Order splitting

Securities Trading: Principles and Procedures
Chapter 14

Outline

- What is order splitting?
- Why is it complicated?
- VWAP strategies.
- Strategies that minimize trading costs.
Order splitting

- Classic problem: A pension or mutual fund needs to buy (or sell) some large quantity by the end of some trading period (typically an hour, day or week).
- Response: split the large quantity (the parent order) into smaller child orders and submitting them to the market over the trading period.
- Complications
  - Market prices are often changing over the trading period.
  - Our earlier trades move the price against us for our later trades.

VWAP Strategies

- The implementation shortfall measure of trading cost for a buy order is
  \[ IS = \text{Avg purchase price} - \text{benchmark price} \]
- As choices for the benchmark price, we considered:
  - The bid-ask midpoint prevailing when the parent order was generated.
  - VWAP the volume-weighted average price over the trading period.
  This benchmark is common: it is easy to compute.
Example: daily VWAP calculation

<table>
<thead>
<tr>
<th>Time</th>
<th>Trade Price</th>
<th>Trade Shares</th>
<th>Price x Shares</th>
<th>Trade Price</th>
<th>Trade Shares</th>
<th>Price x Shares</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:40 AM</td>
<td>$10.10</td>
<td>500</td>
<td>$5,050</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9:45 AM</td>
<td>$10.08</td>
<td>800</td>
<td>$8,064</td>
<td>$10.08</td>
<td>800</td>
<td>$8,064</td>
</tr>
<tr>
<td>10:08 AM</td>
<td>$10.15</td>
<td>200</td>
<td>$2,030</td>
<td>$10.15</td>
<td>200</td>
<td>$2,030</td>
</tr>
<tr>
<td>11:03 AM</td>
<td>$10.12</td>
<td>100</td>
<td>$1,012</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11:58 AM</td>
<td>$10.40</td>
<td>200</td>
<td>$2,080</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1:15 PM</td>
<td>$10.50</td>
<td>300</td>
<td>$3,150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2:20 PM</td>
<td>$10.45</td>
<td>400</td>
<td>$4,180</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3:30 PM</td>
<td>$10.70</td>
<td>200</td>
<td>$2,140</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3:45 PM</td>
<td>$10.61</td>
<td>600</td>
<td>$6,366</td>
<td>$10.61</td>
<td>600</td>
<td>$6,366</td>
</tr>
<tr>
<td>3:50 PM</td>
<td>$10.62</td>
<td>900</td>
<td>$9,558</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

VWAP $10.388
4,200 $43,630
$10.288 1,600 $16,460

Notes:
- Our trades are indicated in bold.
- The VWAP benchmark is computed using all trades in the day (including ours).
- Our VWAP is computed using our trades only.
- If our orders are purchases, $IS = 10.288 - 10.388 = -$0.10$

Analysis

- If our orders are purchases, $IS = 10.288 - 10.388 = -$0.10$
- We bought below VWAP
  - This is good: our aim should be to beat VWAP.
- BUT: Many institutions simply aim to achieve VWAP.
  - VWAP becomes the target trade price.
VWAP Strategies

- Suppose that we want to buy 10,000 shares and the average daily volume (ADV) is 200,000.
- Ideally, we’d like to get $\frac{10,000}{200,000} = 5\%$ of each trade during the day.
- This isn’t practical.
  - Limit order books are handled in price/time priority: we can’t simply claim 5% of each trade.
- Instead, we try to approximate VWAP by trading 5% of volume over small time intervals.

Example

- Divide the 9:30 AM - 4:00 PM trading day into (thirteen) 30-minute intervals: 9:30-10:00, 10:00-10:30, 10:30-11:00, ..., 15:30-16:00.
- Try to trade 5% of the volume in each interval.
- If an interval had a total trading volume of 20,000 shares, we’d try to trade 1,000 shares (of the 20,000).
Daily volume profiles

- If volume were evenly distributed over the day, we’d trade at a constant rate:
  \[
  \frac{10,000}{13} = 769 \text{ shares}
  \]
  in each interval.
- But volume is usually not evenly distributed throughout the day.
  - It is elevated at the start and close, lower in the middle.
  - Typically ...

We collect a sample of trading volume for a recent period and look at 30-minute averages ...

<table>
<thead>
<tr>
<th>Interval</th>
<th>Avg volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:30</td>
<td>26,000</td>
</tr>
<tr>
<td>10:00</td>
<td>20,000</td>
</tr>
<tr>
<td>10:30</td>
<td>18,000</td>
</tr>
<tr>
<td>11:00</td>
<td>12,000</td>
</tr>
<tr>
<td>11:30</td>
<td>12,000</td>
</tr>
<tr>
<td>12:00</td>
<td>12,000</td>
</tr>
<tr>
<td>12:30</td>
<td>12,000</td>
</tr>
<tr>
<td>13:00</td>
<td>12,000</td>
</tr>
<tr>
<td>13:30</td>
<td>12,000</td>
</tr>
<tr>
<td>14:00</td>
<td>12,000</td>
</tr>
<tr>
<td>14:30</td>
<td>12,000</td>
</tr>
<tr>
<td>15:00</td>
<td>16,000</td>
</tr>
<tr>
<td>15:30</td>
<td>24,000</td>
</tr>
</tbody>
</table>

200,000 100%
The basic approach

- Match your rate of trading to the average daily volume profile.
- If our total (parent) order size is 10,000 shares, then:
  - Trade $13\% \times 10,000 = 1,300\, sh$ between 9:30 and 10:00
  - Trade $10\% \times 10,000 = 1,000\, sh$ between 10:00 and 10:30
  - ...
  - Trade $12\% \times 10,000 = 1,200\, sh$ between 13:30 and 14:00

Strategies that minimize trading cost

- Objective: minimize the total outlay for shares that we’re buying
  - ... maximize the total amount received for shares that we’re selling.
- For a purchase, the implementation shortfall relative to the bid-ask midpoint (BAM) is:
  \[
  IS = \text{Avg purchase price} - \text{BAM}
  \]
- BAM is fixed when the parent order is generated.
  - It doesn’t change while the parent order is being split and worked.
  - So strategies that minimize the average purchase price will also minimize IS.
Analysis of a representative problem. We will ...

- ... figure out how to split a purchase of $S_{Total}$ shares over two periods.
- ... assume a linear order price-impact.
- ... investigate the cost and risk of the strategy.
- ... generalize to cases of >2 trading periods.

The two-period order splitting problem

- Objective: buy $S_{Total}$ shares over two periods (1 and 2) at the lowest possible expected cost.
- $S_{Total} = S_1 + S_2$
  - where $S_1$ = shares purchased in period 1 and $S_2$ = shares purchased in period 2.
- The total cost is $C = p_1 S_1 + p_2 S_2$
- It is now time 0. We know $p_0$, but we don’t know $p_1$ or $p_2$.
- Use the impact model to forecast $p_1$ and $p_2$. 
Linear impact model: \( p_t = p_{t-1} + \alpha + \lambda S_t + u_t \)

- At first, to simplify, set the drift \( \alpha = 0 \)
- Compute ahead:
  - \( p_1 = p_0 + \lambda S_1 + u \)
  - \( p_2 = p_1 + \lambda S_2 + u_2 = p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2 \)
- Plug these into the cost expression \( C = p_1 S_1 + p_2 S_2 \)
- To get the expected cost, simplify by setting \( u_1 = u_2 = 0 \)
  - “Looking ahead, we expect \( u_1 \) and \( u_2 \) to be zero.”
- Then \( EC = p_1 S_1 + p_2 S_2 = S_1(p_0 + \lambda S_1) + S_2(p_0 + \lambda S_1 + \lambda S_2) \)
  - where \( S_1 \) and \( S_2 \) are the “unknowns”

We have two unknowns (\( S_1 \) and \( S_2 \)). Eliminate one of them.

- \( EC = p_1 S_1 + p_2 S_2 = S_1(p_0 + \lambda S_1) + S_2(p_0 + \lambda S_1 + \lambda S_2) \)
- Remember that \( S_{Total} = S_1 + S_2 \), so \( S_2 = S_{Total} - S_1 \)
- \( EC = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total}) \)
  - This is one equation in one unknown (\( S_1 \))
Plot of $EC$ (with $S_{Total} = 10,000$ sh; $p_0 = 10$; $\lambda = 0.0001/sh$)

- The total expected cost has a minimum at $S_1 = 5,000$ shares (and $S_2 = 5,000$ shares)

Formally, to find the minimum, set $\frac{d EC}{d S_1} = 0$

- $EC = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total})$
- $\frac{d EC}{d S_1} = 2\lambda S_1 - \lambda S_{Total} = 0$ implies the optimal $S_1^* = \frac{S_{Total}}{2}$.
- In general, with $\alpha = 0$, and trading over $T$ periods,
  
  $S_i^* = \frac{S_{Total}}{T}$
What if $\alpha \neq 0$ (in the two-period problem)?

- Modified optimum: $S_1^* = \frac{\alpha + \lambda S_{\text{Total}}}{2\lambda}$
- With $\alpha > 0$, there is positive drift, so $S_1^*$ rises.
  - Future purchases will be more expensive.
- With $\alpha < 0$, there is negative drift.
  - The price is dropping: buy later.
The expected cost under the optimal trading rule, $EC^*$

- With $\alpha = 0$,
  
  $$EC = \lambda S_1^2 - S_1 \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total})$$

- Under the optimal strategy, $S_1^* = S_{Total}/2$.

- At this optimum, the expected cost is
  
  $$EC^* = \lambda \left( \frac{S_{Total}}{2} \right)^2 - \left( \frac{S_{Total}}{2} \right) \lambda S_{Total} + S_{Total} (p_0 + \lambda S_{Total})$$

  $$= p_0 S_{Total} + \frac{3 \lambda S_{Total}^2}{4}$$

- If we could buy everything at the current price, we’d pay $p_0 S_{Total}$, but because of order-price impact, there’s an additional penalty.

- This penalty goes up as the square of the total size.

The variance of the cost (“risk”) under the optimal trading rule

- The cost is $C = p_1 S_1 + p_2 S_2$, where
  
  - $p_1 = p_0 + \lambda S_1 + u_1$
  - $p_2 = p_1 + \lambda S_2 + u_2 = p_0 + \lambda S_1 + \lambda S_2 + u_1 + u_2$

- The only terms that are random are those that involve $u$s, so
  
  - $C = \cdots + S_1 u_1 + \cdots S_2 (u_1 + u_2) = \cdots + S_{Total} u_1 + S_2 u_2$

- At optimum, $S_1^* = S_2^* = S_{Total}/2$

  - $C^* = \cdots + S_{Total} u_1 + \left( \frac{S_{Total}}{2} \right) u_2$

- $Var(C^*) = \left[ S_{Total}^2 + \left( \frac{S_{Total}}{2} \right)^2 \right] \sigma_u^2 = \frac{5}{4} S_{Total}^2 \sigma_u^2$

- Both the expected cost and the variance go up with $S_{Total}^2$. 
How do $EC^*$ and $Var(C^*)$ depend on $T$ (the trading horizon)?

<table>
<thead>
<tr>
<th>$T$</th>
<th>$EC^*$</th>
<th>$Var(C^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_0 S_{Total} + \lambda S^2_{Total}$</td>
<td>$S^2_{Total} \sigma_u^2$</td>
</tr>
<tr>
<td>2</td>
<td>$p_0 S_{Total} + \frac{3\lambda S^2_{Total}}{4}$</td>
<td>$\frac{5}{4} S^2_{Total} \sigma_u^2$</td>
</tr>
<tr>
<td>3</td>
<td>$p_0 S_{Total} + \frac{2\lambda S^2_{Total}}{3}$</td>
<td>$\frac{14}{9} S^2_{Total} \sigma_u^2$</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>$p_0 S_{Total} + \frac{(T + 1)\lambda S^2_{Total}}{2T}$</td>
<td>$\frac{(1 + T)(1 + 2T)S^2_{Total}}{6T} \sigma_u^2$</td>
</tr>
</tbody>
</table>

Example with $p_0 = $10, $\lambda = 0.0001$, $\sigma_u = 0.01$, $S_{Total} = 10,000$ shares
The risk/return trade-off

- This is sometimes called the “efficient trading frontier” (By analogy with the efficient portfolio frontier.)

Extensions and modifications

- More complex order impacts that have both permanent and temporary (transient) effects.
- Addition of other securities (stocks in the same industry, a market-wide basket, and so on).
- Time variation in parameters.
  - $\alpha$, $\lambda$, and/or $\sigma_u$ depend on time of day.