Outline

- Statistical models of security prices and order impacts
- Given these statistical models, what are the best order-splitting strategies.
- The risk-return trade-off in order splitting.
Statistical models

- The basic models are constructed by starting with a simple model and adding on the features that we need.
- Random-walk model
- Random-walk + drift (“short-term alpha”)
- Impact model: Random-walk + drift + order-price impact

Random-walk model

- Let $t$ represent time (minutes, seconds, milliseconds, ticks ...)
- $p_t$ is the price at the end of interval $t$ (at the end of the minute, second, ...)
  - Usually $p_t$ is the bid-ask midpoint (BAM), but it might be the last sale price.
- $p_t = p_{t-1} + u_t$
  - where $u_t$ is some random disturbance or prediction error that reflects “new information”
    - In expectation this disturbance is zero: $E u_t = 0$.
    - The standard deviation of $u_t$ is $\sigma_u$. 
Recall the example used to analyze implementation shortfall of limit vs. market orders.

- At each step, $u_t = \pm 0.01$ with equal probability.
- $E u_t = \frac{1}{2} \times (0.01) + \frac{1}{2} \times (-0.01) = 0$
- $\sigma_u = \sqrt{\frac{1}{2} \times (0.01)^2 + \frac{1}{2} \times (-0.01)^2} = 0.01$

Looking ahead

- $p_1 = p_0 + u_1$
  $p_2 = p_1 + u_2 = p_0 + u_1 + u_2$
  $p_3 = p_2 + u_3 = p_0 + u_1 + u_2 + u_3$
  ...
  $p_K = p_{K-1} + u_K = p_0 + u_1 + u_2 + \cdots + u_K$
- As of time $t = 0$, all future $u$s are expected to be zero:
  $E p_K = p_0$
What does this model:

\[ p_K = p_{K-1} + u_K = p_0 + u_1 + u_2 + \cdots + u_K \]

say about risk/uncertainty?

As of time \( t = 0 \), the variance of \( p_K \) is:

\[ \text{Var}(p_K) = \text{Var}(u_1) + \cdots + \text{Var}(u_K) \]

(Assuming that the \( u \)s are uncorrelated.)

If the variance of the \( u \)s is constant,

\[ \text{Var}(p_K) = K \times \text{Var}(u) \]

Random-walk with drift (“short-term alpha”)

The basic model \((p_t = p_{t-1} + u_t)\) has no trend (on average)

A trader might believe that there is a predictable trend (from momentum, over-shooting, consensus forecast error, etc.)

\[ p_t = \alpha + p_{t-1} + u_t \]

- \( \alpha \) denotes the trend (either up or down), such as, “+$0.01 per minute”
- “alpha” is used in many finance contexts to denote superior (or, if negative, inferior) performance.

This model is used extensively in option pricing.
Simulated random walk (with/without $\alpha = 0.002$ per tick)

- Non-zero alpha is difficult to detect visually and statistically.

The random-walk with drift: Limit order execution times

- Suppose that the current stock price is $S_0$ and we want to put in a limit order to sell at some price $S_{sell} > S_0$.
- Example: $S_0 = $15 and $S_{sell} = $17.
- How long will it will take for the order to execute?
  - Starting at $15, how long will it take the price to hit $17?  
- Next slide: 3 simulated paths based on $p_t = \alpha + p_{t-1} + u_t$ where
  - $\alpha$ is $0.002$ per second
  - $u_t$ is normally distributed with mean zero and standard deviation $\sigma_u = 0.02$
A transformation

- Suppose $p_t = \alpha + p_{t-1} + u_t$ for $t = 1, 2, \ldots$ seconds
- $p_t - p_{t-1} = \alpha + u_t$
  - “In each one-second time step, the price change is $\alpha$ plus some random disturbance.”
- Keep this framework, but let the time step size get smaller: one second, one-half second, one-quarter second, ...
- In the limit, $p_t$ becomes a Brownian motion process.
Suppose that \( p \) is a Brownian motion with drift \( \alpha > 0 \) per second and variance \( \sigma_u^2 \) per second, and at time zero, \( p = p_0 \).

Let \( t \) be the execution time of a sell limit order (the first hitting time of a barrier located at price \( x - p_0 > 0 \), the limit price).

Then \( t \) is a random variable with probability density function

\[
f(t) = \frac{x - p_0}{\sigma_u \sqrt{2\pi t^3}} e^{-\frac{(x-p_0-\alpha t)^2}{2\sigma_u^2 t}}
\]

The expectation is \( Et = (x - p_0)/\alpha \)

References


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Plot of \( f(t) \) with \( \alpha = 0.002 \) and \( \sigma_u = 0.02 \)

\[
Et = \frac{x-p_0}{\alpha} = \frac{17-15}{0.002} = 1,000 \text{ seconds}
\]

If \( \alpha \leq 0 \), \( Et = \infty \)

(You might get an execution, but don’t count on it.)

Embedded problem: \( S_0 = 20, \alpha = -0.03 \) per minute. We put in a limit order to buy at $19.10. How long do we expect to wait? (Answer in online notes)

Answer: \( \frac{19.10-20.00}{-0.03} = \frac{0.90}{0.03} = 30 \) minutes
What’s useful in predicting short-term alpha? *(revised)*

- Market data (bids, asks, trades) for the stock, the industry (other stocks, same industry), the market index.
- Market data from other markets (bonds, FX, commodities, stock markets in other countries)
- News announcements.
- Traffic on search engines, social media.

The impact model: random-walk + drift + order/price impact

\[ p_t = p_{t-1} + \alpha + \lambda S_t + u_t \]

- \( S_t \) is the net number of shares actively purchased in interval \( t \).
  - the number of shares that lifted the ask less the number of shares that hit the bid.
  - Example: \( S_t = -100 \rightarrow \) “100 shares were sold, net”
- \( \lambda > 0 \) is the impact coefficient.
  - As \( \lambda \) increases, each trade has a larger impact.
- “Purchases drive the price up; sales drive the price down.”
Example

- Suppose that right now \( t = 0 \), the price is \( p_0 = $10 \)
- \( p_t = p_{t-1} + \alpha + \lambda S_t + u_t \) where \( \alpha = 0.01 \) and \( \lambda = 0.0001 \).
- If the next trade is a \( 2,000 \)-share buy order, the predicted value of \( p_1 \) is
  \[ E_{p_1} = 10 + 0.01 + 0.0001 \times 2,000 = $10.21 \]
- If the following trade is a \( 500 \)-share sell order, the predicted value of \( p_2 \) is
  \[ E_{p_2} = 10.21 + 0.01 + 0.0001 \times (-500) = $10.17 \]

Interpretation of \( S_t \)

- In principle \( S_t \) is the net purchase computed over all trades in interval \( t \).
- Including our own trades and trades of others.
- \( S_t = S_{t}^{\text{own}} + S_{t}^{\text{others}} \)
- For forecasting and analysis, we want to use the best available prediction of \( S_{t}^{\text{others}} \).
- Often trading strategies are analyzed assuming that our expectation of others’ trades is \( E S_{t}^{\text{others}} = 0 \)
Interpretation of $\lambda$ (revised)

- The model says that order-price impact is permanent.
- Order price impact arises from the market’s belief that orders *might* be informed.
- All orders move the price in the short-run (minutes, hours), but in the long-run (months, years) the security price is determined by fundamentals.
  - Our orders don’t have any long-term impact.