know $\theta$ at $t = 0$ but knows $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L \leq 1 \leq \theta_H$.
- Retailer has the option of removing the product at any time, “Outlet Option”.

**Retailer’s Optimization:**

$$U(N_0, q) = \max_{\lambda \in \mathcal{A}, \tau \in \mathcal{T}} \mathbb{E}_q \left[ \int_0^\tau e^{-rt} p(\lambda_t) dD(I_\lambda(t)) + e^{-r\tau} R \right]$$

subject to

$$N_t = N_0 - D(I_\lambda(t)),$$

$$dq_t = -\eta(q_t) [dD(I_\lambda(t)) - \lambda_t \bar{\theta}(q_t) dt], \quad q_0 = q.$$

**Optimality Conditions:**

$$\begin{cases} U(n, q) + \Phi\left( \frac{rU(n, q)}{\theta(q)} \right) - \eta(q) U_q(n, q) = U(n - 1, q - \eta(q)) & \text{if } U \geq R \\ U(n, q) + \Phi\left( \frac{rU(n, q)}{\theta(q)} \right) - \eta(q) U_q(n, q) \leq U(n - 1, q - \eta(q)) & \text{if } U = R. \end{cases}$$
**Incomplete Information: Optimal Stopping**

**Proposition.**

a) There is a unique continuously differentiable solution $U(n, \cdot)$ defined on $[0, 1]$ so that $U(n, q) > R$ on $[0, q_n^*)$ and $U(n, q) = R$ on $[q_n^*, 1]$, where $q_n^*$ is the unique solution of

$$R + \Phi \left( \frac{r R}{\theta(q)} \right) = U(n-1, q - \eta(q)).$$

b) $q_n^*$ is increasing in $n$ and satisfies

$$\frac{\theta_H - 1}{\theta_H - \theta_L} \leq q_n^* \xrightarrow{n \to \infty} q_\infty \leq \text{Root} \left\{ \Phi \left( \frac{r R}{\theta(q)} \right) = \frac{\eta(q)}{q} (\theta_H - 1) R \right\} < 1.$$

c) The value function $U(n, q)$

- Is decreasing and convex in $q$ on $[0, 1]$
- Increases in $n$ for all $q \in [0, 1]$ and satisfies

$$\max\{R, V(n, q)\} \leq U(n, q) \leq \max\{R, m(q)\} \quad \text{for all } q \in [0, 1],$$

where

$$m(q) := W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{q_n^*} q.$$

- Converges uniformly (in $q$) to a continuously differentiable function, $U_\infty(q)$. 
Exponential demand rate $\lambda(p) = \Lambda \exp(-\alpha p)$. Data: $\Lambda = 10$, $\alpha = 1$, $r = 1$, $\theta_H = 1.2$, $\theta_L = 0.8$. 
Incomplete Information: Optimal Stopping

**Approximation:**

\[ \tilde{U}(n, q) := \max\{R, W(n, \theta_H) - \frac{(W(n, \theta_H) - R)}{\tilde{q}_n} q\} \]

where \( \tilde{q}_n \) is the unique solution of

\[ R + \Phi\left(\frac{r R}{\theta(q)}\right) = \tilde{U}(n - 1, q - \eta(q)). \]

Exponential demand rate \( \lambda(p) = \Lambda \exp(-\alpha p) \).

Data: \( \Lambda = 10, \alpha = 1, r = 1, \theta_H = 1.2, \theta_L = 0.8. \)
Incomplete Information: Optimal Stopping

**Summary:**

- $U(n, q)$ can be computed using a recursive sequence of ODEs with free-boundary conditions.
- For every $n$ there is a critical belief $q^*_n$ above which it is optimal to stop.
- Again, the sequence $q^*_n$ is increasing with $n$, that is, the retailer is willing to take more risk for larger orders.
- The sequence $q^*_n$ is bounded by
  \[
  \frac{\theta_H - 1}{\theta_H - \theta_L} \leq q^*_n \leq \hat{q} := \text{Root} \left\{ \Phi \left( \frac{r R}{\theta(q)} \right) = \frac{\eta(q)}{q} (\theta_H - 1) R \right\}
  \]
- The “outlet option” increases significantly the expected profits and the range of products $(n, q)$ that are profitable.
  \[
  0 \leq U(n, q) - V(n, q) \leq (1 - \theta_L)^+ R.
  \]
- A simple piece-wise linear approximation works well.
  \[
  \tilde{U}(n, q) := \max\{R, W(n, \theta_H) - \frac{(W(n, \theta_H) - R) \tilde{q}_n}{q}\}
  \]
Concluding Remarks

○ A simple dynamic pricing model for a retailer selling non-perishable products.

○ Captures two common sources of uncertainty:
  – Market size measured by $\theta \in \{\theta_H, \theta_L\}$.
  – Stochastic arrival process of price sensitive customers.

○ Analysis gets simpler using the Fenchel-Legendre transform of $c(\lambda)$ and its properties.

○ We propose a simple approximation (linear and piecewise linear) for the value function and corresponding pricing policy.

○ Some properties of the optimal solution are:
  – Value functions $V(n, q)$ and $U(n, q)$ are decreasing and convex in $q$.
  – The retailer is willing to take more risk ($\uparrow q$) for higher orders ($\uparrow n$).
  – The optimal demand intensity $\lambda^*(n, q) \uparrow q$ and the optimal sales rate $\bar{\theta}(q) \lambda^*(n, q) \downarrow q$.

○ Extension: $R(n) = R + \nu n - K \mathbb{1}(n > 0)$. 

Dynamic Pricing with Demand Learning