Network Games with Incomplete Information Played by “Bayesian” Players

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Chapter 1

Introduction

In competitive markets, nearly identical products or services can have dramatically different fates. The divergent paths of Dropbox and Box, two technology start-ups providing cloud storage services, illustrate the point. As of early 2014, Dropbox had grown to over 300 million users while Box reached just 24 million. This order-of-magnitude difference in user bases translates to valuations that differ by $8 billion. However, these data only understate the disparity in growth between the two companies—this is despite Box having a 2 year head start and making significant gains in recent years.

What drives the take up of new products and services? Diffusion through social connections is at least partly responsible—Dropbox benefited from a wildly successful referral program. The example of Dropbox and Box shows just how hard it can be to predict the outcomes of social diffusions. Beyond this one example, Leskovec et al. (2006) document substantial variation in product diffusion through personal recommendations. In a study looking at books, movies, and music sold by an online retailer, the authors track purchases of half a million products made by 4 million users. Purchase cascades come in all sizes, following a power law distribution. Most look like small trees, while a few become very large. Goel et al. (2012) find similar patterns in the diffusion of social network applications and news stories or videos shared on twitter. Ex ante, there is little to distinguish items that spread from those that fizzle out. Uncovering the factors behind these diffusion processes would not only help businesses better market their products, it would inform better public policy in a variety of domains and aid our understanding of social movements as well.¹

A distinct but related phenomenon is that innovations are often adopted slowly, and different groups adopt innovations at different rates. Even in developed countries, well-established best practices are used inconsistently. After a person survives a heart attack, it is standard practice to prescribe beta blockers to help prevent recurrence. The benefits, in terms of reduced mortality, have been well-documented for over three decades (Yusuf et al., 1985). Nevertheless, many doctors in the United States still do not prescribe beta blockers to heart attack patients. Jencks et al. (2003) find that in the median U.S. state from 2000

¹For instance, the spread of information via Twitter reportedly playing a significant role in the Arab Spring.
to 2001, just 69 percent of ideal patients were prescribed beta blockers. Moreover, they find striking variation between states—in some the rate was as low as 50 percent and in others it was as high as 86 percent. Skinner (2006) estimates economic losses of between $1500 and $5000 per heart attack patient from failure to adhere to known best practices.

In healthcare, the inconsistent application of best practices is not confined to heart attack patients. Jencks et al. (2003) document significant differences between states in giving anti-coagulants after strokes or transient ischemic attacks, mammograms every two years for women over 50, and eye exams every two years for diabetics. In the developing world, far simpler interventions often fail to spread. Economists have invested a great deal of effort studying how to get people to use de-worming pills (Miguel and Kremer, 2004) and mosquito nets (Cohen and Dupas, 2010). Other technologies, from new seed varieties (Griliches, 1957; Munshi, 2004) to computers (Caselli and Coleman, 2001), experience different diffusion patterns in different areas, and adoption rates of different technologies are often correlated within particular groups or regions (Skinner and Staiger, 2005). Improving the dissemination and uptake of innovations could substantially improve health outcomes (Kohn et al., 2000) and reduce wealth disparities between nations (Comin and Hobijn, 2004; Foster and Rosenzweig).

Social influence is an important factor in the spread of new products, services, and technologies. However, the presence of various types of social influence alone cannot explain the different paths that similar products follow or the divergent rates of technology adoption by different groups. The pattern of interactions between individuals—the structure of the underlying social network—likely plays a key role in determining aggregate outcomes.

Despite a large and growing economic literature studying diffusion (Jackson and Yariv, 2007; López-Pintado, 2012) and social learning (Golub and Jackson, 2012; Lobel and Sadler, 2014a) in networks, the role of network structure is still poorly understood. Theoretical work faces a hard methodological challenge on two fronts. First, representing a network is extremely complex, with myriad potential features on which to focus, and outcomes can be sensitive to small changes. Even more challenging is finding a good representation of a person making decisions in the network. In these kinds of situations, rational expectations seem unrealistic, and the heuristics and biases researchers have documented in far simpler settings (Kahneman et al., 1982) may have a significant impact. We might expect simplified reasoning based on some form of categorization (Fryer and Jackson, 2008), and there is recent evidence that simple heuristics describe belief updating far better than Bayesian inference (Chandrasekhar et al., 2012; Corazzini et al., 2012).

At the same time, people’s decisions often demonstrate some degree of strategic thinking. Even in very complex decision problems, researchers find that people engage in free-riding (Bandiera and Rasul, 2006; Kremer and Miguel, 2007), and they are selective with who they tell sensitive information (Banerjee et al., 2012). The combination of simplified decision making and strategic behavior is difficult to explain through existing approaches. Most net-

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2A number of diffusion studies highlight the role of person to person transmission and payoff externalities (Jackson and Yariv, 2010), and recent technology adoption studies focus on social learning as an important driver of adoption (Conley and Udry, 2010; Banerjee et al., 2013).
work models operate at one of two extremes. Authors can analyze strategic behavior by assuming fully rational Bayesian players, or they can obtain sharper, more detailed predictions by assuming a behavioral heuristic. The dilemma between these two approaches has hindered the study of important questions, including those related to product diffusion and technology adoption.

In an effort to better explain the patterns of product and technology adoption that we observe, I first develop a novel way to represent individual decision making within a social network. Much like microeconomists, the players adopt simplified models of the world, abstracting away from less salient details, and they optimize as Bayesian agents within these models. Think of one of these models as a player’s own description of her environment, resulting from a set of choices about what to include and how to allocate her attention. A player’s model may incorporate details of the entire problem as we envision it, or the player may only consider the actions of a few close friends or neighbors within her model. This approach offers a middle path between rational expectations and behavioral heuristics, allowing us to capture realistic features of decisions made in practice in a more tractable framework. Though this solution concept, local Bayesian equilibrium, is motivated primarily by network interactions, its application may extend further. The view of players as microeconomists gives us a way to explain surprises, events not captured in a player’s model of the world, and differences in cognitive ability.

After exploring some of the properties of local Bayesian equilibrium, I apply this concept to understand some possible reasons behind the diffusion patterns we observe. In two separate models, I explore two distinct mechanisms through which people influence one another. The first captures the diffusion of products about which people make fast decisions—think music or movie recommendations from friends, the spread of news stories on social media, or limited time promotional offers for some products and services. These are cases in which we typically do not invest a great deal of effort gathering information, but we often become aware of certain goods because friends or neighbors have adopted. I present an analysis that explains variation in cascade patterns in terms of the underlying network structure. Diffusion approximately follows a branching process that we can define using properties of the network. Depending on the initial seed, the same product may spread to a large fraction of the population, or it may quickly die out. The likelihood and rate of spread depend on both equilibrium behavior and the structure of the network, with high-degree players having an important role in facilitating diffusion. Extensions of recent mathematical results on random graph models allow us to study the role of homophily, clustering, and information as well.

The second model captures cases in which people slowly learn about a new technology over time. Players learn by using the new technology, but they can also learn from neighbors who use the technology. In this setting, the free-rider and encouragement effects (Bolton and Harris, 1999) create complex tradeoffs, and high-degree players exhibit counter-intuitive behavior. The structure of the information sharing network has important implications for long-run adoption. There is a key tradeoff between gathering more information and efficiently sharing information. In general, networks that do a better job disseminating information will gather less, leading to more volatile long-run outcomes. These networks may be too quick
to abandon promising technologies after a few bad early experiences. The analysis suggests that seeding strategies to encourage adoption should focus on individuals who are isolated from one another, ensuring that more independent information is gathered.

These models help us make sense of empirically observed adoption patterns. We see how network structure can account for underlying trends even while the outcome for any particular product or technology remains stochastic. We also obtain additional qualitative predictions that call for more empirical investigation. For instance, the first diffusion model predicts that when products go “viral,” meaning that they spread to a significant fraction of the population, the extent of diffusion should be independent of the number of initial seeds, and the bulk of adoption should concentrate in a narrow time window. The solution concept, local Bayesian equilibrium, provides a broader methodological contribution. By simplifying players decisions, while retaining key aspects of rationality, we can tractably study key economic concepts in network settings. I also argue that this gives us a more realistic representation of human decision making, one that better respects the real cognitive limitations that people face.
Chapter 2

Bounded Rationality and The Network Economist’s Dilemma

At first, surveying economic literature on bounded rationality and on social networks together might appear a strange combination. While bounded rationality has been the subject of much study and heated debate in economics for several decades, the study of social networks has become fashionable only in the last one or two. The respective research communities appear more or less separate. Nevertheless, I believe there is significant common ground. Both literatures are the subject of controversy. Both are fractured, featuring a proliferation of modeling approaches and relatively little consensus. I contend also that the two literatures can help each other achieve greater consensus and further extend the edges of our understanding.

Researchers have long shown concern about the capabilities demanded of economic agents (Clark, 1918), and well-known arguments against the paradigm of rational decision making arose at almost the same time its foundations were laid (Simon, 1955). Various models of “bounded rationality” have been proposed and studied to address these challenges, but despite a vibrant literature, many derived insights have yet to penetrate mainstream economic thought. Though written some years ago, Ariel Rubinstein’s statement still largely holds true:

[T]he usefulness of these models is still far from being established. In fact, I have the impression that many of us feel that the attempts to model bounded rationality have yet to find the right track. It is difficult to pinpoint any economic work not based on fully rational microeconomic behavior that yields results as rich, deep, and interesting as those achieved by standard models assuming full rationality. (Rubinstein, 1998)

For a research program based on bounded rationality to succeed, (boundedly) rational choice models must account for deviations from fully rational behavior that we frequently observe in the real world, linking these deviations to some constraint on the decision maker. A large body of work in the last decade has made advances to this end. Though the formal choice models are varied and may appear quite different, an emerging set of common themes,
describing the types of constraints researchers have deemed relevant, can offer a unifying perspective.

Ultimately, the true test of these models will be their success or failure in explaining economic phenomena relative to other approaches. Some models have seen successful application in economic problems, but to more fully meet Rubinstein’s challenge, the literature requires a domain to showcase rich, deep, and interesting results. An ideal domain would be problematic for the standard paradigm, offering a context in which the critiques of fully rational choice hit particularly hard, and carry its own share of controversy. Most crucially, competing approaches should fall short in this domain. In my view, the study of information and decision in social networks is just such an ideal domain.

The economic literature on social networks has exploded recently. The spike in interest and plethora of associated buzz words might portend a passing fad if the subject matter were not so central to human behavior. The newfound ease of gathering relevant data has spurred empirical research, and theorists have followed closely behind. Much work has focused on understanding basic diffusion processes and coordination games on networks, yielding insights on how behaviors or information spread, how network structures impact aggregate outcomes, and how network position conveys influence. A major challenge for the networks literature is to more broadly apply these models to questions about design or intervention. I argue that existing approaches render such extensions difficult or infeasible, and ideas from the bounded rationality literature may facilitate progress.

Behavior and interactions in networks are fundamentally complex, posing serious issues for agent and modeler alike. Coping with complexity requires model simplification, and theorists studying social networks face a dilemma of what to simplify: the environment or the decision making process. Two extremes—the two horns of the network economist’s dilemma—have shaped the current literature. On one hand, a theorist can model fully rational Bayesian agents in a network. This often poses immense technical difficulty, necessitating strong simplifying assumptions about the environment and raising concerns about realism. On the other hand, one can model mechanistic agents following simple heuristic decision rules. These efforts produce sharper, more complete results, but the cost is a shallow account of incentives, information, and strategy. Neither approach has proved fully satisfactory, and the literature would clearly benefit from a more tractable way to analyze traditional economic concepts in a network setting.

I survey recent theoretical literature on social networks, emphasizing how the network economist’s dilemma has shaped this field and highlighting the distinct contributions of two broad approaches. Work taking the first horn of the dilemma, assuming rational agents, has difficulty dealing with repeated interactions and forward-looking behavior in particular. To get around technical difficulties, researchers typically study static simultaneous move games, or they model players each moving once in sequence. There are rich results in contexts ranging from coordination games to bargaining, but the simplifications necessary to achieve these results can make it difficult to interpret the models in context. Deriving testable implications is a significant challenge, and the intractability of extensions is a barrier to broader application of these models.
The non-Bayesian literature offers a complementary set of findings. Papers taking the second horn of the dilemma can incorporate dynamics and other realistic features of the environment more naturally. Papers assuming simple threshold decision rules or belief averaging procedures have given relatively detailed analyses of diffusion in networks, both of behaviors and information. Researchers have studied how structural aspects of the network affect the speed and extent of diffusion as well as the efficiency of outcomes. Nevertheless, these models face serious limitations because key aspects of human reasoning and strategic behavior are left out. It becomes difficult to analyze the effects of incentives and information, or to anticipate how policy interventions might change behavior.

Before proceeding with the survey, I first explore the role of rationality in economic theory. I advocate a particular interpretation of the concept that allows us to give clear meaning to the term “bounded rationality.” I then consider recent contributions to the bounded rationality literature, calling attention to unifying themes and the relevance of these themes to the study of social networks. The following section examines recent work on social networks in light of the network economist’s dilemma. I consider the contributions and limitations of papers that have taken each horn of the dilemma, emphasizing the open questions that remain. In the concluding discussion, I outline an alternative approach based on Bayesian agents with “local” models of the world, drawing on ideas from the bounded rationality literature to suggest a way between the two horns.

### 2.1 (Bounded) Rationality in Economics

Much economic theory concerns the behavior of the “rational man.” The rational man makes decisions in a methodical, predictable fashion, allowing economists to explore a wide range of tractable models meant to represent a variety of situations. We implicitly hope this abstraction provides a reasonable approximation to the behavior of real individuals trying to satisfy their desires, or at least a helpful guide to making better decisions.

The formal structure adopted in most economic theory is a complete transitive preference ordering. The rational man ranks a set of alternatives, allowing ties in some cases, and this ranking is typically expressed via a utility function. The paradigm is elaborated to deal with uncertainty by supposing the rational man represents beliefs about the world through a probability distribution over underlying states, choosing an action that maximizes the expected value of his utility function with respect to this distribution. Strategic situations are treated as a natural extension; the rational man forms expectations about the behavior of other individuals, which we represent as a probability distribution, and maximizes expected utility given these expectations. In the fullest expression of the paradigm, such expectations are always fulfilled.\(^1\) This theory of rational choice provides an elegant, unified framework to approach choice, uncertainty, and strategy.

For a variety of reasons, many researchers are disillusioned with this framework.\(^2\) Models...
of bounded rationality represent one among many responses to the critiques of rational choice. These models attempt to account for the cognitive limitations of decision makers in a principled way, in essence asking: what would reasonably intelligent people do? This simple idea has proved difficult to translate into economic practice. Disagreement over the proper cognitive constraints to model, the best way to model them, and even the meaning of bounded rationality itself has hampered the pursuit of novel insights.

I argue that the primary reason for this state of affairs is confusion about the object to which the adjective “rationality” should apply in economic models. In a substantive interpretation of rationality, a decision can be rational or not. This view emphasizes the direct link between choice and outcome. An agent whose behavior is substantively rational chooses exactly the action that produces the optimal outcome according to some criterion. Associated models ignore the mechanism used to arrive at the decision, taking as given that the optimal action is chosen and working out the ensuing consequences. In contrast, procedural models of rationality go inside the black box, asking for more detail on how decisions are reached. The procedure itself, rather than the decisions it selects, is what is judged rational or not. A rational procedure is one that generally produces good outcomes. Admittedly, this seems a vague notion and may require a supporting substantive model of rationality to achieve any precision, but a procedural interpretation removes any difficulty in defining bounded rationality. Under a procedural interpretation, the set of feasible procedures represents a bound on rationality, and I contend that only such an interpretation admits a coherent notion of bounded rationality.

After developing a procedural interpretation of rationality in an effort to answer some common criticisms of the bounded rationality literature, I survey recent contributions in light of this interpretation. Although many decision models have emerged, I emphasize the underlying unity of this work through a small set of common themes. This survey is necessarily far from complete, and I have excluded some branches of literature one could describe as dealing with bounded rationality. In particular, I exclude work that takes behavioral biases as a starting point, rather than deriving biases from some constraint on an otherwise rational decision maker. These models fall outside the narrower concept of bounded rationality I am considering. I have also left out work based multiple priors (Gilboa and Schmeidler, 1989) or minimax regret (Stoye, 2011). These models address an agent’s difficulty in formulating a single prior to represent beliefs. I view these as alternative models of substantive, rather than procedural rationality; if interpreted procedurally, many of these models make even greater cognitive demands of agents than the traditional model. Additionally, I constrain myself to decision theoretic issues, setting aside the further complications that game theoretic considerations introduce.

2.1.1 The Role of Rationality

Among the many gems of advice Roy Radner offers his graduate students are the three reasons to write an economic theory paper: the paper ought to explain some stylized fact

(Kahneman et al., 1982; Lipman, 1995; Conlisk, 1996; Rubinstein, 1998).
about the world, it ought to offer someone advice, or it ought to provide other theorists with new tools. The last of these is generally discouraged as the sole purpose of a paper. His guidance highlights an important distinction in the interpretation and purpose of economic theories: some theories are meant to describe the world, while others are meant to prescribe courses of action.  

Nowhere is the distinction more pertinent than in theories of decision. Scholars of decision theory have traditionally focused on normative models, carrying out philosophical exercises to arrive at a notion of a good decision rule. In laying the foundations for a subjective theory of probability, Savage famously emphasized his normative intent:

Pursuing the analogy with logic, the main use I would make of [my postulates] is normative, to police my own decisions for consistency and, where possible, to make complicated decisions depend on simpler ones. (Savage, 1954, pp. 20)

While this suffices for statistics, much of the enterprise of economics lies in the construction of descriptive models, and economists have arguably abused decision theory by appropriating it for this purpose.

Savage’s theory has clearly borne the brunt of this abuse. What was intended as a normative theory to be applied in “small-world” problems has been used as a model of actual human decisions in complex situations. Expected utility underlies the standard presentations of consumer theory and general equilibrium under uncertainty (Mas-Colell et al., 1995) as well as asset pricing theory (Duffie, 2001). Game theory often imposes further structure, assuming common knowledge of a common prior, leading individuals faced with the same information to complete agreement. As Aumann (1987) remarks, this assumption “expresses the view that probabilities should be based on information; that people with different information may legitimately entertain different probabilities, but there is no rational basis for people who have always been fed precisely the same information to do so.” Contrast this with Savage’s perspective:

[I]t is appealing to suppose that, if two individuals in the same situation, having the same tastes and supplied with the same information, act reasonably, they will act in the same way...Personally, I believe that it does not correspond even roughly with reality, but, having at the moment no strong argument behind my pessimism on this point, I do not insist on it. But I do insist that, until the contrary be demonstrated, we must be prepared to find reasoning inadequate to bring about complete agreement. (Savage, 1954, pp. 7)

While Savage is often cited to support the use of expected utility theory in economic models, different arguments are needed to justify the way economics uses his theory.  

I am aware of additional perspectives on the purpose of economic theory, such as the view that it is an arena to explore relationships between concepts, but I believe we can encompass these within a sufficiently broad interpretation of the views I list.  

I emphasize Savage’s (1954) treatment of expected utility, rather than von Neumann and Morgenstern (1947) or Anscombe and Aumann (1963), because Savage provides a far more complete discussion of philosophical issues associated with subjective probability.
Interestingly, the conflation of descriptive and normative theories also flows in the other direction. Simon (1956) coined the term “satisficing” to describe a simple heuristic meant to describe common behavior. Satisficing refers to situations in which alternatives are considered in some exogenous sequence, and the agent accepts the first alternative above some “aspiration level.” Elaborations of the procedure may allow the aspiration level to shift up or down depending on the alternatives considered. While Simon clearly intended this as a descriptive behavioral rule, economists were soon able to justify the procedure within the standard rationality framework in various sequential search problems (e.g. McCall, 1970).

This mixing of theories occurs because economics plays dual roles as a positive and a normative discipline, and the two often feed into each other. The very system under study comprises individuals who actively seek to change the system. This creates complex feedback that is conspicuously absent in many other fields, and theories of decision are necessarily called upon to serve both normative and descriptive purposes. Whether models of rational decision making can effectively fill either role is an important question. When viewed from a normative perspective, rationality is a tool used to arrive at decisions that are good in some sense. If rationality is to serve a descriptive role, then we must claim that individuals generally do make good decisions in the context under study. The traditional paradigm has been heavily criticized in both roles as its predictions often conflict with observation (See Kahneman et al., 1982), and Savage himself noted the impossibility of behaving in accordance with his postulates in all but the simplest situations.

Though in need of revision, rational decision models are still crucial to economic theory. It seems uncontroversial that a normative model should recommend “rational” behavior of some sort, but one could argue we need a different type of model to describe actual decisions. In my view, there is a simple and compelling reason that the same decision model should serve both roles: people learn and adapt to their surroundings. This is not meant to imply that errors never happen, but over time, through trial and error, people arrive at decision procedures that often perform quite well. If it is possible to do better, we should expect an individual to discover that fact given enough time. Conversely, if an individual has ample time to experiment and improve upon decisions in a given situation, yet the behavior we observe appears less than rational, the fault may lie in our definition of rationality. If learning has reached a plateau, we can make a case to call the ensuing behavior rational. A normative model should respect all constraints a decision maker faces, and a failure to discover better alternatives suggests we have ignored an important constraint. If we understand rationality as the result of adaptation, then an appropriately defined model of bounded rationality should serve both normative and descriptive roles, at least in situations in which individuals have the benefit of experience.

Herbert Simon once defined bounded rationality as “rational choice that takes into account the cognitive limitations of the decision-maker—limitations of both knowledge and computational capacity.” (Simon, 1987). This represents an attempt to define a notion of a good decision that is not unrealistically good; the behavior of a boundedly rational agent is not less than rational in any sense, but it is subject to certain constraints that economics has historically neglected. There is a crucial distinction between constraints arising from
the "outer environment" and the "inner environment." Simon (1996) described the outer environment as encompassing external constraints on a decision maker’s feasible actions and the outcomes produced, distinguishing this from the inner environment governing how a decision maker perceives and reasons about the outer environment. Traditionally, economics has focused exclusively on constraints of the outer environment. The goal of the bounded rationality literature is precisely to examine constraints on the inner environment.

Difficulty arises in appropriately defining bounded rationality because substantive views struggle to incorporate constraints on the inner environment. Consider that if we impose such constraints—constraints on the capacity to optimize—a substantive view of rationality demands that our decision maker optimize with respect to these new constraints. This implies the presence of some inner-inner environment that can process this deeper problem and carry out the necessary optimization, defeating the entire purpose of accounting for inner environment constraints. Accepting this, we must constrain the inner-inner environment, and then the inner-inner-inner environment, and so on. If the original costs of decision making warrant modeling, then the costs of optimizing with respect to these costs are likely just as significant and worthy of modeling, and we end up with an infinite regress.

Some work has successfully dealt with similar infinite regress issues in other contexts, but a satisfactory resolution to the “metaoptimization” problem remains elusive. Lipman (1991) suggests reinterpreting the problem as one of defining how an individual perceives a situation. He offers a potential resolution based on fixed points of a map that takes the set of perceived options to the set of perceived ways to resolve uncertainty about those options. While fixed points can exist, this finding depends on restrictive assumptions about how uncertainty is resolved, and the infinite regress stemming from any particular initial perception may not converge to a fixed point. Although inconclusive, this work shows that resolving the infinite regress is far from trivial, and we should not hope for a resolution in all contexts.

While reasoning about cognitive limitations becomes self-referential under a substantive interpretation of rationality, the issue never arises under a procedural interpretation. The decision making procedure, rather than its result, is deemed rational or not, and this procedure is our representation of an agent’s reasoning process. The meaning of a constraint on the inner environment becomes clear as a restriction on the set of feasible decision procedures or a cost associated with using any given procedure. This interpretation requires that we separate rationality as embodied in the decision procedure from the adaptive process that selects this procedure. We still need a criterion for good decisions that guides this adaptive process, but this criterion is distinct from the reasoning an agent actually performs.

A procedural model of rational choice comprises three elements: a description of the outer environment to which behavior adapts, a set of feasible decision procedures constraining the decision maker, and a criterion to evaluate procedures. Rationality means assuming the optimal procedure is implemented, but we do not interpret this selection as part of the reasoning process. In this view, all rationality is bounded by the set of feasible procedures.

The procedural view has implications for the use of rationality in both of its roles. In

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5 Notably, Mertens and Zamir (1985) in their treatment of belief hierarchies.
a positive model, the set of feasible procedures becomes a central modeling choice. This set expresses the extent to which behavior can adapt to the environment, which is distinct from the set of available actions. Restricting the feasible procedures means restricting the operations an agent can perform to transform inputs into decisions. Essentially, it restricts how an agent can condition decisions on information. The appropriate constraints on this set will depend heavily on context, varying based on the type of agent (e.g. individual, firm, government), the technology available to assist in information processing, and possibly the level of experience with a particular problem.

Although I have emphasized the need for experience and practice to justify positive use of rational decision models, what constitutes a learning plateau is open to interpretation. Discussing the evolution of human behavior, Selten (1991) distinguishes between the vastly different time scales along which biological evolution, cultural evolution, and individual learning take place. To these time scales I would add that of technological evolution as the continuing development of information technology clearly enhances our capacity for information processing and reason. The type of adaptation necessary to render a procedure feasible, and the time scale on which such adaptations occur, will impact the bounds of rationality in a given context. The feasible procedures and their associated costs after an hour, a century, and an eon of “practice” may be quite different.

We might also make finer distinctions within the time scale of individual learning. This would allow us to represent contexts in which individuals lack significant experience through more constrained sets of procedures. A novice chess player may “rationally” adopt a very simple approach to the game, while a more experienced player will have learned to play a far more intricate strategy. The procedural interpretation can also furnish a way to effectively apply a theory of rational decision making to wholly unfamiliar circumstances. When faced with a new situation, it seems likely that an agent would resort to a procedure adapted to an environment the agent judges similar. If we can identify the familiar environment to which the procedure is adapted, we have a basis for modeling behavior in the new environment.

The normative implications of the procedural view are more subtle. We are called on to explicitly recognize costs associated with processing information to reach a decision. In simple situations, it might be feasible to adopt a procedure leading to substantively rational choices, but if the available decision procedures are meaningfully restricted, it hardly seems feasible to carry out an analysis to find the optimal procedure. The ability to do so would suggest that the available procedures are not so constrained. Recognizing the existence of a non-rational process at some level, which eliminates the infinite regress of reasoning, a procedural view should emphasize the importance of experimentation—of trial and error—with available procedures in guiding our choices.

Models of bounded rationality have faced a number of criticisms I would like to address in light of the procedural interpretation. One possible charge is that although the idea of accounting for cognitive limitations is a good one, there are many ways to model such limitations, and ultimately the choice is arbitrary. This appears to me a rather weak criticism.

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6Gilboa and Schmeidler (1995) follow this line of reasoning in their development of case-based decision theory.
One could just as easily claim the risk tolerance assumptions in an insurance model are arbitrary, or the particulars of the information structure in a principal-agent model are arbitrary. The paradigm of expected utility itself is open to this charge. Savage once justified his postulates as the largest set of principles on which he could get his intelligent colleagues to agree,\(^7\) which seems an ad hoc process for designing a theory of decision.

A more refined version of this critique notes that models of bounded rationality necessitate a greater number of distinct modeling choices. This creates more opportunities to build conclusions into the model specification and rightly leads to some skepticism. However, this is an issue that arises whenever we attempt to model relatively complex situations. The risk stems from the need to model more intricate details of a problem, not the particular decision rule being adopted.

A good model is motivated by a particular context, and assumptions are made in the effort to faithfully represent the salient aspects of the context. Using a model of bounded rationality is not a free pass to make any assumption: we should always ask what contextual features the assumptions represent and take care in choosing our assumptions. There are typically many ways to represent a given problem with different assumptions, but this does not imply that the choice is arbitrary. Moreover, the study of many different models of the same situation can be instructive, highlighting which assumptions are crucial to certain results and which are inconsequential.

Another challenge concerns the gap this framework creates between an individual’s mental preferences and the choices we can observe. Following the interpretation to its conclusion, observing a person’s choice in a problem does not entitle us, at least not directly, to make any assumption about her preferences. We can infer that the person’s choice resulted from an optimal procedure, but we need more information about the problem context and the feasible procedures to infer preferences. Rubinstein and Salant (2008) offer an eloquent response, arguing that “an essential component of the principle of revealed preference is that one should be able to deduce the parameters of the choice procedure based on behavior. But there is no reason to adopt a position that restricts the scope of the observable information to the set of alternatives and the actual choice.” Rather than posing a problem, this presents an opportunity for a richer application of the principle of revealed preference.\(^8\)

A more serious objection in my view is that no guarantee exists that experimentation and adaptation always lead to optimal tradeoffs between cognitive (or other) costs and goal attainment. For instance, literature studying evolutionary processes raises the prospect of getting stuck in a local optimum. I see at least two possible replies. First, we might argue that such situations are relatively rare, that the situations we are most interested in are not so pathological. Alternatively, we could simply consider additional constraints on the set of feasible decision procedures. The excluded procedures would not represent truly infeasible alternatives, but ones that are ruled out by historical events. Two identical individuals whose learning follows different paths may find themselves with different feasible procedures, just as scholars trained in different disciplines may have difficulty reading and interpreting each

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\(^7\)This is based on an anecdotal account given by Roy Radner.

\(^8\)Caplin and Dean (2011) explore one possible direction to pursue.
other’s work. Nevertheless, we may find contexts in which neither response is appropriate, and we should be prepared to acknowledge this limitation of rational decision models.

A related objection is that I have side stepped the real problem of defining a rational approach to adaptation. Yes people learn and adapt to their environment, but what is the most effective, efficient, or rational way to adapt? Radner (2000) formulates this question more precisely as one of statistical model revision. We are generally vague about our models of the world, and we change our underlying models from time to time because we believe our earlier model may not contain the “true” world. Radner calls models that deal with this vagueness models of “truly bounded rationality.” It is my view that such a notion cannot be given a meaningful definition precisely due to the infinite regress problem. In order to call a model revision procedure optimal, it must be optimal with respect to some grand model that encapsulates all of the models the decision maker could adopt. This begs the question whether the grand model requires revision, and whether there is a rational approach to this revision. Rather than side stepping the problem, I call attention to a fundamental limit of rationality. A non-rational adaptive process must lie behind all rational choice models.

Models of bounded rationality must meet two challenges to be considered successful. First, they must explain through some constraint the persistent features of real decision making that deviate from the standard paradigm. We should understand various framing effects and cognitive biases as optimal responses to some limitation of the decision maker. A more substantive challenge is to meet Rubinstein’s criterion: these models must produce rich, deep insights about the economic world, ultimately providing a useful account of phenomena that the traditional approach has failed to adequately address.

The remainder of this section surveys work that constitutes progress in meeting both challenges. The survey is organized around three emerging themes that correspond to different types of constraints on the inner environment of the decision maker. Coarseness is the most pervasive in the current literature, dealing directly with the inability of agents to condition decisions on all available information. Navigating the choice set acknowledges procedural aspects of how alternatives are perceived and processed, encompassing the large search theory literature as well as new efforts. Finally, delay accounts for the time cost of decision making and the potential inability of agents to update decisions as frequently as they might like.

### 2.1.2 Coarseness

Coarseness is essentially synonymous with problem simplification, describing any work that makes it costly to condition decisions on a finer information partition. Models of coarse decision making can reflect many different constraints; this includes work on categorization, memory limitations, decision procedures implemented by finite automata, and coarse expectations. The underlying assumption is that information processing or recall is somehow costly, so that more sophisticated procedures or models are impractical to implement. The procedure an agent follows must balance the cost of processing against the value of conditioning actions on finer details of the problem situation.

Models of coarse decision making essentially force the agent to optimize with respect to
a naïve model of the real situation, yielding a simplified decision procedure. Agents conduct a similar exercise to that economic theorists often do. Features of the situation that are deemed most salient are selected for inclusion, while details that are judged superfluous get ignored. Decisions will respond to some, but not all relevant aspects of the environment. Perhaps unsurprisingly, these models are capable of replicating many systematic deviations from the traditional model of rational behavior.

Mullainathan (2002) provides a simple example using agents that reason categorically. Suppose an agent wishes to forecast outcomes that depend upon some unknown state of the world. Let $\Omega$ denote the space of outcome relevant states, and let $O$ denote the space of outcomes. For each $\omega \in \Omega$ there is a corresponding distribution $\mu_\omega$ over $O$, and $\nu$ denotes a prior distribution over $\Omega$. The corresponding prior distribution over $O$ can then be expressed as

$$\mu = \int_{\Omega} \mu_\omega d\nu(\omega).$$

Given some data $D$, a Bayesian would update the prior $\nu$ to a posterior $\nu^{(D)}$ according to Bayes’ rule, yielding a posterior distribution over $O$ of

$$\mu^{(D)} = \int_{\Omega} \mu_\omega d\nu^{(D)}(\omega).$$

Now suppose the agent is constrained to make forecasts using one of a small set of feasible posteriors $\{\nu_i\}_{i \leq K}$. We can think of these posteriors as categories, supposing the agent adopts the category most likely to have generated the data. Since many possible data sets must map to the same category, some information is lost. As a consequence, several biases arise in the reasoning of this agent. For instance, given some additional information, if the information is insufficient to induce a category transition the agent underreacts. Alternatively, in some cases in which a transition does occur, the change to a new category may go too far. Moreover, information beyond a certain level of precision adds no value for the agent because there is no category to express this level of precision. The resulting bias not only distorts forecasts, but it distorts incentives to gather information as well.

The impact of limited memory provides a close parallel. Placing limits on memory means limiting the historical events on which an agent can condition decisions, which is formally equivalent to placing observed histories into categories. Dow (1991) provides an early example, studying a model in which agents search for the lowest price on a particular product. Agents have a finite number of memory states, so they are unable to recall the entire history of their search. Optimal behavior involves creating coarse categories to sort possible histories, creating biases in search patterns.

Wilson (2002) builds on earlier work of Hellman and Cover (1970) that explores the implications of categorization for statistical inference. A decision maker with a finite number of memory states observes an information process in discrete time. In each period $t$ that the process continues, a signal $s_t$ is observed. This signal takes values in a finite space $\mathcal{S}$ and is independent and identically distributed conditional on an unknown binary state $\theta \in \{0, 1\}$. After each period, the process terminates with a fixed probability $\eta$, and if this happens,
the decision maker must make a prediction \( x \in \{0,1\} \), earning a payoff of 1 if \( x = \theta \) and 0 otherwise.

The decision maker in this model is restricted to a finite set \( \mathcal{K} \) of memory states. An estimation strategy for the decision maker consists of an initial probability measure \( \mu \) on the set of memory states \( \mathcal{K} \), a transition function \( \sigma : \mathcal{K} \times \mathcal{S} \rightarrow \Delta(\mathcal{K}) \), describing how to update in response to information, and a decision function \( d : \mathcal{K} \rightarrow \{0,1\} \), determining the prediction to make if the information process terminates. Optimal estimation strategies have a number of interesting features, mimicking stylized facts about real human reasoning. Transitions between memory states are stochastic, so different individuals can form different beliefs from the same information. Confirmatory biases are prominent, as the optimal rule dictates that with probability close to one, any signal is ignored once an extreme belief state is reached. Moreover, early signals have outsized influence on the final decision.

Recently, Spiegler (2014) introduces a general framework for modeling individual decision making with an incomplete (or incorrect) understanding of the correlation structure in her environment. The author uses Bayesian networks to represent an agent’s model of world, and a misspecified model leads to biases much as it would in econometric estimation. The framework unifies several earlier approaches and provides a principled way to discuss subjective beliefs about causality. Specifying these beliefs becomes a central modeling choice. Consequently, we become more explicitly aware of how our modeling choices influence outcomes in this decision model.

Collectively, these models demonstrate how common cognitive biases can emerge from models incorporating non-standard constraints on otherwise rational decision makers. This provides an argument that many phenomena often cited as definitive evidence against the hypothesis of rational behavior are actually explainable within a rational framework. Even though the appropriate constraints and the extent to which reasoning is limited will vary greatly between contexts, the consistency with which these biases appear in coarse decision models suggests they are a robust feature of rational decision procedures.

Coarse models of decision making have begun making contributions beyond accounting for cognitive biases, suggesting how these effects may impact market behavior. Piccione and Rubinstein (2003) offer a simple example of a market in which consumers differ in their abilities: some agents have a longer memory of historical prices than others. We can view this as an abstraction of the idea that some consumers are more sophisticated than others—those with longer memories are better able to detect patterns in seller behavior over time. In each period, a consumer must decide first whether to enter the market and learn the current price, paying some small positive cost \( \epsilon \) to do so, and only then whether to purchase the good. The entry decision depends on expectations the consumer forms about the current price. If less sophisticated consumers have a higher willingness to pay, then the ability difference presents an opportunity for price discrimination via cyclical price sequences. As the authors point out, this circumstance has the very reasonable interpretation that more sophisticated consumers have better outside options, or alternatively, price sensitive consumers invest more effort in predicting future prices.

Fryer and Jackson (2008) offer a categorical decision model that endogenizes the selection
of categories, applying their findings to labor market discrimination. In their model, agents sort past experiences into an optimally selected set of categories, with a constraint on the number of categories that may be used. Optimal categorization implies that agents fail to distinguish infrequent events. In a labor market setting, this could mean that minority applicants get treated as average candidates regardless of their actual merits, leading to discrimination in the absence of any malicious intent. Although this may not be particularly surprising, it illustrates how cognitive constraints can explain outcomes that a priori appear inefficient. A second key insight arises from the difficulty of proving general results or deriving comparative statics for optimal categories. The optimal categorization is highly sensitive to the number of available categories and fine details of historical observations. This suggests a level of volatility in coarse decision making that is absent from less constrained decision models.

In the game theory literature, Jehiel (2005) has introduced a solution concept employing a coarse information partition. Consider an extensive form game with perfect recall. An analogy class $\alpha_i$ for player $i$ is a collection of pairs $(j, h)$, where $j$ is a player other than $i$, and $h$ is a decision node in the game, such that for any two pairs $(j, h)$ and $(j', h')$ in $\alpha_i$, the action space of player $j$ at node $h$ is the same as that of player $j'$ at node $h'$. An analogy-based expectation $\beta_i$ specifies a probability distribution over this common action space. Given analogy classes for each player that partition the decision nodes of all other players, an analogy-based expectation equilibrium consists of a strategy profile $\sigma$ and an expectation profile $\beta$ such that two properties are satisfied. First, each player’s strategy is a best response to her analogy-based expectations of other players’ actions. Second, the analogy-based expectations correctly represent the average behavior of each analogy class, given the strategy profile.

Using this solution concept, the author provides resolutions to various finite horizon paradoxes. For instance, in the classic centipede game, agents with analogy-based expectations may pass for many periods, waiting until relatively late in the game to take. This prediction concurs with the corresponding experimental literature. Further work by Jehiel and Koessler (2008) applies analogy-based expectations to classic games of incomplete information. They find that coarse expectations can potentially enhance coordination, and in cheap talk games, more information may actually be transmitted than under rational expectations. These findings suggest a role for cognitive limitations in fostering, rather than hindering, socially preferred outcomes, though alternative coarse solution concepts show this depends heavily on context (Esponda, 2008).

In a more ambitious application, Gul et al. (2012) revisit general equilibrium theory using a notion of coarseness. Consider a static exchange economy with a unit mass of identical households. The aggregate endowment is generated stochastically, with a finite number $n$ of possible realizations. Let $\pi_i$ represent the probability of realizing the $i$th highest aggregate endowment. Prior to the resolution of uncertainty, households can trade state contingent

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9Bartos et al. (2013) study a model in which a decision maker must allocate information gathering effort among candidates, finding that this additional layer in the problem can exacerbate the discriminatory effects of categorization.
claims. A state price vector determines the wealth of each agent. A consumption plan is a vector \( \mathbf{c} \in \mathbb{R}_+^n \), and a household obtains utility

\[
U(\mathbf{c}) = \sum_{i=1}^{n} u(c_i) \pi_i,
\]

where \( u \) is a strictly concave CRRA utility index.

Households in this model are restricted to “crude” consumption plans, meaning that any single household can choose at most \( k < n \) distinct state contingent consumption levels. We can interpret the number of consumption levels as a complexity measure for consumption plans. A more general framework would consider costs associated with complexity; the restriction to crude consumption plans is a special case in which the cost of \( k \) consumption levels is zero, and the cost of additional levels is infinite. Consequently, each household must group endowment realizations into categories, adopting the same consumption level for each state in a given category. An allocation is a probability measure \( \mu \) over the set of crude consumption plans; we interpret \( \mu(\mathbf{c}) \) as the proportion of households with consumption plan \( \mathbf{c} \). An allocation is feasible if total consumption is no greater than the aggregate endowment in all realizations.

A price-allocation pair constitutes a behavioral competitive equilibrium if the allocation is feasible, and all consumption plans in the support of the allocation are optimal, given the prices. With this concept in hand, the authors compare equilibrium outcomes with boundedly rational agents to the outcomes in standard models. A key insight is that for tail realizations of the aggregate endowment, prices are far more extreme than in standard equilibrium models. Intuitively, a small proportion of households distinguish these tail events in their consumption plans, and compensating these households for bearing the aggregate risk requires extreme prices. The agents’ cognitive limitations produce observations we would normally attribute to extreme risk aversion.

The economic literature applying some notion of coarseness extends far beyond the work covered here. The papers I reference illustrate the ability of this concept to capture important deviations from standard rational choice models and to provide new insights on economic problems. A common finding is that outcomes in these models are prone to volatility. We find extreme prices in equilibrium and large variation in beliefs formed from the same information. Behavior becomes harder to predict, but the models also suggest new opportunities for fostering cooperation and generating profit. These are important lessons for researchers studying social networks as coarseness directly addresses the key issue of model complexity. Agents embedded in a network likely act based on coarse models, and we should expect similar phenomena to appear.

### 2.1.3 Navigating the Choice Set

Models that incorporate a notion of navigating the choice set directly address procedural aspects of choice. These models specify how agents perceive the alternatives that are available, whether as a set, as an ordered list, or through some other process. When an agent
must navigate the choice set, alternatives are not all considered simultaneously, and some feasible alternatives may get ignored entirely. Whether certain alternatives are excluded from consideration often leads to significant deviations from standard models.

The essential idea of navigating the choice set is nothing new; it has a long history and has led to some of the most famous and celebrated findings in economic theory. Simon (1956) introduced satisficing nearly six decades ago, making explicit that alternatives were processed as a list and exploring the implications of a simple procedure for doing so. Other authors have applied variations of this procedure in different economic settings. Radner (1975), for instance, studies the implications of satisficing in the allocation of managerial effort. Some notion of satisficing also lies at the heart of the extensive literature on search. We can interpret search models as models of choice from stochastically generated lists, with costs for processing each alternative. Incorporating search frictions into equilibrium models has led to a wealth of insights regarding price and wage dispersion as well as labor market outcomes.10 Typically this work is viewed within the standard paradigm, having little relation to work on bounded rationality. However, at its core this work is about explicitly modeling the procedures through which choices are made and the costs of implementing different procedures. The search theory literature arguably represents the most widely accepted expression of bounded rationality in economics.

At present, there is renewed interest among microeconomic theorists in studying choice from lists. Rubinstein and Salant (2006) undertake an axiomatic study, drawing a parallel with the standard presentation of choice theory. Given a finite set $X$, a list is a non-empty finite sequence of distinct elements from $X$. A choice function from lists selects a unique element from every list. The authors demonstrate equivalence between two properties of such choice functions. The first property is partition independence: partitioning a list into two sublists, choosing an element from each, and then choosing from the resulting two element list, produces the same result as choosing from the original list. The other, equivalent property is list independence of irrelevant alternatives, meaning that deleting any non-chosen item from a list does not affect which element is chosen. Choice functions that satisfy these properties can be uniquely characterized as maximizing a preference ordering, subject to an order priority for indifference classes.

These findings have implications for applying the principle of revealed preference. Even if data are unavailable on the order in which alternatives are considered, the model allows for substantive behavioral predictions. Consider choice correspondences mapping choice sets to those alternatives that are chosen from some ordering of the set. The authors show that for choice functions from lists satisfying partition independence, these choice correspondences satisfy the weak axiom of revealed preference. In fact, any such correspondence is derivable in this way via a partition independent choice function from lists. If the ordering of alternatives is unobservable to the researcher, but is randomly determined according to some distribution, then a deterministic choice function from lists induces a probability distribution over the choice correspondence set. If the choice function is partition independent, this distribution

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10 For some important contributions in this area, see Diamond (1971); Rothschild (1974); Mortensen and Pissarides (1994); Burdett and Mortensen (1998).
satisfies a monotonicity property with respect to the set of alternatives, and this property is directly testable using choice data. Caplin et al. (2011) and Horan (2011) explore further possibilities for using more extensive choice data to elicit preferences when individuals make choices through a sequential search process.

Salant (2011) extends this work to a more detailed study of the effects of cognitive limitations on choice, adding an element of coarseness. The author explicitly considers costs associated with the complexity of a choice rule, where complexity is measured by the number of states an automaton requires to implement the choice rule from a list. Under this metric, rational choice is relatively complex, requiring a number of states that is linear in the number of possible alternatives. A major finding is that any choice rule that is less complex to implement than rational choice must display order effects. In some sense, rational choice is a relatively simple procedure because it is the least complex procedure that is free from order effects. However, this also means that any effort to economize on choice rule complexity must lead to some type of framing effect, suggesting that order dependent choice is an inevitable consequence of our cognitive limitations.

Salant next considers the problem of designing optimal choice procedures when faced with complexity costs. Consider a decision maker who is uncertain about the lists he will face, and consider a simple model of how the lists are generated. The length of the list follows a geometric distribution with parameter $p$, and each element of the list is an independent and identically distributed random draw from the set of alternatives with a given distribution. Suppose the decision maker selects a choice procedure either to maximize expected utility or to minimize maximum regret. In either case, we can describe the optimal procedure as a history dependent satisficing procedure, adopting a particular aspiration level that changes when certain alternatives are encountered. Several behavioral effects emerge when such a procedure is followed. Moving an element to appear earlier in the list, or moving it to the very last position, can increase its likelihood of being chosen. There is also a choice-overload effect, where adding alternatives to the list can actually decrease the utility the decision maker derives. Moreover, if the model is augmented to include a default alternative, situations can arise in which the decision maker favors the default over higher utility alternatives. These results offer a deeper understanding of the possible origins of some framing effects.

Work that incorporates navigating the choice set has so far focused largely on lists, but there are potentially many other useful ways to model how alternatives are perceived. The literature on delegation presents one approach that is important in certain contexts. Some papers on limited attention also fit into this theme, often using it as an explanation for coarse decision making (e.g. Alvarez et al., 2011; Eliaz and Spiegler, 2011). Another possibility is to consider a decision maker who navigates a network of alternatives, rather than a list. This seems particularly relevant to online markets in which pages for one good are linked to pages of related goods, and recommender systems or user reviews may direct individuals towards certain products based on their search history. For this reason, further elaboration of navigating the choice set could prove especially useful to researchers studying behavior in networks.

\footnote{Frankel (2013) provides a recent contribution; see references therein for further reading.}
2.1.4 Delay

When a person or an organization receives new information, there is often a delay before behavior can change in response. There are many sources of delay, including limits on the rate of information processing, the time it takes to communicate, or simply the need to devote attention to more pressing matters first. Models that incorporate delay recognize that time is an integral part of our resource endowment that we must expend in order to carry out activity.

Delay already plays a significant role in the study of organizations. Radner (1993) and Radner and Van Zandt (2001) study simple models of computation meant to represent the processing, and aggregation in particular, of information within an organization. The authors represent the firm as a decentralized network of individual information processors. A processor has an inbox, in which new inputs are received into a queue, and an output register, in which the results of computation are stored. In one unit of time, a processor can add one input to the contents of its output register, replacing the contents of the register with the new result. At any point in time, without any delay, the processor can clear its output register, sending the former contents to a linked processor as a single input. Given a set of input data distributed across the processors, the goal is to aggregate the data into a single output.

Although hierarchies are generally viewed as expressions of centralized authority, they prove remarkably efficient in decentralizing the processing of information. Such decentralization is crucial to the functioning of large enterprises. An important finding in this model is that there is a fundamental limit to the efficiency a network of processors can achieve. As the amount of input information increases, even an unlimited number of processors is unable to prevent increasing delay in the final output, a result described as the “iron law of delay.” The inevitability of increasing delay in large organizations introduces a natural diseconomy of scale. Depending on how costly delay is relative to the value of processing additional inputs, firms may display either economies or diseconomies of scale, with implications for the structure and size of firms.

Outside an organizational context, delay has implications for consumer pricing. Radner (2003) studies a market in which consumers exhibit a delayed response to price changes for a particular good. In any given period of time, only some fraction of the consumers pay attention and reevaluate their habitual purchasing decisions. Demand for the good becomes “viscous,” adjusting slowly to seller price changes. Taking the set of consumers as a continuum with unit mass, we can view the aggregate demand for the good as a continuous process \( X(t) \). Let \( w \) denote the common long-run willingness-to-pay among the agents, let \( k \) denote a “viscosity” parameter, and let \( P(t) \) denote the price a monopolist charges at time \( t \). The following differential equation gives a simple model of the evolution of the demand process \( X(t) \)

\[
X'(t) = \begin{cases} 
  k (w - P(t)) (1 - X(t)) & \text{if } 0 \leq P(t) \leq w \\
  -k (P(t) - w) X(t) & \text{if } w \leq P(t) \leq m,
\end{cases}
\]

where \( m \) is an exogenous upper bound on the price the monopolist can charge. Con-
sumers flow between purchasing and not purchasing as the price moves below or above their willingness-to-pay. The flow rate is proportional to the mass of consumers who could potentially change behavior and to the difference between the willingness-to-pay and the current price. For a broad range of parameters, the optimal strategy for the monopolist is a target penetration policy. The monopolist discounts the product until market penetration reaches the target, charging the consumer willingness-to-pay thereafter. This model suggests that delay in consumer responses is an important driver of business strategies that “invest” in building a customer base.

The concept of delay also offers key insights in the finance literature. A study a model in which consumers exhibit delayed responses to asset price changes, finding that this delay may explain the equity premium puzzle. In the model, each consumer holds a checking account and a continuously rebalanced mutual fund account. Consumption requires funds in the checking account, which is replenished by drawing on the mutual fund every $D$ units of time. Between withdrawals, consumers cannot monitor the mutual fund accounts, so their consumption plans cannot adjust with asset prices during this time. The analysis shows how this can account for upwardly biased estimates of risk aversion and a corresponding high equity premium.

Recent work has done much to address the charge that the bounded rationality literature has failed to offer the rich, deep insights demanded of economic theory. Many cognitive biases and framing effects are explainable as rational responses to cognitive constraints. The emerging themes of coarseness, navigating the choice set, and delay provide important insights in economic problems.

The economics of social networks offers a ripe domain to further elaborate each of these themes and strengthen the case for using models of bounded rationality. The next section explores this research in more depth, highlighting how available decision models have shaped the literature. The organizing theme is the dilemma between the intractability of maintaining a fully rational decision model and the lack of economic intuition offered by heuristic rules. The accomplishments of the bounded rationality literature so far suggest that a middle path is possible, and I argue there are rich opportunities for researchers who pursue this direction.

2.2 Theory of Social Networks

Economic theories of behavior in social networks face an important dilemma. On one hand, researchers can model fully rational Bayesian agents. These agents have detailed beliefs over the structure of a large network in which they are embedded; they face a complex task, and so does the modeler, raising issues of both tractability and realism. When an agent responds to information, she must consider the actions and beliefs of her neighbors, her neighbors’ neighbors, and so on. If her vantage point in the network is limited, this reasoning process must also consider numerous hypothetical agents who may or may not have links to others in the network. Obtaining any results often requires strong assumptions, and findings are typically incomplete.

Bayesian behavior can also seem unrealistic as a positive model of behavior in such
complicated settings. Complete rationality is demanding of agents in even simple situations, but when we consider behavior in networks, the usual concerns are magnified. One of the strongest cases for modeling rational agents in economics is that adaptation and learning through experience should lead individuals to follow heuristics that roughly approximate rational behavior. We need not assume individuals are actually Bayesians, only that they behave “as if” they are Bayesians. Conlisk (1996) argues that this justification depends upon agents receiving accurate and timely feedback in response to their decisions, feedback that allows them to link action to consequence. In a complex network, this link is difficult to discern, and there is empirical research suggesting that simpler heuristics are far better descriptions of actual human behavior in networks (e.g. Chandrasekhar et al., 2012).

Deeper mathematical issues can arise when we model “large” societies. When individuals are uncertain about the structure of a network, the priors they adopt must include beliefs about the links, preferences, and beliefs of a potentially unbounded number of other agents in the network. The corresponding parameter space is large in a fundamental sense that poses problems for consistent estimation and use of information. Well-known results in measure theory and topology\(^{12}\) imply that any probability measure places probability one on a countable union of compact sets. Consequently, in a large enough parameter space—for instance, one with countably many continuous-valued parameters—any prior is supported on a tiny portion of the space. If the true parameter values lie outside this support, estimates are inconsistent, and it is unclear how to appropriately use information.

All of this appears a strong case against fully Bayesian models of behavior in networks, but the alternative at present also has unappealing features. The other horn of the dilemma is to model agents following a simple heuristic, eliminating any notion of optimization. These models are easier to analyze and produce sharper results than their Bayesian counterparts. The price is that incentives, information, externalities, and strategy either disappear, or are dealt with only on a shallow level. Key economic concepts are difficult to incorporate into these models, so we are forced to leave important questions unaddressed.

The social networks literature would greatly benefit from a middle path between these two extremes. If we imagine a theorist’s production possibilities frontier, with the complexity of the modeling environment on one axis and the complexity of agent decision making on the other, it seems strange to focus production efforts at the corners. New insights could emerge if we move to an interior point. Such a middle path has so far proved elusive, but the bounded rationality literature may play a role in resolving the network economist’s dilemma. A decision framework is needed that respects cognitive limitations of agents with incomplete information about a network while still providing a coherent way to use information.

The remainder of this section surveys economic theory on social networks. I use the network economist’s dilemma to structure my survey, highlighting the distinct contributions of each approach. To contain the survey within manageable bounds, I restrict attention to games and processes on exogeneous networks. As this literature is still vast, I focus on a few canonical examples within each approach; the inclusion or exclusion of particular papers should not be interpreted as a signal of value. I emphasize important open questions and

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the role bounded rationality could play in furnishing answers.

2.2.1 The First Horn: Bayesian Agents

Bayesian agents formulate beliefs and plans accounting for the behavior of all other agents and all possible contingencies. The analysis of Bayesian behavior becomes quite complicated with even very simple network structures. Consequently, models of behavior in networks that retain the assumption of fully rational agents must pay a price in other assumptions to ensure a tractable solution. At present, there are two popular approaches. One is to model a one-shot, simultaneous move game, which we may interpret as the steady state of a (generally unspecified) dynamic process. Results often focus on the influence individual agents exert on the final outcome and the efficiency of these outcomes. A second approach is to model sequential decision making in which each individual agent only makes one decision. This allows the study of dynamics while eliminating complicated feedback from the model. Analysis is simplified in both cases because there is no repeated interaction, and future concerns will not affect today’s decisions.

Sundararajan (2007) provides a simple example of a static model, studying agents who make a binary product adoption decision. A finite collection of agents $N$ is represented as the vertices of an undirected graph $G$. The graph is drawn from a commonly known distribution with certain symmetry properties that facilitate the analysis; if an edge exists between two agents, the agents are neighbors. All of the agents must decide simultaneously whether to adopt a particular product at some positive cost. The utility derived from adoption is increasing in the number of adopting neighbors and in the agent’s stochastically generated valuation type. An agent’s degree and valuation type completely characterize equilibrium behavior in this model, with finer details of the network structure irrelevant to individual decisions. The solution is an intuitive threshold strategy in which agents with sufficiently high valuation types adopt, and the adoption thresholds are non-increasing in degree. Intuitively, higher degree agents are more likely to adopt and derive higher utility in expectation because they expect to have more adopting neighbors.

These results illustrate an important theme in the networks literature, that of centrality. Researchers view measures of centrality as important indicators of power and influence, and some measure of centrality often emerges in network models to characterize equilibrium behavior. In this case, degree centrality is the appropriate measure. Several other papers explore the impact of degree distributions on outcomes in network games (e.g. Jackson and Yariv, 2007). Recently, Galeotti et al. (2010) provide a comprehensive treatment of static network games with incomplete information in which agents’ degrees can summarize their information about the network.

Taking a different approach to static games, Ballester et al. (2006) consider a network game with complete information and a continuous action space. The authors propose a model in which an agent $i$ chooses a continuous effort level $x_i$, obtaining the linear-quadratic payoff

$$u_i(x) = \alpha x_i + \frac{1}{2} \sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j,$$
where $\alpha > 0$ and $\sigma_{ii} < 0$. The structure of the influence network is represented through the matrix of cross-derivatives $\sigma_{ij}$. The corresponding simultaneous move game has a unique Nash equilibrium that is characterized by the Bonacich centrality of the agents. Unlike degree centrality, Bonacich centrality is not a localized measure; it depends on details of the entire network. Intuitively, we can think of Bonacich centrality as the proportion of time a random walk on the graph would spend at a particular vertex. Agents with high centrality are well-connected to other high centrality agents.

While this measure describes the benefit conferred on a particular agent by virtue of her network position, the authors find that a different centrality measure is needed to characterize the solution to a social planning problem. If we remove (or add) an agent to the network with the goal of decreasing (or increasing) the aggregate activity level, we need to account not only for the focal agent’s Bonacich centrality, but the impact of the agent’s presence on the centrality of others. The authors characterize the “key player” through a measure of intercentrality that appropriately accounts for these changes. This work highlights complex issues that arise in network models. Not only is behavior highly sensitive to model assumptions, but the appropriate centrality measure can change depending on the application even within the same model.

In a unique contribution, Manea (2011) examines the impact of network structure on bargaining power. Agents are situated in a network represented by an undirected graph. In each period, a linked pair of agents is selected at random to engage in a bargaining game. One agent makes an offer to the other to divide a unit surplus between them. If the other agent accepts, the two receive these payoffs and exit the game, being replaced by identical agents at the same location in the network. If there is no agreement, the agents remain in the game, waiting for the opportunity to make a different deal. Agents discount future payoffs at a common rate $\delta \in (0, 1)$, and each agent in the pair is equally likely to be the proposer. The fundamental tradeoff an agent makes is whether to hold out for a better offer. The author characterizes bargaining power through a measure called the shortage ratio. Loosely, the shortage ratio captures asymmetry in the number of trading partners available to different groups. Equilibrium behavior induces subnetworks in which groups of agents exert oligopoly power over captive trading partners. This work offers insight into the interdependence of influential network positions, suggesting extension to a more detailed study of the impact of network structure on exchange dynamics and pricing.

One limitation of static models is that most work focuses on relatively few information structures: either agents (or the modeler) are assumed to have full knowledge of the network, or all uncertainty about the network structure is captured in the degree distribution. Assuming complete information seems a natural starting point for analysis, but in most cases this is unrealistic. Moreover, in a large network, making use of this information in applications could prove intractable. For instance, Candogan et al. (2012) study monopoly pricing in a model similar to that of Ballester et al. (2006). In this case, optimal pricing strategies are NP-hard to compute, and the authors propose computationally efficient approximations. Work focusing on degree distributions has the advantage of generating results based on aggregate parameters that are relatively easy to estimate, but there are other high
level features of networks that may be important. Real world networks display clustering and “small world” properties with remarkable regularity (Barabási and Albert, 2002). These are important features of networks, but Bayesian models have little to say about their impact.

A related challenge is to better integrate theory on influence with empirical work. There are at least two important phenomena in the empirical literature that have received little attention in Bayesian models. First, an extensive literature studies the effects of tie strength and the diversity of connections available to an individual. Researchers have examined the relative importance of different types of connections, emphasizing the tradeoffs between forming stronger ties and forming more diverse ties (e.g. Aral and Van Alstyne, 2011). A second focus in the empirical literature has been disentangling homophily from influence in observational data (e.g. Aral et al., 2009). When we see neighboring individuals adopting similar behaviors, one may have influenced the other to engage in that behavior, or the two may simply share similar preferences that led them to behave similarly. Agents and links in theoretical models are typically homogeneous, preventing any analysis of related questions. A more complete theoretical treatment would enhance our understanding of these phenomena and may also provide valuable guidance for future empirical research.

To a limited extent, papers adopting dynamic sequential models have addressed these issues, but this approach makes interpretation difficult. The sequential framework is particularly popular in the Bayesian social learning literature, in which researchers study the diffusion and aggregation of decentralized information. Banerjee (1992) and Bikhchandani et al. (1992) provide the earliest examples in this area, demonstrating the presence of information externalities in a sequential learning model. In the basic model, there is an unknown binary state \( \theta \in \{0, 1\} \), and each agent \( n \in \mathbb{N} \) makes a single irreversible decision \( x_n \in \{0, 1\} \). Each agent earns a unit of utility if and only if her decision matches the state. Agents choose their actions in sequence. Each agent observes a private signal that is independent of other signals conditional on the state \( \theta \), and each agent observes the choices, but not the signals, of all previous agents. If the history of choices is more informative than an agent’s signal, individual rationality leads to an externality. If the observed history of choices strongly favors one state, later agents will ignore their private information, and therefore their choices reveal no new information to the agents that come after them. With positive probability, an inefficient “herd” can develop in which all agents beyond a certain point ignore their private signals and select the worst action.

In the last decade, these sequential social learning models have begun to incorporate network structure. Banerjee and Fudenberg (2004) and Smith and Sorensen (2008) study cases in which agents sample historical actions instead of observing the entire history. Acemoglu et al. (2011) introduce a more general model of learning in stochastic networks. These analyses characterize long run learning outcomes via the connectivity of the network. The specific technical condition varies depending on the model, but the intuitive messages are similar. Herding outcomes can happen if signals are bounded in strength, but if some agents receive arbitrarily strong signals, a well-connected network will eventually aggregate the dispersed information.

In more recent work, Lobel and Sadler (2014a,b) provide a richer understanding of how
robust these results are to asymmetric information about the network structure and explore how homophily affects learning. Their findings reveal an important connection between the diversity–bandwidth tradeoff and homophily. In an observational learning model, homophilous connections provide the most useful information to an agent, so these connections individually exert more influence on behavior. However, the strongest aggregation results depend upon agents having enough diverse connections. Intuitively, homophilous ties are individually more informative but quickly start providing redundant information, while diverse ties provide smaller amounts of relatively independent information. Which type of ties are more beneficial will depend on the density of available connections.

The sequential framework allows some study of complex network structures and heterogeneous agents, but it remains unclear how repeated interactions will affect these results outside of special cases. Recently, Mossel et al. (2012) develop techniques to study a particular model of Bayesian social learning in a general network with repeated interactions, finding that aggregation occurs asymptotically in a large network as long as certain “egalitarian” properties are satisfied. This is a significant contribution, demonstrating that much of the intuition from other models carries over—at least in this case. However, the mathematical sophistication required to achieve this highlights key challenges for this literature going forward. Extensions may prove intractable, and the assumption of fully rational agents becomes less tenable.

Even without repeated interactions, these findings are difficult to apply in a broader context. For instance, some authors study monopoly pricing when consumers sequentially learn from each other about the quality of a good (Bose et al., 2008; Ifrach et al., 2013), but the results are highly dependent on full observation. A similarly underdeveloped area concerns the impact of costly information acquisition when agents are strategic. Bolton and Harris (1999) provide a comprehensive treatment of “strategic experimentation” in a repeated full observation setting, and Mueller-Frank and Pai (2013) consider costly information acquisition in the sequential social learning model with complete observation. However, more complex observation structures are essentially unstudied due to technical difficulties. A framework that simplifies the decision making process of the agents while still capturing strategic interactions could facilitate a more complete treatment of these problems.

Each approach to modeling fully Bayesian behavior in a network suffers from important limitations. Dynamics are difficult to tractably model, and the simplifying assumptions this literature makes create a barrier to applications. Moreover, fully Bayesian reasoning appears unrealistic in many of these models, and it seems likely that a simpler framework could capture the more salient aspects of human decisions. The non-Bayesian literature I explore next offers a complementary set of findings, removing some gaps in our understanding based on the Bayesian literature. Nevertheless, we shall see that even the union of these two approaches leaves much open for future research.

2.2.2 The Second Horn: Heuristic Rules

Heuristic decision rules make it easier to describe and analyze dynamic behavior and repeated decision making. Depending on the context, the heuristics under study are motivated
as a form of myopic best response, an empirically realistic descriptive rule, or as some intuitively appealing rule-of-thumb. A large segment of the literature studying heuristic decision rules in networks considers the diffusion of behavior among agents playing localized coordination games: an agent plays the same game with each of several neighbors, but must adopt the same strategy with each neighbor, and it is advantageous to choose the same action as a neighbor. A common approach is to assume that agents play best responses to the action profile from the previous period, leading to behavior we can describe via a simple decision threshold. The ensuing dynamics often resemble those in infectious disease models.

Morris (2000) produced one of the benchmark papers in this area, analyzing a model in which individuals adopt a behavior deterministically if and only if at least a proportion $q$ of their neighbors adopted the behavior in the prior period. The author considers a network represented as an infinite undirected graph, in which links between nodes indicate that agents are neighbors of one another, and each agent has at most $M$ neighbors for some finite bound $M$. The central question in the paper is whether the network can exhibit contagion. Suppose there are two available actions, labeled 0 and 1. Contagion occurs if action 1 can spread through best response dynamics from some finite group to the entire population. For any given network, the contagion threshold is the highest value of $q$ such that contagion can occur. We can characterize this threshold through a notion of group cohesion. A group of agents $X$ is $p$-cohesive if for each agent in $X$, a proportion at least $p$ of the agent’s neighbors are also in $X$. The contagion threshold is the smallest $p$ such that every co-finite group contains an infinite $1 - p$-cohesive subgroup. This finding captures the intuition that highly clustered groups can act as barriers to diffusion.

A pair of regularity conditions offers additional insight on the structure of networks that facilitate diffusion. For a group $X$, define $\Gamma_n(X)$ as the set of agents reachable within $n$ steps of $X$. If the cardinality of $\Gamma_n(X)$ does not grow exponentially fast for any finite $X$, there is low neighbor growth. Low neighbor growth indicates that a significant proportion of new neighbors at each step are shared neighbors of many agents in the previous step. When combined with a more technical condition called $\delta$-uniformity, low neighbor growth promotes contagion. Intuitively, if neighbor growth is too fast, behavior spreads quickly at first, but adopting agents eventually make up a small proportion of the neighbors of potential future adopters, halting the spread. This indicates that some clustering is necessary to encourage widespread diffusion.

More recent papers elaborate on these findings. López-Pintado (2006) and Jackson and Rogers (2007) extend the analysis to randomly generated networks, studying the effect of the degree distribution on the contagion threshold and the extent of contagion. López-Pintado (2008) explores the sensitivity of these results to the specification of the diffusion function. López-Pintado (2012) and Jackson and López-Pintado (2013) study the effects of other structural properties of the network on diffusion, such as correlations between an individual’s number of in-degrees and out-degrees and the effects of homophily along certain attributes. These papers provide a rich view of how various contextual factors might interact with network structure to promote or inhibit diffusion of behavior in a network. Other authors have applied these insights to problems ranging from immunization (Galeotti and
Rogers, 2012) to marketing (Galeotti and Goyal, 2009).

Jackson and Yariv (2007) take a different approach to modeling diffusion through payoff externalities, incorporating rudimentary future expectations into their heuristic. In this model, a degree distribution characterizes an infinite network. Individual agents are endowed with a certain number of links according to this distribution, and these links are paired randomly. Thus, a typical neighbor will have higher degree than a typical agent in the network since representation in neighborhoods is weighted by degree. Each agent can adopt action 0, leading to net utility of zero, or the agent can adopt action 1 at some stochastically determined idiosyncratic cost, with utility that is increasing in the number of neighbors who adopt action 1. Agents best respond to expectations about neighbors’ future actions, supposing that each neighbor’s action will be an independent random draw from the current global action profile.

The results offer a rich set of comparative statics, linking features of the degree distribution to the rate of adoption, the extent of adoption, the stability of long-run outcomes, and the shape of the adoption curve. Findings are qualitatively similar to those of Sundararajan (2007), but this work also provides a rare instance of an explicit dynamic process that converges to a Bayes-Nash equilibrium of a static game. Each equilibrium of the dynamic process corresponds to a Bayes-Nash equilibrium in a corresponding static version of the game, and the stability of equilibria according to this process provides one selection criterion.

Despite advances, there remain unsatisfying aspects of this literature. First, these models often depend on deterministic adoption processes, or they represent the network through a mean-field approximation. This leads to deterministic outcomes, but empirical work has shown that similar contagions often have different trajectories within the same network (Goel et al., 2012). A model that accounts for these stochastic outcomes could help evaluate risk-return tradeoffs to policies that manipulate diffusion in a network.

The relatively shallow account of individual choice is also a concern because it prevents these models from addressing deeper economic questions. For instance, with the growing use of referral incentives in marketing efforts, many researchers are interested in how this affects an individual’s propensity to spread behaviors in a network and how it impacts broader patterns of diffusion (e.g. Pfeiffer and Zheleva, 2012). The lack of strong micro-foundations in many theoretical models makes it difficult to study such incentives. Researchers may also be interested in how limited knowledge of the overall social network, both by marketers and by consumers, impacts the effectiveness of these marketing strategies. Strategic interactions and information asymmetries are likely important in this context as well as others, and a more detailed, less mechanistic model of choice is necessary to capture their effects.

Another scenario conspicuously absent in these diffusion models is a case with multiple behaviors or products competing for market share. This is a large hole as many interesting domains involve a choice between several similar products, each displaying some local network effect. For instance, we might like to understand the effects of the network on long-run market structure: is the network vulnerable to lock-in (Arthur, 1989), or can multiple products co-exist? Designing strategies to penetrate an established market or protect an existing lead
requires a richer model. Developing a realistic and tractable choice model to study these questions could provide depth to the literature and offer more robust predictions.

Although heuristic rules often preclude standard representations of information, there is a vibrant literature studying information diffusion in networks that takes the second horn of the network economist’s dilemma. Several papers adopt a belief averaging heuristic originating from DeGroot (1974). Suppose there are two possible underlying states, so we can represent the beliefs of an agent as a number \( p \in [0, 1] \) denoting the probability the agent assigns to one of the states. Given a set of \( N \) agents, we can model their interactions through a stochastic matrix \( T \). Agents update their beliefs by taking a weighted average of the current beliefs of other agents. The entries in row \( i \) of the matrix represent how much weight agent \( i \) places on the beliefs of other agents. The matrix also implicitly represents a directed graph of connections between the agents: there is a directed edge from agent \( i \) to agent \( j \) if and only if \( T_{ij} > 0 \). If \( p^{(t)} \) is a real-valued vector representing the beliefs of the agents at time \( t \), the belief vector at time \( t + 1 \) is

\[
p^{(t+1)} = Tp^{(t)}.
\]

In this formulation, beliefs converge to a unique limit as long as \( T \) represents a connected graph satisfying an aperiodicity condition. The limiting belief of the agents is the dot product of the initial belief vector and the unique left eigenvector of \( T \) corresponding to the eigenvalue 1, giving rise to the notion of eigenvalue centrality as a measure of influence.

Within this framework Golub and Jackson (2010) explore the relationship between network structure and the convergence of beliefs. If the initial belief vector arises from a set of independent and identically distributed signals, we can view the belief averaging heuristic as a naïve updating procedure. As the authors point out, the procedure is not necessarily so naïve; under certain assumptions about signal precision, this heuristic coincides with the first period of Bayesian updating. In a large network in which no small group accounts for a significant fraction of links, the limit belief is a good approximation to the belief obtained through complete aggregation of the initial signals. The second largest eigenvalue of the interaction matrix helps characterize the rate of belief convergence, providing additional detail on how the network structure affects the spread of information.

Golub and Jackson (2012) extend this model to study the impact of homophily on learning rates. They consider a randomly generated network partitioned into distinct groups of agents. Agents in one group are linked to agents in another with given independent probabilities. The random network exhibits homophily if agents are more likely to connect with members of their own group. At least two important contributions follow. First, the authors characterize convergence rates not through the full matrix representing the network, but through the smaller matrix of inter-group link probabilities. This greatly reduces the dimensionality of the problem, which facilitates numerical applications. Second, this representation allows the authors to show that increased homophily, meaning increased intra-group clustering, slows down the convergence of beliefs in a sufficiently dense network.

Some authors have retained elements of Bayesian reasoning in information diffusion models while eliminating strategic considerations to simplify the analysis. Bala and Goyal (1998) made a seminal contribution in this area, examining a repeated interaction game with myopic
Bayesian agents. In their model, there is an unknown state $\theta$ with finitely many possible values, and there is a finite set of possible actions $X$. Each action $x \in X$ is associated with a distribution of outcomes conditional on the underlying state $\theta$. In each period, an agent takes an action $x \in X$ and observes the outcome, from which information on $\theta$ may be inferred. A discrete set of agents is linked together in a network, and agents observe the actions and outcomes obtained by their neighbors in every period.

Agents in the model are myopic in two senses. First, in each period an agent selects the action with the highest expected utility given the current belief, without regard to the value of gathering information about other actions. Second, agents update their beliefs based only on the observed outcomes of their neighbors. They do not attempt to infer additional information from the choices their neighbors make. Thus, there is no updating of beliefs about the structure of the network. The key results follow from an imitation principle: each agent must achieve long-run expected utility at least as high as any neighbor because imitation is an available strategy. Consequently, in any connected network, all agents achieve the same long-run expected utility. A large connected network will aggregate information as long as there are no small groups that command disproportionate attention from the rest of the network. Jadbabaie et al. (2012) derive similar results using a hybrid approach in which agents receive new signals each period, updating beliefs according to Bayes’ rule, but are influenced by their neighbors via the DeGroot belief averaging mechanism.

This body of research has produced surprisingly consistent results, highlighting the negative influence of overly influential groups, but finding that beliefs generally converge to a more or less correct limit in well-connected networks. This insight is robust to numerous specifications and is even similar in spirit to many comparable results in the Bayesian literature. However, this may not be cause for comfort. It is curious just how easy it appears for agents in a network to reach a correct consensus in Bayesian and non-Bayesian models alike because this seems at odds with reality in many contexts. An important task for the social learning literature is to explain the persistence of disagreement as a function of the network. With the already extensive treatment of both fully rational and heuristic decision rules in the literature, a new approach is needed to answer this question.

The networks literature based on heuristic decision rules has been greatly successful in providing descriptive insights, but without a more detailed model of choice, it seems difficult to use this framework to understand the effects of interventions. Both the Bayesian and the heuristic literatures have yielded deep insights, but many interesting questions are all but untouched. Many of these questions are design questions, asking how to incentivize behavior to achieve certain goals. This necessitates a choice model with many of the features of rational choice, but the complexity of the problem environments mandates some simplification.

2.3 Remarks

As of this moment, the economic literatures on bounded rationality and on social networks constitute two largely distinct areas of research. While there has been much recent progress in both, a closer union presents exciting opportunities for future research. Despite the extensive
work of the last decade, my sense is that economic research on social networks is in its infancy. The structure of these networks affects our job market opportunities, our negotiating power in exchanges, the information that informs our decisions, and the competitive structure of markets, among many other things. Exploring these phenomena is important, and it will require a modified approach to modeling individual choice. Finding a middle path between fully Bayesian models and heuristic decision rules would resolve the network economist’s dilemma, allowing broader theoretical study of decision making in networks.

One possible approach is to use agents who base their decisions on “local” models of the world, meaning that they only model the behavior of their immediate neighbors, rather than behavior in the entire network. Many authors who study Bayesian models already implicitly take this approach when they make simplifying assumptions about priors. An important feature in the model of Sundararajan (2007) is that the equilibrium result does not require players to form expectations about the behavior of others who are far away in the network. Every Bayes-Nash equilibrium is equivalent to a fulfilled expectations equilibrium in which expectations are formed only for a player’s neighbors. If instead of beginning with a prior on the graph and deriving local decision rules, we directly specify players’ local beliefs we may obtain a more flexible framework to build more detailed models.

In the model of Sundararajan (2007), players’ strategies are functions only of their degrees and valuation types. Given a strategy profile, a player can form expectations about her neighbors’ degrees and valuation types in order to predict all payoff-relevant behavior. Hence, we can define a coherent equilibrium notion without reference to the graph $G$: we need only specify a player’s beliefs about her neighbors’ degrees and valuation types. Many graphs, or distributions over graphs, may be consistent with the same local beliefs about neighbors. In some sense, explicitly defining beliefs on $G$ over specifies the model for the purpose of predicting individual behavior. Furthermore, there is no reason we need to restrict ourselves to representing a player as a degree and valuation pair. We could just as easily define a local equilibrium notion for a more general type space of players. This suggests a novel coarse equilibrium concept with the potential for wide application.

If successful, this and similar efforts will cast bounded rationality in a new light, solidifying its position as a central pillar of economic theory. I have advocated a particular view on the role of rationality in economics, emphasizing the evolutionary nature of rational behavior, while acknowledging the limits inherent in this perspective. If we embrace an evolutionary view, then it becomes sensible to describe separately the environment to which behavior is adapted, and the environment in which behavior occurs. Such an approach offers greater freedom to theorists on one hand, but imposes greater responsibility to carefully consider assumptions on the other.
Chapter 3

Complex Networks and Local Bayesian Equilibrium

Economic understanding of peer influence and the spread of information in networks has come a long way in the last decade, but many questions remain. Finding a satisfactory way to represent individual decisions presents a challenge for theoretical work. When people are influencing and informing one another through a complex network of interactions, the bounds of their rationality are important. These bounds, expressed through many well-documented psychological biases and heuristics (e.g. Kahneman et al., 1982), can have a profound aggregate impact on behavior and welfare.\(^1\) Both for this reason and due to the complexity of networks, economic models of social interactions often rely on heuristic decision rules.

Nevertheless, strategic effects, and other features of rational decision making, frequently appear in empirical studies of social networks. When gathering information is costly, people free-ride on the effort of their friends and neighbors (Bandiera and Rasul, 2006; Kremer and Miguel, 2007). When deciding whether to share a sensitive piece of information, people avoid telling those who are well-connected (Banerjee et al., 2012). Many human decisions reflect a combination of rational elements with simple heuristics, particularly in social interactions, but we lack a compelling framework to represent this in theoretical models. Developing such a framework would aid our understanding of how social ties influence economic outcomes.

For instance, though it is a heavily studied social phenomenon, we have only a rudimentary understanding of the formation and spread of beliefs and opinions. Among the most pressing gaps is an explanation for disagreement. From early work on agreeing to disagree to more recent studies of social learning, theories based on a wide range of decision rules and network structures predict robust long-run consensus.\(^2\) However, casual observation repeatedly contradicts this prediction. There are long standing disagreements in many domains—politics, ethics, climate and medical science—even on matters of fact. Resolving

\(^1\)See for instance, Sargent (1994) and Spiegler (2011).

\(^2\)Consensus occurs in settings with Bayesian agents (Banerjee, 1992; Acemoglu et al., 2011; Mossel et al., 2012), individuals following simple heuristics (DeMarzo et al., 2003; Golub and Jackson, 2010), and mixtures of the two (Bala and Goyal, 1998; Jadambaie et al., 2012).
this and related questions about social influence requires a new approach.

My main contribution is a novel solution concept—local Bayesian equilibrium—tailored for studying games on networks. This concept expresses a hierarchical approach to economic modeling. Any problem that we study exists in a larger context. When we write a model, we impose a boundary between problem and context. Our players typically put this boundary in the same place that we do, and they view the situation at the same level of detail. However, it often makes sense for our players to employ simpler models. Faced with the complexity of a large network of interactions, a player may adopt a more limited perspective. The central idea of local Bayesian equilibrium is that players take a common approach to problem simplification: they view the network as a collection of local neighborhoods rather than as an interconnected whole.

Within a network, a local neighborhood is simply a rooted graph with limited depth—pick a player and think about all of the people no more than $k$ links away. The simplest example takes $k = 1$, including only those to whom the player is directly linked. If these are the only people who are payoff relevant, then our player can plot her strategy without forming expectations about those further away in the network. Our player might have beliefs about how many neighbors each of her neighbors has, but she need not worry about overlap in the neighborhoods of her neighbors, the size of the entire network, or how far away most players are from her. Many networks with various sizes and features are consistent with the same collection of local neighborhoods.

I consider players who are rational within the confines of their local models—think of them as Bayesians with incomplete or misspecified representations of the world. In this framework, players naturally follow certain heuristic decision rules, like making choices as a function of their degrees, yet they also display strategic effects like free riding. In dynamic settings, there is a sense in which limiting the size of the neighborhood that a player models implies a limit on the player’s memory. Hence, biases that emerge from having bounded memory can also reflect reasoning of bounded depth in a network. Going beyond network applications, local Bayesian equilibrium offers a new approach to modeling bounded rationality. Players with different cognitive abilities may include different levels of detail in their local models, and we can study the advantages associated with having a more complete view of the world.

After illustrating the concept through examples, I use local Bayesian equilibrium to address disagreement. I examine two versions of the common prior assumption, one at the level of the network and one at the level of local neighborhoods, and I show that common priors over the network can generate strongly conflicting local beliefs. Since many results on reaching consensus depend upon common priors, this suggests one explanation for disagreement. The broader pattern of interactions between players, which is not fully captured in the local models, might preclude reaching consensus. What appears as a lack of common priors within players’ local models could actually reflect features of the network.

The notion of local equilibrium grew from recent attempts to establish a general, tractable framework for analyzing network games. At least implicitly, various network game models already use local Bayesian equilibrium. In a one-shot simultaneous move game with incomplete information, a carefully chosen prior over the network leads to strategies that are
functions of player degrees. Equilibrium behavior is equivalent to a local Bayesian equilibrium for players who only form expectations about their immediate neighbors. Sundararajan (2007) and Galeotti et al. (2010) use this approach to study games with local payoff externalities, and Lobel et al. (2015) apply similar techniques to analyze consumers responding to a referral program. Local Bayesian equilibrium generalizes this approach to allow wider application, including in dynamic settings.

An appealing feature of local reasoning is that it gives us a middle ground between fully rational models and heuristic choice rules. We can study information and strategic effects without requiring our players to perform supernatural feats of computation. As a nice bonus, we gain some of the tractability enjoyed by simpler rules of thumb. There is a question whether this approach to simplifying decisions is the right one for network games. A small but growing number of studies suggest that people do rely on local network features to guide their decisions. Chandrasekhar et al. (2012) and Corazzini et al. (2012) find evidence that a belief averaging heuristic provides a better description of opinion dynamics than Bayesian updating. In a laboratory experiment, Gallo and Yan (2014) find that the deviations subjects make from Nash equilibrium predictions are consistent with following a degree-based heuristic. While more study is needed, it seems likely that a framework based on local reasoning can furnish more accurate predictions than other approaches in network games.

3.1 The Player as a Microeconomist

Imagine a player who mimics, in the way she views and thinks about the world around her, a microeconomist. This player has in mind an abstract representation of her environment, one that is limited in its scope and stripped down to its essential features. Taking this representation as given, rational behavior is feasible: the player makes inferences according to Bayes’ rule and acts optimally.

In her representation, our player includes a set of other players, states of the world, beliefs about those states, actions, and of course payoffs. These elements—players, states, beliefs, actions, and payoffs—result from a set of choices about what to include. Whose actions should she model and predict? How large is the state space? How much detail does a state of nature specify? Is this a one-shot game, or are future and past actions important as well? What determines her payoff? Does she have an outside option? What are reasonable prior beliefs? What do other people know? The answers to these questions demarcate the boundary of our player’s model of the world and limit the complexity of her decision problem.

As economists, when we think about a problem, one we can represent as a game, we typically assume that our players make the same modeling choices that we do. The people corresponding to our players in the real world have a distinct vantage point, but we suppose that they consider salient the same contextual features, the same other players, and the same unknown variables. The problem we study and the problem our players solve share the same level of detail and the same level of complexity.

I start from the assertion that our players need not share our view of their situation. Our
players may have different concerns than we do, and they may have more limited cognitive resources to expend. It often seems natural for players to work within a simpler framework. However, this does not mean they follow some behavioral heuristic. Our players can exhibit rational behavior within a model that omits some of the details we include.

3.1.1 Small Worlds

A player $i$ makes a choice under uncertainty, and her payoff depends on the actions of a small number of others—call these people her neighbors, and denote the set of neighbors $N_i$. Our player categorizes people into types $\theta \in \Theta$, and she has in mind a strategy profile mapping types to actions. Our player also has in mind a joint distribution over states and her neighbors’ types, which together with the strategy profile defines a distribution over states and her neighbors’ actions. Together with the payoff function, this distribution defines our player’s expected utility maximization problem.

Every type describes a decision problem of this form, and we define an equilibrium for the set of types. As economists, we have in mind a population of players with overlapping neighborhoods, which form a network. In this network, we categorize people into the same set of types. There could be many players of each type, or just a few. Many networks with varying features and sizes can arise from the same collection of types. The players need not represent the network in their decision making process; they only need to form beliefs about their neighbors’ types.

Each type $\theta \in \Theta$ corresponds to a particular type game $G_\theta$, which comprises a set of other players, actions, states, beliefs, and payoff functions $\{N_\theta, S, \Omega_\theta, F_\theta, U_\theta\}$. Think of $G_\theta$ as a prototype for a player’s decision problem. A particular player $i$ is of type $\theta$ if her decision problem is isomorphic to $G_\theta$. The set $N_\theta$ represents the neighborhood of a player with type $\theta$. If player $i$ is of type $\theta$, there is a bijection $f : N_i \rightarrow N_\theta$. Each neighbor in turn possesses a type, and we use $t_{N_\theta} = f(t_{N_i})$ for the tuple of types corresponding to the image of $N_i$ under $f$.

The state space $\Omega_\theta$ captures how a player of type $\theta$ views uncertainty about the world. A player $i$ chooses a set of states $\Omega_i$ to include in her model, representing the features she deems most payoff relevant. If $i$ is of type $\theta$, there is a bijection between $\Omega_i$ and $\Omega_\theta$. The distribution $F_\theta$, a joint distribution over $\Omega_\theta$ and $\Theta^{|N_\theta|}$, represents the beliefs of a player with type $\theta$. Player $i$ has beliefs $F_i$ over the underlying state and the types of her neighbors, and if she is of type $\theta$, then we can choose the bijections $f : N_i \rightarrow N_\theta$ and $g : \Omega_i \rightarrow \Omega_\theta$ such that $F_i(X,Y) = F_\theta(f(X),g(Y))$ for any measurable $(X,Y) \subseteq \Theta^{|N_i|} \times \Omega_i$.

The space of actions $S$ is common to all types, representing the union of all possible actions available to each type. The function $U_\theta : \Omega_\theta \times S^{|N_\theta|} \times S \rightarrow \mathbb{R}$ gives the payoffs for a player of type $\theta$. Write $s_i \in S_i$ for player $i$’s action, and write $s_{N_i} = \prod_{j \in N_i} S_j$ for the tuple of neighbor actions. There is an injective mapping $h : \bigcup_j S_j \rightarrow S$, which is common to all

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3We could also think of it as an equivalence class of decision problems.
players, such that for any player $i$ of type $\theta$ we have

$$U_i(\omega_i, s_{N_i}, s_i) = U_\theta(g(\omega_i), h(s_{f(N_i)}), h(s_i)),$$

where $U_i$ is player $i$’s utility function. Given this correspondence, a player’s type captures all essential aspects of her decision problem, and henceforth I refer to players by type.

A strategy profile is a mapping $\sigma : \Theta \to S$ giving an action for each possible type. If a player of type $\theta$ chooses action $s$ in response to $\sigma$, her expected payoff is

$$u(\theta, \sigma, s) = \mathbb{E}_{F_\theta}[U_\theta(\omega_\theta, \sigma(t_{N_\theta}), s)].$$

(3.1)

A strategy profile $\sigma$ is a local Bayesian equilibrium if for each type $\theta$ and each feasible $s \in S$ we have

$$u(\theta, \sigma, \sigma(\theta)) \geq u(\theta, \sigma, s).$$

(3.2)

A local Bayesian equilibrium is formally equivalent to a Nash equilibrium of a standard game in which $\Theta$ is the set of players. Equation (3.1) defines the payoff for player $\theta$ given the profile of actions $\sigma$ of the other players. A profile $\sigma$ is a Nash equilibrium in this game if and only if it is a local Bayesian equilibrium in our game. The standard existence theorem for Nash equilibrium then implies the existence of a local Bayesian equilibrium.

**Theorem 1.** If $S$ and $\Theta$ are finite, a local Bayesian equilibrium exists in mixed strategies.

Although this looks like a static presentation of local Bayesian equilibrium, there are natural ways to apply the solution concept in dynamic settings. Suppose that each player in a large network makes a single decision, but the timing of decisions depends on players’ actions and the structure of the network. For example, a player’s choice might prompt her neighbors to decide shortly thereafter. From the local perspective of a player, this would look like a static game. As economists, we might take an interest in the pattern and timing of choices throughout the network. Alternatively, if players have bounded memory, or find it costly to fully account for intertemporal effects of their decisions, we can adapt this definition to serve a dynamic setting. I present examples later to illustrate these approaches, though this certainly does not exhaust all possibilities.

### 3.1.2 Interpreting Local Bayesian Equilibrium

Type games represent players’ local models within a larger network. The players categorize their neighbors to simplify the decision making process, treating each neighbor as a representative of a type. A type network, a directed graph in which we label each node with a type $\theta \in \Theta$, describes the population of players. Nodes are players, and a directed edge indicates a link to a neighbor. We identify a player’s neighbors in the network with the neighborhood $N_\theta$. Players distinguish between neighbors only up to their types, while the type network describes exactly who is whose neighbor.

Harsanyi (1967–68) introduced a different kind of “type” to succinctly represent belief hierarchies in games with incomplete information. Our types play a similar role, providing a
compact representation of a distinct belief hierarchy. In a Harsanyi game, types capture the beliefs of players who are mutually aware of one another. Articulating these beliefs results in statements like, “I believe that you believe that I believe...” etc. Our types capture an incomplete set of beliefs about the network. The corresponding belief articulation is a statement like, “I believe that you believe that your neighbors (whoever they are) believe that their neighbors believe...” etc. I say an incomplete set of beliefs because a player never forms her own beliefs about neighbors of neighbors: she has beliefs about her neighbors, beliefs about her neighbors’ beliefs about their neighbors, but no distinct beliefs about her neighbors’ neighbors. In comparison with the belief hierarchy for the full network game, in which every player thinks about every other player in the type network, this belief hierarchy has substantially fewer elements. Our belief hierarchy, if fully articulated, is still a complex object, but it is far simpler than the complete hierarchy of beliefs.

How might people learn to play a local equilibrium? Real networks are dynamic; people interact repeatedly with different and changing sets of others over time. A person forms expectations about her present situation and the people with whom she interacts based on similar past instances. Beliefs in the present can express the historical frequency of certain kinds of interactions with certain kinds of people. Over time, observing these frequencies, people should adapt to follow roughly optimal actions. If different people have different patterns of interactions with different kinds of people, this suggests a natural way for conflicting beliefs to arise: incompatible priors in one instance reflect different past experiences.\(^4\) Section 3.3 formalizes an argument based on this intuition, showing that under a version of the common prior assumption, different network positions can produce arbitrary differences in local beliefs.

If we focus on the role of types as a coarse categorization of other players, local equilibrium presents a tool, going beyond network models, to study bounded rationality in games. Different types can correspond to more or less complete models of the same situation. A more detailed type game could include a larger set of states, a larger set of players, some of whom are unknown to some other players, or beliefs that place positive probability on a larger set of types. I present a simple example to illustrate this possibility at the end of the next section.

### 3.1.3 Locality

What does it mean, more precisely, for players to adopt local models? Think about a natural metric on the nodes of a network, in which the distance between any two nodes is the length of the shortest path between them. A player’s model is local if it does not require, either explicitly or implicitly, that the player hold beliefs about others who are more than \(k\) links away. The notion of a rooted graph allows us to talk about neighborhoods of varying

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\(^4\)One intuitive way to think about the selection of a model or a prior is as a choice made with a case-based decision rule (Gilboa and Schmeidler, 1995). Different experiences would surely lead to different priors when entering an interaction. Even if two people have the same set of experiences, if they use different similarity functions to evaluate cases, they would end up with different expectations.
depths around a player.\(^5\)

**Definition 1.** A **rooted graph** is a pair \((G, x)\), where \(G\) is a directed graph and \(x\) is a node in \(G\). A **rooted graph of depth** \(k\) is a rooted graph \((G, x)\) such that no node in \(G\) is more than \(k\) links away from \(x\).

A rooted graph is simply a graph with a particular node singled out for attention. Think of the graph originating and growing away from the root node. A neighborhood around \(x\) is a subgraph of \((G, x)\) in which we discard nodes and links more than a fixed distance away from \(x\). A neighborhood around a player is a rooted subgraph together with types labeling each node.

Since players only model their immediate neighbors, any neighborhood around a player is observationally equivalent to a tree network. We can use this tree network as a standard form to represent neighborhoods. To obtain this standard form, start with a particular player \(i\) at the root, and iteratively construct neighborhoods with increasing depth. For the first level, create nodes corresponding to each neighbor of player \(i\), label them with their types, and draw directed edges. Repeat the process for every neighbor at each subsequent level. The result is a type network structured as an infinite tree. Every rooted type network maps to exactly one such tree, but many rooted type networks may map to the same tree.\(^6\)

**Definition 2.** The **\(k\)-local neighborhood** \((G_k, i)\) of a player \(i\) is the tree-shaped type network, constructed as described in the above paragraph, carried to the \(k\)th level.

To illustrate how different observationally equivalent networks can appear, consider the two networks in figure 3.1. Suppose there is only one type of player, and this type always has two neighbors. The undirected circle (a) and the directed infinite tree (b) are observationally equivalent. If each player takes an action in each of a discrete set of time periods, any particular player will observe the same distribution of actions in both cases.

Players’ beliefs are explicitly defined on only their 1-local neighborhoods—over the players to whom they are directly linked. For \(k > 1\), there are two ways to derive implicit beliefs about \(k\)-local neighborhoods. The most obvious is to leverage neighbors’ beliefs: a player \(i\) has beliefs about her neighbors’ types, and a neighbor’s type defines her beliefs about her neighbors. However, the distribution we obtain over neighbors of neighbors does not correspond to any belief of player \(i\). Recalling our earlier discussion of incomplete belief hierarchies, beliefs about neighbors’ beliefs about their neighbors are not the same as beliefs about neighbors of neighbors.

The other way that implicit beliefs about \(k\)-local neighborhoods can arise is due to dynamics. Suppose players act simultaneously in each of two periods. In the first period, player \(i\) has beliefs about her neighbors’ types and the strategy profile, and she chooses an action based on these beliefs. At the start of the second period, she has observed the first

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\(^5\)I adopt the same notation for rooted graphs as Mossel et al. (2012).

\(^6\)This is analogous to our discussion of incomplete belief hierarchies in the previous section. The collection of local neighborhoods provides only a partial description of the network. This partial description defines an equivalence class of networks in which players observe the same distribution of actions among their neighbors.
period actions of her neighbors, which provides information about their neighbors. To form beliefs about what her neighbors will do in the second period, she must form beliefs about what they observed in the first. This implies beliefs about the actions of players in her 2-local neighborhood, and hence beliefs about the types of players in her 2-local neighborhood.

As more time passes, players must model larger neighborhoods. To formalize this idea, fix the graph $G$ that links a population of players. Suppose players choose actions simultaneously in each period $t \in \{0, 1, 2, \ldots\}$, and each player has at least two feasible actions in every period. Let $H_{t,d} = (S^d)^t$ denote the set of histories a player with $d$ neighbors can observe at the beginning of period $t$. The information set for a player of type $\theta$ in period $t$ includes an element $H_t \in H_{t,d}$.

**Definition 3.** A player $i$ can distinguish $k$-local neighborhoods at time $t$ if for any two distinct $k$-local neighborhoods $(G_k, i)$ and $(G'_k, i)$, there exists a strategy profile $\sigma$ such that the histories $H_t$ and $H'_t$ corresponding to these $k$-local neighborhoods are distinct.

**Theorem 2.** Players can distinguish $k$-local neighborhoods at time $k$.

**Proof.** Induct on $k$; the base case $k = 1$ is immediate. Any two distinct 1-local neighborhoods contain different collections of types, and we can choose a strategy profile in which a different vector of actions will result in the first period. Suppose the conclusion holds up to $k - 1$ and consider two distinct $k$-local neighborhoods of player $i$. This implies that at least one neighbor of player $i$ has distinct $k - 1$-local neighborhoods in the two networks and can therefore distinguish them. Choose a strategy profile allowing this neighbor to distinguish the two $k - 1$-local neighborhoods after $k - 1$ periods, and give her a strategy that selects different actions in period $k$.

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Mossel et al. (2012) prove a related result, showing in their social learning model that players’ strategies in period $k$ can depend upon only their $k$-local neighborhoods. This result is a partial converse.
Theorem 2 presents a challenge for dynamic applications of local Bayesian equilibrium. On the one hand, our standardized representation of neighborhoods as trees reduces the complexity of reasoning in a network. On the other, after many periods pass, players’ models cease to be “local” in any meaningful sense. In an infinite game, players must think about the entire tree.

If we take seriously the idea that players should think locally, then Theorem 2 indicates an important connection between locality and memory. The more a player remembers about the past, or the more her strategy conditions actions on the past, the larger the neighborhood she implicitly models. Local models are incompatible with dynamically consistent beliefs. One approach to deal with this issue is to limit players’ memory. If a player recalls the actions her neighbors took only over the prior \(k-1\) periods, then we can fix a prior on her \(k\)-local neighborhood that allows her to reason consistently in the current period. In essence, the player carves out a \(k\)-period problem from a larger dynamic context, solving a model that is “local” both in the network and in time.

### 3.1.4 Relationship to Other Solution Concepts

In recent years, the game theory literature has seen several novel solution concepts designed to capture players’ cognitive limitations. Analogy based expectations equilibrium (Jehiel, 2005) is an influential approach based on categorical thinking. Analogy based expectations describe a coarse way of forming beliefs about what other players in a game will do. In a finite extensive form game with perfect recall, each player \(i\) partitions other players’ decision nodes into analogy classes. If two decision nodes are in the same element of the partition, the action spaces at those nodes are identical. An analogy based expectation is a function \(\beta(\alpha)\) that maps each element \(\alpha\) of a player’s analogy partition to a probability distribution over the corresponding action space. Each player \(i\) optimizes under the assumption that at every decision node in the analogy class \(\alpha\), the other players choose the same action distribution \(\beta(\alpha)\). In equilibrium, the distribution \(\beta(\alpha)\) matches the distribution of actions taken at decision nodes in \(\alpha\) during actual play. Through this solution concept, the author shows how categorical thinking can resolve finite horizon paradoxes and explain some behavioral deviations from rational models.

Though our motivations are somewhat different, local Bayesian equilibrium and analogy based expectations equilibrium take a similar approach to modeling bounded rationality. In fact, with an appropriate set of types, we can formally represent an analogy based expectations equilibrium as a local Bayesian equilibrium. In the simplest case, imagine a two player game in which nature first chooses one of two publicly observed states \(\omega \in \{1, 2\}\) with equal probability, and both players then take simultaneous actions. In the extensive form game, each player has two decision nodes, corresponding to each possible state. If both players separate each other’s decision nodes into two analogy classes, then we have a standard rational expectations equilibrium. Consider the case in which player one groups player two’s decision nodes into a single analogy class, while player two separates the decision nodes of
player one.\footnote{Jehiel (2005) analyzes a game of this form to aid the reader’s understanding of analogy based expectations.}

To recast this in the language of local Bayesian equilibrium, consider a type network with two players and a type space with four possible types—player one is of type $\theta \in \{\theta_1, \theta_2\}$ and player two is of type $\theta' \in \{\theta'_1, \theta'_2\}$. All types have exactly one neighbor and a common state space. A player of type $\theta_i$ is certain the state is $\omega = i$ but believes her neighbor is equally likely to have type $\theta'_1$ or type $\theta'_2$. A player of type $\theta'_i$ is certain the state is $\omega = i$ and is certain her neighbor is of type $\theta_i$. If we match the action spaces and payoff functions appropriately, the local Bayesian equilibria of this game will coincide exactly with the analogy based expectations equilibria in the original game. With some care, we can use local Bayesian equilibrium to encompass the general case as well.

Cursed equilibrium (Eyster and Rabin, 2011) represents a somewhat different approach to bounded rationality, drawing directly on the literature that studies psychological biases. In a cursed equilibrium, players partly ignore correlations in the information that other players receive. If we think carefully about the definition the authors present, we realize that cursed equilibrium falls within the standard definition of Bayes-Nash equilibrium—what is unusual is the lack of a common prior. The “cursedness” parameter offers a simple way to vary the extent of disagreement between players’ priors. There is extensive empirical support for the behavioral effects the authors seek to capture, but we could interpret this evidence as support for a lack of common priors, rather than a lack of rationality. While some economists have argued strongly in favor of the common prior assumption (e.g. Aumann, 1987), section 3.3 argues that, at least in some situations, we should seriously consider this alternative explanation.

Unawareness structures, introduced by Heifetz et al. (2006), offer a useful complement to the framework proposed here. Even outside of network settings, local Bayesian equilibrium can make sense as a tool to study bounded rationality and differences in cognitive abilities. One way to express such differences is through the state spaces $\Omega_\theta$. A more sophisticated player would have a larger state space capturing more detail, while a less sophisticated player would work with a less complete state space. Unawareness structures give us a principled way to relate these distinct state spaces to one another. Conversely, the types I have introduced can offer a compact way to represent unawareness structures in practice.

There are many reasons and many approaches for studying cognitive limitations and biases in games. Network games provide the immediate motivation for Local Bayesian equilibrium, but this solution concept can also offer a way to unify other frameworks. Types give us a flexible way to represent categorical thinking with various degrees of detail. The examples in the next section demonstrate the use of this framework and illustrate a sample of potential applications.
3.2 Examples

3.2.1 A Public Goods Provision Game

We often learn about new products or technologies from our friends, and the information we gain from friends’ experiences influences our choices.\footnote{Researchers have documented these effects in numerous settings, from development (Conley and Udry, 2010; Banerjee et al., 2013) to investments (Duflo and Saez, 2003) to health plans (Sorensen, 2006) to consumer goods (Mobius et al., 2011).} Since gathering information is costly, a social learning about innovations faces a public goods provision problem, and recent studies find evidence of free riding in information sharing networks (Bandiera and Rasul, 2006; Kremer and Miguel, 2007). The relationship between network structure and incentives to gather information is an active area of research (Ballester et al., 2006; Bramoullé and Kranton, 2007; Elliott and Golub, 2013), and local Bayesian equilibrium presents a helpful new perspective to explore related questions.

A population of \( n \) players, connected in an undirected graph \( G \), faces a public goods provision problem.\footnote{This example is based on the model of Bramoullé and Kranton (2007).} Each player \( i \) chooses an investment level \( e_i \), incurring a cost \( ce_i \) with \( c > 0 \). Payoffs depend on the investment a player makes and the investments her neighbors make. Let \( G_i \) denote the set of players to whom player \( i \) is linked, with \( i \in G_i \) by convention, and let \( \tau_i = \sum_{j \in G_i} e_j \). Player \( i \)’s payoff at the end of the game is \( u(\tau_i) - ce_i \), where \( u \) is an increasing, concave utility function that is common to all players. For concreteness, from here on assume \( u(x) = \sqrt{x} \) and \( c = \frac{1}{2} \). First order conditions imply that the marginal value of effort for player \( i \) is greater than the marginal cost of effort if and only if \( \tau_i < 1 \).

We look at two different ways the players can model this situation. First, suppose the players share our perspective: they have perfect knowledge of the network structure \( G \), know their exact locations within the network, and coordinate on an equilibrium in pure strategies. We consider standard Bayes-Nash equilibria of the game as a benchmark. The results of Bramoullé and Kranton (2007) imply that there is always an equilibrium which is specialized, meaning that all players choose effort levels either 0 or 1, and if any equilibrium is stable in a particular sense, it must be specialized.

Alternatively, maybe the players are unable to reason about the entire network, and they adopt local models instead. We consider a particularly simple set of local models in which players observe their degrees, and there is exactly one type corresponding to each possible degree. In the type game \( G_\theta \) for a degree \( d \) player, the set of neighbors \( N_\theta \) contains \( d \) other players, and the state space \( \Omega_\theta \) is a singleton. The action space \( S = \mathbb{R} \) is the set of possible investment levels, and the utility function \( U_\theta \) is simply \( u(\bar{e}) \), where \( \bar{e} \) is the sum of neighbors’ and own investment. Our final task is to choose a distribution \( F_\theta \) that represents beliefs about neighbors’ types. Since there is a one-to-one correspondence between types and degrees, we only need to specify beliefs about neighbors’ degrees. Fix a degree distribution \( F \in \Delta(\mathbb{N}) \), and suppose a player believes that each of her neighbors has degree drawn independently according to \( F \). Such beliefs might arise from some knowledge of the empirical degree distribution. A local Bayesian equilibrium corresponds to a symmetric
equilibrium in the model of Galeotti et al. (2010), and their results tell us that equilibrium effort is decreasing in degree.

Comparing outcomes of the two models in a few simple networks reveals important differences. In the square network of figure 3.2 for instance, the additional knowledge players have in the first version of the model enables welfare improving coordination. If players know the network and the strategies of the other players, then there are exactly two stable Bayes-Nash equilibria. Each equilibrium involves players at opposite corners investing effort 1 and the other two players investing 0. If players are constrained to use local models, then we are restricted to symmetric strategies: in equilibrium, all players invest effort $\frac{1}{3}$. More total effort is exerted in the first version, resulting in greater welfare. As Bramoullé and Kranton (2007) note, specialization can often lead to Pareto improvements, but this requires the ability to coordinate throughout the network. Such coordination is typically possible within a small group of mutual friends, but it may be infeasible in larger networks.

![Figure 3.2: The square network](image)

Which networks lead to better welfare outcomes can also depend on how players model the situation. The two networks in figure 3.3—the first divided into two components and the second a line of five players—have the same degree distribution. Players who adopt local models cannot distinguish between the two: if we fix the degree distribution $F$ that describes beliefs about neighbors, the local Bayesian equilibria are the same in both networks. While equilibrium actions are the same, the network structure will impact welfare. Players with one neighbor exert more effort in equilibrium than those with two; since the payoff function is concave, the first network, in which a player’s neighbors have the same degree as she does, leads to higher welfare.

If players have complete knowledge of the network and coordinate their strategies, the reverse is true. In the network with two components, all stable equilibria involve a total effort of 2: each player receives a benefit of 1, leading to aggregate welfare of 4. In the line network, the unique stable equilibrium involves the two players on the ends and the player in the middle exerting effort 1. A simple calculation verifies that the line network leads to higher aggregate welfare. This example illustrates why the way we model individual behavior in a network really matters. If we assume people are more sophisticated than they actually are, we can misjudge which networks lead to better welfare, so being cognizant of
3.2.2 Customer Referrals

An important way that companies acquire new customers is through personal referrals. As social media and online data collection continue to expand, it is becoming easier to track referrals and incentivize current customers to recruit friends. Having observed the different ways that companies implement their referral programs, Lobel et al. (2015) recently took up the problem of referral incentive design. A key simplification, focusing on a static interaction between a firm and a focal consumer, allows the authors to gain traction on a very difficult network problem. Taking this as a starting point, local Bayesian equilibrium allows us to leverage the same representation of the consumer problem and go a step further.

A monopolist produces a product at no marginal cost and uses a referral program to market the product to consumers.\footnote{We describe the setup of Lobel et al. (2015), restricting the firm to offer a linear payment function.} At any moment, only some consumers are aware of the firm’s product. When a consumer makes a purchase, she has the option to refer any number of her friends, and the firm may compensate her for doing so. The firm charges a price $p$ for its product, and pays consumers $w$ for every referral that subsequently results in a purchase. Consumers have independent valuations for the product drawn according to $V$, which is supported on $[0, 1]$, and making a referral costs a consumer $\delta > 0$. Time is discrete, and if a consumer purchases and refers friends in one period, we assume those friends make decisions in the next period. Each consumer buys at most one unit of the firm’s product, and the firm wishes to maximize its profits.

We need two representations of the referral network: one from the perspective of the consumers and one from the perspective of the firm. Suppose a consumer observes her valuation $v \in [0, 1]$ for the product and the number of friends $d$ to whom she can send...
referrals. I specify a type $\theta$ for each valuation and degree pair $(v, d)$. As in the previous example, the neighborhood $N_\theta$ contains $d$ neighbors, and the state space $\Omega_\theta$ is a singleton. The action space $S$ is the set of pairs $(b, r)$, where $b \in \{0, 1\}$ is a purchase indicator, and $r \in \mathbb{N}$ is the number of referrals a player sends. If $N_r \leq r$ is the number of neighbors who purchase in response to a referral, the payoff to a type $\theta$ player is

$$U_\theta = b(v - p - r\delta + N_r w).$$

For the beliefs $F_\theta$, suppose a player with type $\theta$ believes that each neighbor has valuation drawn independently according to $V$ and degree drawn independently according to another distribution $D$. This gives a distribution over neighbors’ valuations and degrees, which corresponds to a distribution over types. We now have a complete description of the type games, and the local Bayesian equilibria are exactly the equilibria of the consumer game in Lobel et al. (2015).

Given consumers’ local models, we have considerable flexibility to specify the firm’s model of the network. An especially well-informed firm might know the graph $G$ that describes the network and observe which individuals are initially aware of its product. A less well-informed firm might have a distribution over networks in mind. In either case, after choosing a referral reward $w$, the firm can compute an equilibrium and predict a distribution over diffusion outcomes. The results of Sadler (2015b), which explore the relationship between network structure and diffusion patterns, suggest one way to approach this analysis. Several extensions, like a firm that can choose to seed specific consumers with special offers, are also possible in this framework.

This referral game illustrates how we can embed a relatively simple static game within a dynamic context. Each individual consumer makes a single decision at one particular moment. After purchasing and referring some number of friends, payoffs are realized within a single period. Locally, a consumer’s problem looks static. She just needs to predict how her immediate neighbors will respond to a referral. The firm, on the other hand, faces a dynamic diffusion problem. Having a simple way to represent consumer behavior in this context, while still accounting for influence and strategic effects, can help address novel and challenging problems.

### 3.2.3 Persuasion Bias

Psychological studies routinely find that mere repetition of information can influence people’s opinions (Tversky and Kahneman, 1973; Hawkins and Hoch, 1992). DeGroot (1974) introduced a model that captures this effect through a simple belief updating rule. There is an unknown parameter $x$, which takes values in a convex subset $X$ of some vector space, and each of $n$ individuals wishes to estimate this parameter. Individuals update their estimates at discrete times $t \in \mathbb{N}$. A non-negative matrix $W$, with rows summing to 1, describes the updating rule. Writing $x_i(t)$ for individual $i$’s estimate at time $t$, estimates at time $t + 1$ are given by

$$x_i(t + 1) = \sum_{j=1}^{n} W_{ij} x_j(t).$$
Individual $i$’s updated estimate is a weighted average of others’ estimates in the previous period.

The DeGroot updating rule has become popular in recent work on opinion dynamics. For instance, DeMarzo et al. (2003) use the procedure in their model of “persuasion bias” to help explain polarized opinions, and Golub and Jackson (2010, 2012) study the role of network structure and homophily in more detail. One of the reasons the DeGroot procedure is so appealing is its tractability. Standard results for Markov chains allow a fairly complete characterization of belief evolution in terms of the spectrum and eigenvectors of $W$. There is empirical support for this updating rule (Chandrasekhar et al., 2012; Corazzini et al., 2012), but the literature lacks an explanation why people who have ample opportunity to learn continue following a simple procedure that is prone to error.

How might this updating rule naturally arise? Suppose $X = \mathbb{R}^k$, and each individual $i$ begins with a private signal $x + \epsilon_i$, where the parameter $x$ is normally distributed, and the $\epsilon_i$ are independent, mean-zero, and normally distributed errors. If priors are very diffuse—approaching a normal distribution with infinite variance in the limit—then posterior expectations on $x$ approach $x_i(0) = x + \epsilon_i$. If individual $i$ then observes the estimates of some others $x_j(0)$, the first period of Bayesian updating corresponds to a linear weighted average (See DeGroot, 2004, Chapters 9 and 10). Bayesian updating in future periods, after observing neighbors’ estimates, corresponds to a weighted average with adjusted weights that depend on the network structure (Mossel and Tamuz, 2010). Hence, at least within a single period, we can think of linear averaging as a rational approach to estimation with a very diffuse prior.

Recalling the analysis of section 3.1.3, we can explain the DeGroot procedure as a result of local modeling. If individuals only model their immediate neighbors, they cannot condition their updating rules on the history. Suppose each individual $i$ does not remember how she arrived at her current estimate $x_i(t)$. Instead, she takes this estimate as her signal in a one-period model. She has a set of neighbors and a belief about their signal precisions, which gives her a set of linear weights. Our player will use the same weights in each successive period because she retains the same one-period model that she has carved out of the dynamic context. Instead of viewing the chain of updates as a single contiguous decision problem, she treats each update as an isolated belief revision.

One reason to find this explanation appealing is that, in real life, our successive belief updates on any particular topic are separated by many unrelated decisions. These interruptions make it more difficult to treat updates as connected parts of a larger problem; treating them in isolation saves cognitive resources. We might expected the constant weights to represent some average over what rational updating would look like across periods, which would be consistent with observations that people underreact to early information and overreact later.

### 3.2.4 A Cournot Duopoly Game

I present one last example, which exhibits just a trivial network structure, to illustrate how local Bayesian equilibrium can offer insight in more diverse contexts. Consider a textbook Cournot duopoly game with two firms producing identical goods at no cost. In the relevant
market, demand is high or low with equal probability, corresponding to the inverse demand curves \( p = H - q \) and \( p = L - q \) respectively, with \( 0 < L < H \). Each firm \( i \in \{1,2\} \) receives a conditionally independent binary signal \( s_i \in \{l,h\} \), which satisfies

\[
\mathbb{P}(s_i = h | H) = \mathbb{P}(s_i = l | L) = g > \frac{1}{2}.
\]

After observing the signal, each firm chooses a level of production, the price adjusts to clear the market, and payoffs are realized. In this basic duopoly game, the unique symmetric equilibrium has each firm produce \( q_L \) in response to a low signal and \( q_H \) in response to a high signal, with

\[
q_L = \frac{H + L}{6} - \frac{(2g - 1)(H - L)}{(2g - 1)^2 + 2} \quad \text{and} \quad q_H = \frac{H + L}{6} + \frac{(2g - 1)(H - L)}{(2g - 1)^2 + 2}.
\]

In the case of a fully informative signal this reduces to \( q_L = \frac{L}{3} \) and \( q_H = \frac{H}{3} \), and with an uninformative signal it reduces to \( q_L = q_H = \frac{H + L}{6} \).

Now suppose the above model is only approximately true. There is another market that provides a partial substitute for the product our two firms are selling. Depending on the price in the other market, a portion \( \delta < L \) of consumers may switch from one to the other. Demand in the other market is high or low with equal probability, independent of demand in our market. If there is high demand in the other market, then a high price results, driving \( \delta \) additional consumers into our market. If there is low demand in the other market, then a low price attracts \( \delta \) consumers away from our market. The real inverse demand curves are

\[
p = H \pm \delta - q \quad \text{or} \quad p = L \pm \delta - q
\]

according to the demand realizations in each market.

We consider two types of firms, a less sophisticated firm \( \theta_1 \) and a more sophisticated firm \( \theta_2 \). Each type competes against one other firm, choosing a quantity and profiting based on the realized demand. A type \( \theta_1 \) firm adopts our first model of the market, ignoring the effect of the second market. This firm believes that inverse demand is either \( p = L - q \) or \( p = H - q \) and believes that its competitor is also type \( \theta_1 \). Type \( \theta_2 \) firms are aware of the second market and can perfectly observe demand in the second market before making a production decision. A type \( \theta_2 \) firm believes that its competitor is the less sophisticated type \( \theta_1 \). Suppose firm 1 is the less sophisticated type, and firm 2 is the more sophisticated type.

With different views of the problem, our two firms take different actions. Since firm 1 holds the simpler view of its decision problem, it produces \( q_L \) in response to a low signal and \( q_H \) in response to a high one. Firm 2, anticipating the behavior of firm 1, will produce \( q_L \pm \frac{\delta}{2} \) or \( q_H \pm \frac{\delta}{2} \) according to its signal and the realization in the second market. Firm 2 makes a higher profit in expectation, obtaining a slight advantage from its deeper understanding of the market.

The advantage of firm 2 is distinct from simply having more information. Rather than using information that is unavailable to firm 1, firm 2 is solving a fundamentally different decision problem. In principle, firm 1 can observe demand in the second market just as well as firm 2 can, but firm 1 is either unaware of the other market’s relevance, or for some
other reason it is unable to act on this information. Firm 1 may observe over time that firm 2 is making higher profits, but it is unclear how firm 1 might come to recognize the importance of the second market. Firm 1 may not even be able to tell whether firm 2 has extra information or is just making errors. Elaborations of this approach could offer new insights on the sources of competitive advantage among firms.

3.3 Network Structure and Common Priors

Nearly all game theoretic models employ some version of the common prior assumption. The structure of the game, and a distribution over possible realizations within that structure, is common knowledge, implying a priori agreement among the players about every relevant detail. This is not without cause; Harsanyi (1967-68) argued that we should generally work with “consistent” beliefs, and Aumann (1987) makes a case why it is natural to assume common priors. More recently, the social learning literature has furnished many arguments, across many settings with different decision procedures, why groups should reach consensus in the long run (e.g. Bala and Goyal, 1998; Golub and Jackson, 2010; Jadbabaie et al., 2012; Mossel et al., 2012). Nevertheless, there are reasoned objections to the common prior assumption (e.g. Brandenburger and Dekel, 1993; Morris, 1995; Gul, 1998). More importantly, we often observe strong disagreements, even on matters of fact (e.g. man-made climate change, the safety of vaccines). Such disagreements call for a more compelling explanation. A lack of common priors could provide such an explanation, and Local Bayesian equilibrium can provide guidance on when and why we should expect to find conflicting priors.

We explore common priors on two distinct levels: the level of local neighborhoods and the level of the type network as a whole. I impose a version of the common prior assumption at the level of the type network, requiring that the local beliefs \( F_\theta \) for each type are consistent with one process that generates the type network and the underlying state. Fix a set of types \( \Theta \), and suppose that all types share the same state space \( \Omega \). Nature realizes a state \( \omega \in \Omega \) and a type network \( G \) from a joint distribution \( F \) over states and type networks. Select a node uniformly at random in the realized type network. Conditional on selecting a type \( \theta \) node, let \( F_{|\theta} \) denote the distribution over states \( \omega \) and tuples \( t_{|\theta} \) of neighbors’ types. Throughout this section, we assume that there is some \( F \) such that \( F_\theta = F_{|\theta} \) for all types \( \theta \in \Theta \) (i.e. the local beliefs of each type arise from a common prior over the process that generates type networks).

**Assumption 1.** There is a distribution over states and type networks \( F \) such that for each type \( \theta \in \Theta \), the local beliefs \( F_\theta \) agree with the conditional distribution \( F_{|\theta} \).

Our principal question in this section is: what restriction, if any, does Assumption 1 place on the local beliefs that players can hold?

At the level of local neighborhoods, behaving in accordance with a common prior means that behavior within each type game is consistent with a commonly held set of beliefs. This means we can map each type game \( G_\theta \) to a behaviorally equivalent Bayesian game with a common prior.
Definition 4. The local beliefs \( F_\theta \), for each \( \theta \in \Theta \), are **consistent with a common prior** if:

1. There is some \( N \) such that \( N_\theta = N \) for all \( \theta \in \Theta \)

2. There is a joint probability distribution \( \mu \) on \( \Theta^{N+1} \times \Omega \) such that for each type \( \theta \in \Theta \) and each \( N+1 \)-tuple \( t \) we have

\[
\mu \left( (t, \omega) \mid t_0 = \theta \right) = F_\theta \left( (t_{-0}, \omega) \right),
\]

where \( t_{-0} \) is the tuple \( t \) excluding the first entry.

If local beliefs are consistent with a common prior, then all players have more or less the same local view of the world. We can model behavior at the local level as if everyone starts with a common prior in a standard Bayesian game. Whenever we isolate an economic problem from its broader context and study it as a game, we carve a set of players from a larger network of interactions, and we endow these players with a common local perspective. Implicitly, we assume that our use of a common prior does not affect how the game fits into its context, but in fact, we impose a heavy restriction.

**Assumption 2 (Complete Network).** There is a fixed integer \( N \) such that all type games \( G_\theta \) contain \( N \) players, and local beliefs are derived from a distribution in which nature always selects a complete graph with \( N \) nodes.

**Theorem 3.** If local beliefs satisfy the complete network assumption, then local beliefs are consistent with a common prior. If local beliefs are consistent with a common prior, then these beliefs can be obtained under the complete network assumption.

**Proof.** The first claim is immediate: nature’s distribution under the complete network assumption is the common prior \( \mu \) from our definition. If local beliefs are consistent with a common prior, then we can construct a suitable distribution for nature to draw from. Let the type network consist of a complete graph with \( N \) nodes in all realizations, and suppose types and states are generated according to the distribution \( \mu \) from the definition. This trivially generates the desired local beliefs under the complete network assumption.

Beliefs that are consistent with a common prior make sense when the local models actually encompass the full network. If the network is incomplete, or the larger context is important, then local models may exhibit inconsistencies. How different can players’ local beliefs be under Assumption 1? Perhaps surprisingly, in a directed graph Assumption 1 imposes essentially no restriction. Any local beliefs can arise as a natural reflection of the network structure.

A simple example provides intuition for the general result. In a two player coordination game, each player has two possible actions, up or down. There are two possible states \( \omega \in \{0, 1\} \), which determine the payoffs from matching versus not matching. In state 0, the players each earn 1 unit of utility for matching actions and zero otherwise, while in state 1
this is reversed. Suppose players are of two types \( \theta \in \{0, 1\} \) with the following beliefs: a player of type \( \theta \) believes the state is \( \theta \) with probability \( 1 - \frac{1}{N} \) and that the other player is of type \( 1 - \theta \) with certainty. For all \( N > 2 \), these beliefs are inconsistent with a common prior.

Nevertheless, these beliefs are consistent with Assumption 1. Suppose nature first realizes a state \( \omega \in \{0, 1\} \), with each state equally likely, and then realizes a type network with \( N \) nodes, each with a single directed edge. Conditional on a realized state \( \omega \), there are \( N - 1 \) players of type \( \theta = \omega \) and one player of type \( \theta = 1 - \omega \). All players of type \( \omega \) form a directed link to the lone player with type \( 1 - \omega \), while this last player randomly links to one of the others. The conditional beliefs \( F_{|\theta} \) are precisely those in the previous paragraph. Local belief inconsistencies encode information about the structure of the broader network.

**Theorem 4.** Suppose both the state space \( \Omega \) and the type space \( \Theta \) are finite. Given any local beliefs \( \{F_{\theta}\}_{\theta \in \Theta} \) and any \( \varepsilon > 0 \), there is a joint distribution \( F \) over \( \Omega \) and finite type networks such that the total variation distance between \( F_{|\theta} \) and \( F_{\theta} \) is less than \( \varepsilon \) for every \( \theta \in \Theta \).

**Proof.** A player of type \( \theta \) believes the state is \( \omega \) with probability \( p_{\theta, \omega} \) and that her neighbors have the vector of types \( t \) with probability \( q_{\theta, t} \). Given nature’s distribution \( F \) over states and type networks, we use \( \hat{p}_{\theta, \omega} \) and \( \hat{q}_{\theta, t} \) to denote the corresponding probabilities under the conditional distribution \( F_{|\theta} \). We will construct for each \( n \in \mathbb{N} \) a distribution \( F \) over \( \Omega \) and finite type networks such that \( |p_{\theta, \omega} - \hat{p}_{\theta, \omega}| \) and \( |q_{\theta, t} - \hat{q}_{\theta, t}| \) converge to zero uniformly as \( n \) grows.

Define for each \( \omega \in \Omega \)
\[
p_{\omega} = \frac{1}{|\Theta|} \sum_{\theta \in \Theta} p_{\theta, \omega},
\]
and let nature first realize a state according to the distribution given by \( \{p_{\omega}\}_{\omega \in \Omega} \). Let \( d \) be an upper bound on the number of neighbors any type has, and define \( f^{(n)}_{\theta, \omega} \) by
\[
f^{(n)}_{\theta, \omega} = \begin{cases} 1 & \text{if } \frac{n p_{\theta, \omega}}{p_{\omega}} < \frac{3}{2} \\ k & \text{if } k - \frac{1}{2} \leq \frac{n p_{\theta, \omega}}{p_{\omega}} < k + \frac{1}{2} \end{cases}.
\]
Conditional on a realization \( \omega \), the population contains \( d f^{(n)}_{\theta, \omega} \) players of type \( \theta \). Hence, by Bayes’ rule we have
\[
\hat{p}_{\theta, \omega} = \frac{d f^{(n)}_{\theta, \omega} \cdot p_{\omega}}{\sum_{\omega \in \Omega} d f^{(n)}_{\theta, \omega} \cdot p_{\omega}}.
\]
By construction, we have \( |f^{(n)}_{\theta, \omega} - \frac{n p_{\theta, \omega}}{p_{\omega}}| \leq 1 \), implying
\[
\frac{n d p_{\theta, \omega} - d p_{\omega}}{(n + 1)d} \leq \hat{p}_{\theta, \omega} \leq \frac{n d p_{\theta, \omega} + d p_{\omega}}{(n - 1)d}.
\]
For large \( n \), both the upper and lower bounds converge to \( p_{\theta, \omega} \). To complete the proof, observe that in any realization, there are always at least \( d \) players of any particular type. Hence, it is always possible for a player of any type \( \theta \) to realize any vector of types \( t \) among her neighbors. Once nature has realized a state \( \omega \) and a population as above, we can generate directed edges leading away from each player exactly according to the distribution \( q_{\theta, t} \). \( \square \)
3.4 Remarks

Typically, the people most salient to our everyday decisions are our closest friends and neighbors. We often rely on these connections for information and attempt to coordinate our choices with theirs. The information available to us, and our ability to coordinate, depends on a vast network of social interactions, one we have no hope to fully appreciate in our thought processes. Nevertheless, we can still think carefully about what our neighbors do. Limiting a player’s thinking to her local neighborhood, which we formalize using local Bayesian equilibrium, gives us a realistic way to represent these cognitive limitations.

Many approaches to modeling bounded rationality use some form of coarse reasoning. An important contribution of this literature, from models of categorization (Jehiel, 2005; Fryer and Jackson, 2008) to bounded memory (Wilson, 2002; Salant, 2011), is to explain psychological biases within a rational framework. Local reasoning is another type of coarse reasoning that relies on both categorization and bounded memory. Categorizing players into types allows us to formalize the idea of a local model, and Section 3.1.3 highlights the incompatibility of local models with long memory. Though beyond the scope of my analysis, the success of earlier models of coarse reasoning suggests that local Bayesian equilibrium may help us rationalize certain biases in the context of a network game.

In a world with common priors, the persistence of disagreement after repeated interactions is a mystery. However, whether the common prior assumption makes sense in any particular interaction is sensitive to the larger context. Local Bayesian equilibrium helps to illustrate this point: if we impose a common prior on the type network, we make no restriction on players’ local beliefs. If two players have distinct neighborhoods—choosing different places for the boundary between model and context—conflicting local beliefs are a natural consequence.

One challenge for applications of local Bayesian equilibrium is dealing with a relatively large set of modeling choices. The flexibility of modeling the network more or less independently from individual decisions is a double-edged sword, inviting charges of arbitrariness. The difficulty is compounded in dynamic settings, where we must also make choices about players’ memory and future expectations. Of course, models based on behavioral heuristics are subject to the same criticism, but these approaches rarely claim to represent any sort of rational decision process. Instead, authors motivate these heuristics as a simple description of empirically observed behavior. Ideally, an application of local Bayesian equilibrium would adopt a simple set of empirically motivated assumptions—we could think of players as following a relatively sophisticated heuristic. If done well, we can hope for richer descriptions of behavior in networks that provide a rational basis for simple choice rules. At the very least, local Bayesian equilibrium expands a theorist’s production possibilities frontier by providing an approach in between fully rational decisions and simple rules of thumb.

In many network problems, we cannot reasonably say that a player’s payoff depends only on her neighbors’ actions. Prices, for instance, depend upon the aggregate actions of all individuals who participate in the relevant market. There are many cases in which we might like to combine local reasoning with information on aggregate population statistics. In its current formulation, local Bayesian equilibrium does not permit this, but extending our approach to do so presents a promising avenue for future work.
New technologies often require time to reach everyone who would benefit from them, even if the benefits are clear. Early studies of technology adoption documented the now ubiquitous S-shaped adoption curve in agriculture (Griliches, 1957) as well as other industries (Mansfield, 1961). For a wide range of innovations, from antiulcer drugs (Berndt et al., 2003) to telecommunications services (Gruber and Verboven, 2001), adoption is initially slow, picks up speed, and then levels off. Many of these studies reveal significant spatial and temporal correlations in adoption—adoptions by individuals and firms are typically clustered in space and time—and more recent work demonstrates that peer influence is at least partly behind these correlations (e.g. Tucker, 2008; Banerjee et al., 2013). This suggests that adoption spreads through a diffusion process: early adopters influence friends and neighbors to adopt as well, and over time a new product gets transmitted through the network of connections between individuals and firms.

Technology adoption provides just one example: diffusion processes are pervasive in economic and social networks. Our friends influence what we buy (Aral and Walker, 2011), how we take care of our health (Rao et al., 2014), and how we save for the future (Duflo and Saez, 2003). Products, fashions, behaviors, news, and ideas all travel through social ties and personal interactions. For this reason, many fundamental questions about technology adoption, development, public health, marketing, and social norms boil down to questions about an underlying diffusion process. How fast and to what extent will a behavior spread? How does the nature of interactions affect this? What is the role of the broader network? How can we encourage or discourage wider diffusion?

The patterns we observe in the empirical literature suggest two important facts about social diffusions. First, there are attributes of particular groups and regions, attributes that persist over time, which lead some groups to adopt innovations faster and more completely than others. For instance, Caselli and Coleman (2001) examine the factors behind different levels of adoption for personal computers in different countries, and Skinner and Staiger (2005) find that adoption rates of different technologies across different points in time are positively correlated within the same geographic region. Second, the extent to which any particular product diffuses is at least partly random. Leskovec et al. (2006) track purchases and personal recommendations of products sold by an online retailer, and they find that
even among very similar products, purchase cascades come in all sizes, following a power law distribution. Goel et al. (2012) find similar patterns in the diffusion of news stories and videos through social media. In the present paper, I seek to explain these phenomena as the natural result of varying network structures and local externalities.

The peer influence behind diffusion can come from two different sources. One source of influence is information. Through my friends, I could learn about a new book release, the quality of a new smartphone, or how to more effectively grow tomatoes in my garden. Influence can also come from a variety of complementarities. Some goods, like telephones, are simply more useful when more people have them. Others, like books, music, or movies, can be more enjoyable when they are part of a shared experience. The presence of either makes adoption a strategic choice, but the two mechanisms affect diffusion patterns differently.

Our analysis considers both mechanisms of influence to some extent. The role of information is captured through awareness of a product: players are either aware or unaware of the opportunity to purchase a product, and they can become aware as a result of friends’ purchases. This approach has the benefit of simplicity, but it neglects information that is gained gradually, like knowledge of how to most effectively use a new technology. We will also consider local payoff externalities, which depend on the number of neighbors that adopt. Although not all types of diffusions can fit into this framework, it benefits from simplicity and can still apply in many settings.

The basic setup involves a population of players connected in a random network with random seeds that are initially aware of a new product. If a player adopts the product, her neighbors become aware of it and make adoption decisions in the following period. Payoffs can depend on neighbors’ adoption choices, so a player must form expectations about neighbors’ behavior to reach an optimal decision. I use the concept of local Bayesian equilibrium (Sadler, 2015a) to analyze individual behavior, and I represent the network using a configuration model (e.g. Britton et al., 2006). We focus on questions about whether adoption spreads, how far it spreads, and how quickly it spreads through the network. Specifically, in the limit as the population becomes large, we look at the portion of the network that eventually adopts. Our analysis relies on relatively recent mathematical results, but there is a surprisingly clear intuition based on a branching process approximation. In the configuration model, the network rooted on any particular node looks locally like a standard branching process with independent offspring. The degree distribution of the network, together with an equilibrium strategy profile, allows us to specify an approximating branching process, which we can use to describe diffusion outcomes in large networks.

Since diffusion outcomes follow a branching process, outcomes are inherently stochastic, but the distribution of outcomes depends crucially on properties of the network structure. There is a discontinuity in outcomes depending on whether the approximating branching process is supercritical or subcritical. The subcritical case encompasses most of the patterns we typically observe, in which cascades of adoption are predominantly small and shallow. Adoption can never spread from a single seed to a significant portion of the network, but if a positive fraction of the population is aware at the start, we can derive a closed form expression for the fraction of the population that adopts in the large network limit. The
supercritical case captures the rare events in which a product “goes viral.” Even in this case, a large cascade is not certain. Adoption can spread from one seed to a large fraction of the network, or fizzle out quickly, depending on who the initial seed is.

The criticality threshold depends on the first two moments of the network degree distribution. Higher variance in the degree distribution encourages more diffusion, highlighting the importance of central hubs. Surprisingly, higher density in the degree distribution, holding variance fixed, may lead to either more or less diffusion. The same parameter that determines this threshold also characterizes the rate of diffusion if we have a large cascade. When we have a large cascade from a single seed, the vast majority of adoption takes place within a narrow time window that scales logarithmically with the population size. The exponent of this logarithm increases with the variance of the degree distribution, indicating faster spread.

The real power of this framework is the ease with which we can extend the basic model. A technical contribution is to generalize the key branching process approximation result to a multitype configuration model. This provides a way to capture more realistic features of networks—like homophily and clustering—and study the role of information. I present several examples to illustrate the kinds of insights we can obtain. Moreover, since the branching process approximation allows us to capture key properties of a diffusion process with relatively few parameters, it seems likely that this framework could adapt to inform econometric analysis.

Finally, I suggest one approach to model players who strategically wait to adopt. This significantly complicates individual decisions because there is option value to waiting, and adoption by some neighbors may provide information on the likely behavior of others. Although general claims are difficult to make in this extension, I present an example illustrating how informational effects can lead to equilibrium strategies that are non-monotonic in the number of adopting neighbors. If the network is generated through a basic configuration model, diffusion patterns depend on players who are willing to adopt after just one neighbor has adopted. This suggests a different approach to encouraging diffusion than in the model without waiting and that clustering may contribute positively to diffusion in this context.

Related Literature

Much of the theoretical economics literature on diffusion focuses on reversible decisions. Choosing between two behaviors or two technologies, individuals can switch from one to the other and back again at any time; analysis of these models centers on steady state adoption and stability. López-Pintado (2006, 2008, 2012), Jackson and Yariv (2007), Jackson and Rogers (2007), and Jackson and López-Pintado (2013) all use this approach to study how networks affect diffusion, and other authors have applied these insights to better understand how to encourage or discourage diffusion (Galeotti and Goyal, 2009; Galeotti and Rogers, 2012).¹ Many of these models also employ a mean-field representation of the network, which benefits from tractability but typically leads to deterministic dynamics. By retaining a discrete network structure, we obtain stochastic outcomes that better reflect the cascade

¹Not all diffusion models focus on the role of network structure. For instance, Young (2009) explores how different diffusion mechanisms influence the shape of the adoption curve.
patterns we observe in the empirical literature (e.g. Leskovec et al., 2006; Goel et al., 2012).

I make two important departures from much of this literature. I study irreversible adoption decisions: as soon as a player adopts, she adopts for all time. This is a natural assumption in some cases, like the spread of information—once I share a piece of information with a friend, I cannot subsequently render him ignorant of it—or when adoption entails a significant upfront investment and low costs for continuing use. However, even outside of these cases, irreversible adoption can reasonably approximate short-run adoption dynamics. Switching is rarely costless, and the short-run is often economically meaningful. A second departure involves the way players make decisions. To simplify the analysis, most diffusion models employ some version of myopic best response. Through the concept of local Bayesian equilibrium, we can adapt results on equilibria in static network games (Sundararajan, 2007; Galeotti et al., 2010) to apply in a dynamic context. This gives us a richer description of individual behavior, which allows us to look more closely at information and strategic effects.

Watts (2002) and Campbell (2013) are among the closest precursors to this particular diffusion model. Watts (2002) studies cascades in a random network generated according to a configuration model. Assuming that each node will adopt after some exogenous fraction of neighbors does so, the resulting cascade patterns depend crucially on nodes that adopt after a single neighbor does. Campbell (2013) applies results on percolation in the configuration model to study optimal pricing when individuals learn about a good through world-of-mouth. Beyond its richer representation of individual behavior, the present paper offers an alternative approach to studying properties of the configuration model. Using a branching process approximation, we can completely characterize the component structure of the network, or any subnetwork. This provides a powerful and intuitive set of tools to study the extent and rate of diffusion in more detail.

4.1 Diffusion in a Random Network

Before starting with a formal model, think of an everyday example of diffusion. Say a person notices a news story one of her friends posts on social media. She reads the story, finds it interesting, and decides to repost it herself. Her other friends subsequently click on the link, repeat the pattern, and lead more people to read the article. Despite the simplicity of this example, there are already myriad complications to consider in any analysis.

There are at least two types of externalities that are important in sharing news stories. One is an attention externality—I might not become aware of a particular story unless a friend posts it first—and another results from consumption complementarities—I may enjoy reading a story more if I can discuss it with friends later. There are also at least three decisions to make—whether to pay attention to a friend’s social media posts, whether to read this particular article, and whether to repost it. A person might also actively choose when to take each of these actions.

I present a model meant to capture the essential features of this and related diffusion processes. I include both attention and payoff externalities, but I necessarily simplify the decisions players make. Players make binary adoption decisions that automatically grab
the attention of their neighbors—I become aware of a product if a friend adopts it, and my friends all become aware if I adopt it. I also assume in this section that players make their adoption decisions immediately after they become aware of the opportunity, without observing whether and how many friends have already adopted. This seems a decent approximation for decisions we make quickly, without counting how many of our friends have adopted, or for cases in which observing friends’ choices is difficult. Sharing news stories, word-of-mouth spread of certain consumer products, daily deals on social media, and even the spread of some fashions can reasonably fit this representation.

The game we study comprises a population of $N$ players who interact in a random undirected network. During the game, a player may have the opportunity to adopt some product or behavior. Most are initially unaware of the opportunity, but a player becomes aware if a neighbor adopts. One random player is initially aware of the opportunity to adopt. Time is discrete, and players choose whether to adopt the period after they become aware. A player’s payoff at the end depends on whether she adopts and may depend on how many of her neighbors adopt.

We describe a player using her valuation type $v \in [0,1]$ and her number of neighbors $d \in \mathbb{N}$. We refer to $d$ as the player’s degree. On becoming aware of the opportunity to adopt, a player chooses an action $x \in \{0,1\}$. Action 1 means the player buys the product or adopts the behavior. The pair $(v,d)$ is all of the information available to a player, and strategies will be functions $\sigma(v,d)$ mapping pairs $(v,d)$ to actions in $\{0,1\}$. Players have incomplete information about their neighbors and about the network, so we must describe beliefs to guide their decisions. A player assumes that others’ valuation types are drawn independently according to the distribution $V$. The degree distribution $D$ similarly describes her beliefs about the network: she assumes that other players have degrees drawn independently according to $D$. I assume throughout that $D$ has finite mean and variance, and that $\mathbb{P}(D \geq 3) > 0$, though not all results depend on finite variance. We use $V$ and $D$ interchangeably to denote the distribution functions and random variables distributed according to $V$ and $D$ respectively. Since the network is undirected, a random neighbor is likely to have higher degree than a random player. Players believe the degree of a random neighbor follows the size-biased distribution $D'$ satisfying

$$
\mathbb{P}(D' = d) = \frac{d \mathbb{P}(D = d)}{\mathbb{E}[D]}.
$$

The payoff from adoption is increasing in a player’s valuation type $v$ and non-decreasing in the number of neighbors who adopt $A$. A player of type $(v,d)$ choosing action $x$ earns

$$
u(x,v,d,A) = x [f(v,d,A) - c],
$$

where $f \geq 0$ is increasing in $v$ and non-decreasing in $A$ for each $d$, and $c$ is a fixed positive cost. Non-adoption leads to a default payoff, which we normalize to zero. We assume that for some positive mass of pairs $(v,d)$, adoption is never profitable. Our focus is on equilibria in symmetric strategies, meaning that all players with the same type $(v,d)$, if they become aware of the opportunity to adopt, choose the same action.
The distributions \( V \) and \( D \), together with a symmetric strategy profile \( \sigma(v, d) \), allow a player to form expectations about whether her neighbors will adopt. If she chooses to adopt, our player supposes that each neighbor will independently adopt with probability

\[
P_\sigma = \mathbb{E}[\sigma(V, D')].
\]

If the player has degree \( d \), her number of adopting neighbors \( A \) follows a binomial distribution with \( d \) trials and success probability \( P_\sigma \).\(^2\) The profile \( \sigma(v, d) \) is a local Bayesian equilibrium if for each \( (v, d) \) and each \( x \in \{0, 1\} \) we have

\[
\mathbb{E}_\sigma[u(\sigma(v, d), v, d, A)] \geq \mathbb{E}_\sigma[u(x, v, d, A)].
\]

Since adoption causes each neighbor to become aware of the choice opportunity, analysis of individual decisions here mirrors that in many one-shot simultaneous move games (e.g. Sundararajan, 2007; Galeotti et al., 2010), and we shall leverage existing results. The diffusion dynamics, as a function of the equilibrium, will occupy most of our effort.

We represent the broader network using a configuration model. The network is generated stochastically by first realizing a sequence of degrees, each drawn independently according to \( D \), and then selecting a graph uniformly at random from the set of simple undirected graphs with the given degree sequence.\(^3\) Each player is then assigned a valuation type drawn independently according to \( V \). The game begins with a realization of the network and a single player who is aware of the product. The players subsequently follow an equilibrium strategy profile, which causes adoption to diffuse through some part of the network. Time is discrete, with the initial player acting in period one, and others acting in the period immediately after they are prompted by a neighbor’s action. Our analysis of diffusion outcomes will focus on the extent of adoption when \( N \) becomes large, the likelihood that a significant fraction of the population adopts, the length of time required for diffusion, and welfare.

The beliefs ascribed to the players are local in the sense that we specify beliefs—degree and valuation distributions—about neighbors without reference to the larger network. Importantly, local beliefs are sufficient to define a notion of equilibrium. In this case, the local Bayesian equilibrium essentially coincides with a rational expectations equilibrium in which players model the entire network. We may find slight deviations between these local beliefs and the actual distribution of player degrees, but the law of large numbers guarantees these differences will vanish in a large network. In more complex situations, including some we consider later, we cannot maintain this precise correspondence, but the notion of a local equilibrium still gives us a framework to tractably analyze strategic behavior.

\(^2\)An astute reader might object that an aware player knows for sure that at least one neighbor has adopted. I make this assumption to simplify the representation of players’ decisions, and in particular so that I do not need to distinguish the strategies of seeded players from those of unseeded players. If we made the distinction, unseeded players would have \( A \) as one plus a binomial distribution with \( d - 1 \) trials, but this would not affect the qualitative results.

\(^3\)We must condition on a realized degree sequence such that this set is non-empty. The actual degree distribution among the players for finite \( N \) will not perfectly agree with \( D \), but it will converge to \( D \) as \( N \) grows.
This simple representation of the network facilitates a transparent exposition, but it also permits several extensions that can more accurately depict real situations. We implicitly assume that individual beliefs about the degree distribution coincide with the actual distribution of degrees in the network, but we can just as easily analyze equilibrium behavior resulting from subjective beliefs about neighbors’ degrees or valuations. Adjusting the awareness externality—adding multiple seeds, inducing transmission with only some probability—is surprisingly simple. I discuss these extensions at the end of this section. I address the role of homophily, clustering, information, and waiting to adopt in later sections.

### 4.1.1 Individual Decisions

Clearly, a player’s best reply is non-decreasing in $P_\sigma$, and $P_\sigma$ in turn is increasing in the strategy profile $\sigma$. Hence, we can apply Tarksi’s fixed point theorem to establish the existence of minimal and maximal equilibria. Equilibrium strategies are threshold strategies: a player with degree $d$ adopts only if her valuation type is above the threshold $v_d(P_\sigma)$. The neighbor adoption probability $P_\sigma$ gives us a complete Pareto ordering on the set of equilibria. Since payoffs are non-decreasing in the number of adopting neighbors, they are non-decreasing in $P_\sigma$.

**Proposition 1.** A local Bayesian equilibrium exists, and the set of equilibria is Pareto ordered by neighbor adoption probabilities.

Given the formal equivalence to a one shot game, we can apply the results of Galeotti et al. (2010) to derive comparative statics. Suppose $D$ and $D^*$ are two degree distributions. If for any equilibrium $\sigma^*$ for the distribution $D^*$ there exists an equilibrium $\sigma \geq \sigma^*$ for the distribution $D$, we say that adoption is higher under $D$. If the payoffs in equation (4.2) take the form $f(v,d,A) = g(v,A)$ for some $g$ and all degrees $d$, then higher degree players have lower valuation thresholds $v_d(P_\sigma)$ in any equilibrium. This implies that first order stochastic dominant shifts in the degree distribution must lead to higher adoption.

**Proposition 2.** Suppose $f(v,d,A) = g(v,A)$ for some $g$ and all degrees $d$. Then in any equilibrium $\sigma$ the decision thresholds $v_d(P_\sigma)$ are non-increasing in $d$. Moreover, if $D$ and $D^*$ are two degree distributions such that $D$ first order stochastically dominates $D^*$, then adoption is higher under $D$.

**Proof.** This is a direct application of Propositions 2 and 7 of Galeotti et al. (2010).

If there are multiple equilibria, the comparative statics that Proposition 2 offers are not perfectly satisfying: the lowest equilibrium in a dense network might be lower than the highest equilibrium in a sparse one. One way to address this is through equilibrium selection. For instance, Sundararajan (2007) studies a closely related one shot game and provides a selection argument based on coalition proofness in favor of the maximal equilibrium. However, if we take seriously the idea that individuals do not interact with those more distant in the network and have limited capacity to reason about the broader population, then coalition proofness is a problematic criterion for selection.
Alternatively, we may look for conditions that guarantee a unique equilibrium. Suppose that \( V \) is a continuous distribution. Given any strategy profile \( \sigma \) inducing neighbor adoption probability \( P_\sigma \), we can define the best reply neighbor adoption probability

\[
\phi(P_\sigma) = \sum_{d \in \mathbb{N}} \mathbb{P}(D' = d)\mathbb{P}(V > v_d(P_\sigma)).
\]

(4.3)

A simple condition guaranteeing a unique equilibrium is that the function \( \phi(P) - P \) has a single zero. If we have \( \phi'(P) < 1 \) almost everywhere, the condition is satisfied, and equilibrium is unique. Intuitively, if strategic complementarities are not too strong, so that best replies change little in response to an increase in \( P_\sigma \), then we have a unique equilibrium.

An important case violating the conditions of Proposition 2 occurs if payoffs depend not on the number of adopting neighbors but on the proportion. That is, the payoffs from adoption follow \( f(v, d, A) = g(v, A/d) \) for some \( g \) and all \( d \). This type of payoff structure is likely to arise in situations where individuals are deciding whether to switch to a new technology from an existing one when both feature payoff externalities. The new technology may be attractive only if a majority of one’s neighbors also switch.

A simple example shows that equilibrium strategies need not increase in degree. Suppose that a random neighbor is equally likely to have degree one or degree two, that the cost of adoption is \( c = 1 \), and that adoption pays off 3 if all neighbors adopt and 0 otherwise for all valuation types. In one equilibrium strategy profile, all degree 1 players adopt and all degree 2 players refrain. We can directly compute the expected payoff to adoption for each: degree 1 players earn \( 3 \cdot \frac{1}{2} - 1 = \frac{1}{2} \) in expectation, while degree 2 players would earn \( 3 \cdot \frac{1}{4} - 1 = -\frac{1}{4} \).

The existence of non-monotone equilibria depends on having strong complementarities that produce increasing returns. If the payoff function \( g(v, A/d) \) is concave, then all equilibria are non-decreasing in degree.

**Proposition 3.** Suppose \( f(v, d, A) = g(v, A/d) \) for all \( d \) and some weakly concave \( g \). Then in any equilibrium \( \sigma \) the decision thresholds \( v_d(P_\sigma) \) are non-increasing in \( d \). Moreover, if \( D \) and \( D^* \) are two degree distributions such that \( D \) first order stochastically dominates \( D^* \), then adoption is higher under \( D \).

**Proof.** Since all players face the same neighbor adoption probability \( P_\sigma \), the expected fraction of neighbors who will adopt is the same for any degree. For higher degree players, the distribution of the fraction of adopting neighbors has lower variance. Jensen’s inequality then implies that higher degree players derive a higher expected payoff from adoption. The second claim follows immediately.

With a firm grasp on individual decisions, we now turn to the aggregate consequences of individual behavior.

**4.1.2 The Extent of Diffusion**

Understanding the diffusion process is equivalent to understanding the relationship between three distinct networks. The *social network*, together with an equilibrium strategy
profile, determines the network of potential adopters. This is the network of players who would choose action 1 if given the opportunity. The network of potential adopters, together with the initial seed, determines the network of actual adopters.

The network of potential adopters follows a configuration model with a degree distribution that is sparser than that in the social network as a whole. Not all players would adopt if given the chance, and among those who would, not all of their neighbors would adopt if they did. A player is in the network of potential adopters with probability

$$\rho_\sigma = \mathbb{E}[\sigma(V, D)].$$

If \(v_d\) is the lowest valuation type with \(d\) neighbors that adopts in equilibrium, then a random potential adopter has the degree distribution \(D_\sigma\) with

$$\mathbb{P}(D_\sigma = d) = \frac{\mathbb{P}(D = d) \mathbb{P}(V > v_d)}{\sum_{k \in \mathbb{N}} \mathbb{P}(D = k) \mathbb{P}(V > v_k)}.$$

A random neighbor who is a potential adopter has the size-biased degree distribution \(D'_\sigma\) defined analogously to \(D'\) in equation (4.1). We also need to define the forward degree distribution \(D''_\sigma\) with

$$\mathbb{P}(D''_\sigma = d) = \mathbb{P}(D'_\sigma = d + 1).$$

This is the number of new links that a random neighbor has.

There is one more step to obtain the degree distribution in the network of potential adopters. Each neighbor of a potential adopter is a potential adopter with probability \(P_\sigma\). Conditional on a degree \(d\) in the social network, a player’s degree in the network of potential adopters follows a binomial distribution with \(d\) trials and success probability \(P_\sigma\). Let \(D_p\) denote a distribution in which we draw a degree according to \(D\), but we only keep each link with independent probability \(p\). In the network of potential adopters, the degree distribution is \(D_{\sigma, P_\sigma}\), and a random neighbor has additional links distributed according to \(D''_{\sigma, P_\sigma}\). As an exercise, check that mapping to the forward distribution and retaining links with independent probability \(p\) commute; that is, \((D_p)^{\prime\prime} \overset{D}{\rightarrow} (D'')_p\).

Questions about the network of actual adopters are questions about the connected components in the network of potential adopters. The extent of adoption is precisely the size of the connected component that contains the initial seed. Is this component negligible when compared with the size of the network at large? If not, how big is it?

More formally, given a degree distribution \(D\) and an equilibrium strategy profile \(\sigma\), what proportion of the population chooses action 1? We look at the limit in a large network to answer this question. Let \(X_N\) denote the (random) final number of adopters in a network with \(N\) players. Now define

$$\alpha = \lim_{N \to \infty} \frac{X_N}{N},$$

4 A convenient feature of the configuration model, due to the independence in the linking process, is that any selection of a subgraph that we make, which is independent from the matching of edges, results in another configuration model. When I talk about the degree distribution for a particular subnetwork as a function of the degree distribution \(D\), I am always referring to the asymptotic distribution as \(N\) becomes large. In any finite network there will be small differences, which are inconsequential in large network limits.
where convergence is in distribution. The random variable $\alpha$ measures the extent of diffusion from a single seed in a large network. If $\alpha$ is zero, then diffusion halts abruptly and never gets beyond a small portion of the network. We can think of this outcome as a failed product, a video on youtube that does not go viral, or another behavior adopted in only some niche subcommunity. If $\alpha$ takes a positive value, then adoption spreads to a non-trivial fraction of the population via person to person transmission.

Our analysis of $\alpha$ is based on a branching process approximation. If the network is large, then conditional on seeding a potential adopter, the early stages of diffusion approximately follow a two stage branching process $T_\sigma$ in which the root has offspring distributed according to $D_{\sigma,P_\sigma}$, and all subsequent nodes have independent offspring distributed according to $D''_{\sigma,P_\sigma}$. Amazingly, this branching process allows us to completely characterize the component structure in the network of potential adopters.

**Theorem 5.** Suppose $\mathbb{P}(D_{\sigma,P_\sigma} \geq 3) > 0$. Then

$$\lim_{N \to \infty} \mathbb{P}(X_N = k) = \mathbb{P}(|T_\sigma| = k) \text{ for each } k \in \mathbb{N}, \text{ and}$$

$$\alpha = \begin{cases} \mathbb{P}(|T_\sigma| = \infty) & \text{with probability } \mathbb{P}(|T_\sigma| = \infty) \\ 0 & \text{with probability } \mathbb{P}(|T_\sigma| < \infty). \end{cases}$$

**Proof.** This is a special case of Theorem 7 proved later.

Intuitively, having $\alpha = 0$ corresponds to the event that the branching process $T_\sigma$ goes extinct. We can apply standard results to compute the probability of this event using generating functions. The probability generating function for $D_{\sigma,P_\sigma}$ is

$$g_\sigma(x) = \mathbb{E}[x^{D_{\sigma,P_\sigma}}] = \sum_{k=0}^{\infty} \mathbb{P}(D_{\sigma,P_\sigma} = k)x^k.$$ 

For the forward distribution $D''_{\sigma,P_\sigma}$, the probability generating function is $g'_\sigma(x)$. If $T_\sigma$ has a positive survival probability, there is a unique solution $\xi_\sigma \in (0, 1)$ satisfying $g'_\sigma(\xi) = \mathbb{E}[D_{\sigma,P_\sigma}]\xi$. Our two stage branching process $T_\sigma$ goes extinct with probability $g_\sigma(\xi_\sigma)$. If the branching process survives, our initial seed is contained in a large component of the network. As long as some players have 3 or more neighbors with positive probability, there can only ever be one large component, and its size converges in the limit to a portion $1 - g_\sigma(\xi_\sigma)$ of potential adopters.

Remarkably, this means that $\alpha$ can only ever take one positive value: whenever adoption spreads, the extent is the same. Since potential adopters make up a portion $\rho_\sigma$ of the social network, the random variable $\alpha$ takes the value $\rho_\sigma(1 - g_\sigma(\xi_\sigma))$ with probability $\rho_\sigma(1 - g_\sigma(\xi_\sigma))$ and zero otherwise. Since we have assumed some players will never adopt, this means $P_\sigma < 1$, and there is always a positive chance that the behavior quickly stops spreading. Moreover, the giant component will never cover the entire network of potential adopters. As a result, even when adoption spreads far, the adopting component is only a fraction of all potential adopters.
adopters. If the branching process is critical or subcritical, meaning the network is relatively sparse, then diffusion will always halt before adoption spreads beyond a small group.

We can efficiently summarize a great deal of information about the diffusion process using the average forward degree. Define

\[ \nu_\sigma = \mathbb{E}[D_{\sigma}^\nu] = \frac{\text{Var}[D_{\sigma}]}{\mathbb{E}[D_{\sigma}]} + \mathbb{E}[D_{\sigma}] - 1. \]  

(4.5)

Every node besides the root in the branching process \( T_\sigma \) has \( P_\sigma \nu_\sigma \) offspring in expectation. Elementary results on branching processes tell us that \( T_\sigma \) survives with positive probability if and only if \( P_\sigma \nu_\sigma > 1 \).

**Corollary 1.** We have \( \alpha > 0 \) with positive probability if and only if \( P_\sigma \nu_\sigma > 1 \).

The parameter \( P_\sigma \nu_\sigma \) offers a concise way to think about how the extent of diffusion changes with equilibrium behavior and network structure. An increase in equilibrium strategies can only increase the likelihood and extent of widespread adoption: adding nodes and links to the network of potential adopters can only increase the size of the giant component. However, there are two distinct effects when we make such a shift. First is the direct effect of an increase in \( P_\sigma \): we get more diffusion because this makes the network of potential adopters more dense. There is also an indirect effect as the degree distribution \( D_{\sigma} \) of a potential adopter can shift. The average forward degree \( \nu_\sigma \) of a potential adopter may increase or shrink, which can either amplify or dampen the change in \( P_\sigma \).

This parameter can also inform an analysis of changes in the degree distribution. Suppose we make a FOSD shift in \( D \), increasing the density of the network. Under the conditions of either Proposition 2 or Proposition 3, equilibrium adoption should be higher. However, since equilibrium strategies are monotone in degree, the players who were adopting before were the high-degree players. We increase the population of potential adopters, but we likely reduce the average degree of a potential adopter since the new potential adopters are relatively low-degree players. Of course, changes in the degree distribution also impact diffusion independently from changes in equilibrium behavior. In particular, we see that \( \nu_\sigma \) is strictly increasing in the variance of the degree distribution, highlighting that a network that is centralized around a few high-degree hubs facilitates more diffusion. Increasing the density while holding variance fixed has an ambiguous effect. Intuitively, what happens is that we become less likely to reach one of the highest degree nodes because the low-degree nodes are more connected amongst themselves.

The precise impact of an increase in equilibrium behavior will depend upon fine details of the degree distribution \( D \), the degrees of potential adopters under the strategy profile \( \sigma \), and the degrees of the new potential adopters. We can, however, make a general claim about the relative impact of adding players with different degrees to the network of potential adopters.

**Proposition 4.** Suppose \( \sigma \) and \( \sigma^* \) are two strategy profiles such that \( \rho_\sigma = \rho_{\sigma^*} \). If \( D_\sigma \) FOSD \( D_{\sigma^*} \), then

\[ \rho_\sigma (1 - g_\sigma(\xi_\sigma)) \geq \rho_{\sigma^*} (1 - g_{\sigma^*}(\xi_{\sigma^*})) \]
Proof. While the population of potential adopters is the same under the two strategy profiles, under $\sigma$ this population consists of higher degree players. Since higher degree players are disproportionately represented as neighbors, we necessarily have $P_\sigma \geq P_{\sigma^*}$. It follows that $D_{\sigma,P_\sigma}$ FOSD $D_{\sigma^*,P_{\sigma^*}}$, and the conclusion follows immediately from the definition of the generating function.

Having higher degree players in the network of potential adopters increases the likelihood and extent of diffusion. A shift in the equilibrium will have a larger impact on the aggregate outcome if the change is concentrated in the behavior of the highest degree players.

Our analysis so far suggests at least two immediate extensions. We can use the same techniques to model diffusion if a player’s adoption decision prompts neighbors to become aware of the game only with some probability. This would simply result in a sparser network of potential adopters. A more interesting extension is to allow multiple initial seeds. If we start with $k$ randomly placed initial seeds, we essentially get $k$ independent chances to start one in the giant component. This means that the likelihood of having a significant cascade increases. However, since the size of the giant component remains the same, the maximum extent of diffusion remains the same. In the supercritical case, our random variable $\alpha$ takes the value zero with probability $[1 - \rho_\sigma (1 - g_\sigma(\xi_\sigma))]^k$ and the value $\rho_\sigma (1 - g_\sigma(\xi_\sigma))$ otherwise.

Often, the patterns of diffusion we see are not large cascades emanating from a single seed. More common are much smaller trees that quickly stop, but these can still account for a significant portion of total adoption if there are many seeds. If $P_\sigma \nu_\sigma < 1$, so the network of potential adopters is too sparse for supercritical diffusion to occur, we can still ask how much diffusion we get if we seed a positive fraction of the network. Suppose that $P_\sigma \nu_\sigma < 1$, and each node in the network is initially aware of the product with independent probability $q$. What portion of the network $\alpha$ will eventually adopt?

Theorem 5 gives us the tools we need to answer this question. Let $\tau_k = \mathbb{P}(|T_\sigma| = k)$ denote the probability that the total size of the tree $T_\sigma$ is $k$. Since the branching process is subcritical, it goes extinct with probability one, and we have $\sum_{k=1}^{\infty} \tau_k = 1$. By Theorem 5 a random player in the network of potential adopters belongs to a connected component of size $k$ with probability $\tau_k$ in the limit. Hence, we can compute the probability that a player is in a connected component with a seeded player; this is precisely the extent of diffusion $\alpha$.

**Corollary 2.** Suppose $P_\sigma \nu_\sigma < 1$ and players are initially aware with independent probability $q$. The limiting fraction of the network that adopts is

$$\alpha = \rho_\sigma \sum_{k=1}^{\infty} \tau_k \left(1 - (1-q)^k\right).$$

Proof. If any player in a connected component is seeded, all players in that component will adopt. For a component of size $k$, the probability that at least one player is seeded is $1 - (1-q)^k$. Asymptotically, a random player is a potential adopter with probability $\rho_\sigma$, and conditional on being a potential adopter is in a connected component of size $k$ with probability $\rho_k$. The result follows.\qed
Corollary 3. Let $\bar{k} = \sum_{k=1}^{\infty} \tau_k k$ denote the expected size of the connected component containing a random potential adopter. We have
\[ \alpha \leq \rho_\sigma \left( 1 - (1 - q)^{\bar{k}} \right). \]

Proof. This is immediate from the previous Corollary and Jensen’s inequality. □

Even without a large cascade, diffusion can contribute significantly to the adoption of a new product or technology. These results provide a basis to compare the amount of diffusion we get in different equilibria or when seeding different portions of a network. We can also apply this finding to a range of design problems. The extent of adoption is increasing and concave in the portion we seed, so if a planner incurs convex costs to seed awareness, there is a unique optimum, and deriving comparative statics is straightforward.

### 4.1.3 The Rate of Diffusion

In many instances, the full extent of diffusion is reached only quite slowly, so the rate and timing of diffusion may be more important to aggregate welfare than the long run extent. How long does it take for most adopters to become aware of the opportunity? How long until diffusion is complete? In the supercritical case, conditional on $\alpha > 0$, these are questions about averages distances and the diameter of the giant component in the network of potential adopters. In the configuration model, the diameter of the giant component grows proportionally to $\ln \sigma$, and the distance between two random nodes typically falls in a narrow range. For our diffusion model, this means that the time it takes for a behavior to diffuse is proportional to the logarithm of the population size, and the transition from having only a tiny fraction of the population adopting to having a proportion $\alpha - \epsilon$ adopting occurs very rapidly.

To make this precise, let $X_N(t)$ denote the number of adopters as of time $t$, and define
\[ \mu = \sum_{k=1}^{\infty} k \xi_k^{\bar{k} - 1} \mathbb{P}(D_{\sigma, \nu_\sigma} = k). \]

We are interested in the time it takes to complete the diffusion process, namely the random variable
\[ \tau = \min \{ t : X_N(t) = X_N \}, \]
as well as the evolution of the ratio $\gamma_N(t) = \frac{X_N(t)}{X_N}$ over time.

Theorem 6. Conditional on $\alpha > 0$, the ratio $\frac{\tau}{\ln \sigma}$ converges in probability to
\[ \frac{1}{\ln \nu_\sigma} + \frac{2}{|\ln \mu|}. \]

Moreover, for any $\epsilon > 0$ we have
\[ \lim_{N \to \infty} \mathbb{P} \left( \gamma_N(\log \nu_\sigma + \epsilon, N) > \epsilon \right) = 0, \]
and
\[
\lim_{N \to \infty} \mathbb{P} \left( \gamma_N \left( \log_{P_\sigma \nu_\sigma} - \epsilon \right) N < 1 - \epsilon \right) = 0.
\]

**Proof.** The first claim follows from Theorem 5.1 in Fernholz and Ramachandran (2007). Asymptotically, the diameter of the configuration model converges to 
\[
\frac{1}{\ln P_\sigma \nu_\sigma} + \frac{2}{|\ln \mu|},
\]
implying the result. The second claim follows from Theorem 1.1 in van der Hofstad et al. (2005), which shows that the distribution of distances between random nodes in the giant component becomes concentrated around \(\log_{P_\sigma \nu_\sigma} N\). The second of these two theorems in fact implies a stronger result, namely that the expected number of periods it takes for \(\gamma_N(t)\) to grow from \(\epsilon\) to \(1 - \epsilon\) is uniformly bounded.

To gain some intuition for the result, recall that \(P_\sigma \nu_\sigma\) is the expected number of offspring for a typical node. If \(Z_t\) is the number of individuals in the \(t\)th generation of \(T_\sigma\), then the martingale convergence theorem implies that \(\frac{Z_t}{(P_\sigma \nu_\sigma)^t}\) converges almost surely. Since \(X_N(t)\) behaves like \(Z_t\), at least until adoption has spread to some positive fraction of the network, the number of adopting individuals grows roughly like \((P_\sigma \nu_\sigma)^t\).

The first part of Theorem 6 uses results on the diameter of the configuration model to establish a firm upper bound on the time it takes for diffusion to run its course. In a large network, the distance between any two members of the giant component is not much more than \(\log_{P_\sigma \nu_\sigma} N\). The second part is more striking: essentially all of the diffusion takes place in narrow time frame concentrated around \(t = \log_{P_\sigma \nu_\sigma} N\). This reflects that the distribution of distances between two nodes in the giant component is concentrated around this value. From one initial seed it takes roughly time \(\log_{P_\sigma \nu_\sigma} N\) for a non-trivial fraction of the population to adopt. After that threshold is reached, it takes only finitely many time steps for a large fraction to subsequently adopt. On a logarithmic time scale, the length of time it takes to go from a fraction \(\epsilon\) to a fraction \(1 - \epsilon\) of adoptions vanishes.

We can derive comparative statics on diffusion rates based on the same parameter that determines the critical threshold for a large cascade. The larger \(P_\sigma \nu_\sigma\), the faster adoption will diffuse. The dependence of \(\nu_\sigma\) on degree variance again highlights the important role of high-degree hubs in spreading adoption. Analogous to Proposition 4, we find a monotone relationship between the degree of players who adopt in equilibrium and the rate of diffusion.

**Proposition 5.** Suppose \(\sigma\) and \(\sigma^*\) are two strategy profiles such that \(\rho_\sigma = \rho_{\sigma^*}\). If \(D_\sigma\) FOSD \(D_{\sigma^*}\), then \(\nu_\sigma \geq \nu_{\sigma^*}\), so diffusion is faster under \(\sigma\).

**Proof.** As before, we must have \(D_{\sigma, P_\sigma}\) FOSD \(D_{\sigma^*, P_{\sigma^*}}\), which implies \(P_\sigma \nu_\sigma \geq P_{\sigma^*} \nu_{\sigma^*}\). 

Although I have focused on the more interesting supercritical case, standard branching process results can also tell us about the time it takes diffusion to run its course in the subcritical case. I will spare the details, except to note that the distribution of \(\tau\) converges to the extinction time distribution for the branching process \(T_\sigma\). If \(P_\sigma \nu_\sigma < 1\), the probability that \(\tau > t\) converges to a constant times \((P_\sigma \nu_\sigma)^t\).
4.1.4 Summary

The results of this section show that network structure can help explain the diffusion patterns we observe empirically. The predominantly small cascades observed by Goel et al. (2012) are consistent with subcritical diffusion, while the occasional viral adoption corresponds to the supercritical case. In both the subcritical and the supercritical case, the aggregate outcome of diffusion is stochastic, and similar products may follow drastically different patterns even when the underlying network is the same. The parameter $P_\sigma \nu_\sigma$ also suggests a structural explanation for the importance of highly central individuals. Market mavens and opinion leaders derive at least some of their influence from network position rather than from having better information.

Our results also suggest a number of additional patterns, particularly in the case of viral adoption. When the number of seeds is small relative to the size of the network, the long-run level of adoption is essentially independent of the number of initial seeds. Whether there is one seed or a dozen, the long-run level of adoption, if it is positive, will equal the size of the giant component in the network of potential adopters. If a giant component exists, additional seeds increase the likelihood that adoption will spread, but not the extent. If there is no giant component, we must seed a positive fraction of the network to get a non-trivial level of adoption.

The network may amplify or dampen changes in individual behavior, depending on which individuals are becoming more or less likely to adopt. Both the extent and the rate of diffusion respond more when higher degree players change their behavior. Increasing connectivity in the network unambiguously increases the extent and rate of adoption. Under mild conditions, upward shifts in the degree distribution produce upward shifts in equilibrium behavior, amplifying this change. Finally, we find that the vast majority of diffusion occurs during a relatively short window of time. When significant adoption originates from a small set of seeds, much time may pass before adoption spreads to a significant fraction of the population, but after that point adoption rapidly takes off.

We are able to summarize a surprising amount of information about diffusion patterns through the parameter $P_\sigma \nu_\sigma$ determined in equilibrium. This should facilitate actual estimation of models based on this framework. From data on adoptions and network structure, one could estimate the first two moments of the degree distribution and adoption probabilities to predict diffusion patterns. Alternatively, from data on diffusion patterns one could estimate $P_\sigma \nu_\sigma$. Information on either adoption probabilities or the degree distribution could recover the other parameter. Such an approach may run into trouble if the actual structure of a network is not well-approximated by the configuration model, but as I hope to illustrate in the next section, we can extend the model to incorporate other network features while retaining the central insights of the branching process approximation.
4.2 Homophily, Clustering, and Information

One of the most ubiquitous features of social networks is that people are disproportionately linked to those who are like themselves. Sociologists call this phenomenon homophily, and many studies document homophily along numerous dimensions, including race, religion, political beliefs, and other interests (McPherson et al., 2001). Real networks also tend to exhibit a significant amount of clustering; friends typically have friends in common. An important limitation of the analysis thus far is its reliance on independent degrees and valuations. How might homophily and clustering affect diffusion?

A related omission is the presumed anonymity of neighbors. In the previous sections, a player had essentially no information about any of her neighbors. Not only is this unrealistic, we typically have some knowledge of our friends’ preferences, but it significantly hampers the application of insights from diffusion models. If we are trying to encourage the spread of a product or behavior, the individuals in a social network may have useful local information that we can leverage. Understanding how people’s knowledge about one another impacts their behavior and aggregate adoption is the first step towards designing more effective interventions.

Addressing these limitations requires a more general representation of the network connecting individuals, one that allows multiple types of nodes, each with a different distribution of neighbors. I introduce a multitype configuration model and characterize the distribution of component sizes based on a multitype branching process approximation. This will allow us to derive conclusions about the extent of diffusion analogous to those in the previous section.

As in the standard configuration model, we describe a random process for generating a network connecting $N$ nodes. Let $\{1, 2, ..., \Theta\}$ denote a finite set of types and let $p_\theta$ denote the probability that a random node has type $\theta$. A node $i$ of type $\theta$ realizes a tuple of links $d = (d_1, d_2, ..., d_\Theta)$ according to the distribution $D_\theta$. A particular tuple indicates that the node has $d_1$ neighbors of type 1, $d_2$ neighbors of type 2, etc. For each node in our network, we first realize a type according to $p = \{p_\theta\}_{\theta=1}^\Theta$, and then we realize sets of link stubs according to $D = \{D_\theta\}_{\theta=1}^\Theta$. Each stub is labeled with the type of node from which it originates and the type of node to which it can link. Finally, we pair compatible links uniformly at random. The multitype configuration model is the distribution of networks generated by this process, conditional on a realization that leaves no unmatched links and produces a simple graph.

Let $e_i$ denote a tuple of zeros with a one in the $i$th coordinate. I require that for any two types $\theta$ and $\theta'$ that $p_\theta E[D_\theta \cdot e_{\theta'}] = p_{\theta'} E[D_{\theta'} \cdot e_\theta]$, so the number of stubs linking type $\theta$ nodes to type $\theta'$ nodes matches the number linking type $\theta'$ nodes to type $\theta$ nodes in expectation. For simplicity, I also impose an irreducibility condition: for any two types $\theta$ and $\theta'$, we can realize a path connecting them with positive probability.

The branching process $T$ that approximates the network rooted at a random node is again realized in two stages. The type of the root is first realized according to $p$, and it has offspring generated according to $D$. For all subsequent generations, we characterize a node using its type and the parent’s type. For a child of type $\theta$ with a parent of type $\theta'$, the degree
tiple follows the size-biased distribution \( D'_{\theta, \theta'} \) defined by
\[
\mathbb{P}(D'_{\theta, \theta'} = d) = \frac{d\theta'}{\mathbb{E}[D_\theta \cdot \epsilon_{\theta'}]}. 
\]

Hence, the offspring distribution of the child is \( D''_{\theta, \theta'} = D'_{\theta, \theta'} - \epsilon_{\theta'} \).

The branching process \( T \) completely captures the asymptotic component size distribution for our multitype configuration model. If \( G \) is the realized network, let \( N_k(G) \) denote the number of nodes in a component of size \( k \), and let \( L_1(G) \) and \( L_2(G) \) denote the sizes of the largest and second largest components respectively. Our main result is the following.

**Theorem 7.** In the multitype configuration model, suppose that at least one type \( \theta \) has three or more neighbors with positive probability. As the network size \( N \) goes to infinity we have

\[
\frac{N_k(G)}{N} \xrightarrow{p} \mathbb{P}(|T| = k),
\]

\[
\frac{L_1(G)}{N} \xrightarrow{p} \mathbb{P}(|T| = \infty), \text{ and}
\]

\[
\frac{L_2(G)}{N} \xrightarrow{p} 0.
\]

**Proof.** See Appendix. \(\)

In the rest of this section, I apply this result to better understand how homophily, clustering, and information impact diffusion.

### 4.2.1 Homophily

Suppose the valuation distribution \( V \) is symmetric around \( \frac{1}{2} \) with no atoms; let \( V^- \) and \( V^+ \) denote the distribution \( V \) conditional on a realization below and above \( \frac{1}{2} \) respectively. Local beliefs about the network are still captured through the degree distribution \( D \), but players with low (high) valuations are more likely to link to others with low (high) valuations. We use \( h \in \left[ \frac{1}{2}, 1 \right] \) to parameterize homophily in the network. If a player’s valuation is below (above) \( \frac{1}{2} \), then she believes each neighbor has valuation drawn from \( V^- (V^+) \) with probability \( h \) and from \( V^+ (V^-) \) with probability \( 1 - h \).

We can represent this homophily in the broader network using a configuration model with two types. I refer to players with valuations less than \( \frac{1}{2} \) as low types, and those with valuations greater than \( \frac{1}{2} \) as high types. For the configuration model with \( N \) players, we first realize a sequence of \( N \) types, low and high, each with equal probability. Low (high) types have valuations drawn according to \( V^- (V^+) \) with probability \( h \) and from \( V^+ (V^-) \) with probability \( 1 - h \).

For each stub attached to a low (high) type, we label it low (high) with independent probability \( h \) and high (low) otherwise. Stubs labeled low (high) must link to players of low (high) type. Consequently, players have own-type degrees distributed according to \( D_h \), and other-type degrees distributed according to \( D_{1-h} \).
A strategy profile is still a mapping from valuation and degree pairs \((v, d)\) to adoption decisions. After a player adopts, a random low-type neighbor will adopt with probability 
\[
P^-\sigma = \mathbb{E} [\sigma(V^-, D')]
\]
and a random high-type neighbor will adopt with probability 
\[
P^+\sigma = \mathbb{E} [\sigma(V^+, D')].
\]
Due to the homophily in the network, a low-type player believes that each of her neighbors will adopt independently with probability 
\[
hP^-\sigma + (1 - h)P^+\sigma,
\]
and a high-type player believes that each neighbor will adopt with probability 
\[
hP^+\sigma + (1 - h)P^-\sigma.
\]
The number of adopting neighbors still follows a binomial distribution, conditional on a player’s degree, but low and high types face different success probabilities. The existence of an equilibrium follows from the same argument as before, and we always have an equilibrium in threshold strategies in which low-type players are less likely to adopt than high-type players.

Proposition 6. In the diffusion model with homophily, a local Bayesian equilibrium exists with 
\[
P^-\sigma \leq P^+\sigma.
\]
In any equilibrium \(\sigma\) with 
\[
P^-\sigma \leq P^+\sigma,
\]
there is a collection of thresholds \(\{v_d\}_{d \in \mathbb{N}}\) such that players of degree \(d\) adopt whenever \(v > v_d\) and do not adopt whenever \(v < v_d\).

Proof. Consider the set of strategy profiles satisfying 
\[
P^-\sigma \leq P^+\sigma.
\]
Since \(h \geq \frac{1}{2}\), this means that high type players face a higher neighbor adoption probability than low-type players. Consequently, the best reply strategy must induce more adoption by high-type players than low-type players, and the best reply will be a threshold strategy as before. The best reply map is clearly monotone, so Tarski’s fixed point theorem immediately implies the result.

When we vary the homophily parameter \(h\), the effect on equilibrium behavior is ambiguous because of two effects that are potentially in conflict. First, there is the direct effect of having more own-type neighbors. For high types, this effect encourages more adoption because it tends to increase the neighbor adoption probability, while for low types the direct effect is the opposite. However, there is also an indirect effect due to the change in other-type behavior. If high types increase their adoption enough, low types will gain more from adoption due to their high-type neighbors and might increase their adoption as well. Similarly, if low types reduce their adoption enough, this could counteract the direct effect for high types.

A simple example illustrates that more homophily can either increase or decrease individual propensities to adopt. Suppose payoffs are linear in the proportion of adopting neighbors, that is
\[
f(v, d, A) = kA/d.
\]
With this payoff function, the thresholds players use will be independent of their degrees: there is a single threshold \(v_\sigma\) such that players with \(v > v_\sigma\) adopt, and players with \(v < v_\sigma\) refrain. Suppose the model parameters are such that in equilibrium \(v_\sigma > \frac{1}{2}\), and suppose
we increase \( h \). No low-type players were adopting before, so there is no indirect effect on high types: the equilibrium threshold will lower, and high-types will be more likely to adopt. Conversely, if \( v_\sigma < \frac{1}{2} \), then raising \( h \) induces no indirect effect on low types, and \( v_\sigma \) increases in equilibrium.

To apply Theorem 7 to characterize the extent of diffusion, we first need to specify the configuration model for the network of potential adopters. Low types in the network of potential adopters have degrees distributed according to \( D^- \) with

\[
\mathbb{P}(D^- = d) = \frac{\mathbb{P}(D = d)\mathbb{P}(V^- > v_d)}{\sum_{k \in \mathbb{N}} \mathbb{P}(D = k)\mathbb{P}(V^- > v_k)},
\]

and high types in the network of potential adopters have degrees distributed according to \( D^+ \) with

\[
\mathbb{P}(D^+ = d) = \frac{\mathbb{P}(D = d)\mathbb{P}(V^+ > v_d)}{\sum_{k \in \mathbb{N}} \mathbb{P}(D = k)\mathbb{P}(V^+ > v_k)}.
\]

For a low-type potential adopter, each neighbor is a low-type potential adopter with probability \( hP^- \) and a high-type potential adopter with probability \( (1 - h)P^+ \). Similarly, for a high-type potential adopter, each neighbor is a high-type potential adopter with probability \( hP^+ \) and a low-type potential adopter with probability \( (1 - h)P^- \). In the network of potential adopters then, a low-type player first realizes a degree according to \( D^- \), each link is labeled low or with probabilities \( hP^- \) and \( (1 - h)P^+ \), and otherwise the link is removed. We can analogously define the degree distribution for high-type potential adopters. These degree distributions define a configuration model for the network of potential adopters.

A nice feature of this particular model of homophily is that the forward distribution of the approximating branching process \( T_\sigma \) does not depend on the parent node's type. Since degrees are realized independently according to \( D \), and the type assigned to each stub is assigned independently, a low-type node with a low-type parent will have the same forward degree distribution as a low-type node with a high-type parent. If \( D_{\sigma}^- \) and \( D_{\sigma}^+ \) are the forward distributions as defined in the previous section, then for each node after the root, a low-type node has low-type children distributed according to \( D_{\sigma,hP^-}^- \) and high-type children distributed according to \( D_{\sigma,hP^+}^- \). Similarly, a high-type node has low-type children distributed according to \( D_{\sigma,(1-h)P^-}^+ \) and high-type children distributed according to \( D_{\sigma,(1-h)P^+}^+ \). Define the quantities

\[
\nu^- = \mathbb{E}[D_{\sigma}^-] = \frac{\mathbb{V}[D_{\sigma}^+]}{\mathbb{E}[D_{\sigma}^-]} + \mathbb{E}[D_{\sigma}^-] - 1 \quad \text{and} \quad \nu^+ = \mathbb{E}[D_{\sigma}^+] = \frac{\mathbb{V}[D_{\sigma}^-]}{\mathbb{E}[D_{\sigma}^+]} + \mathbb{E}[D_{\sigma}^+] - 1.
\]

We can determine the critical behavior of \( T_\sigma \) from the mean-offspring matrix \( \{m_{ij}\} \), where \( m_{ij} \) is the expected number of type \( j \) children for a type \( i \) parent according to the forward distribution. In this case, writing down the mean-offspring matrix is fairly easy:

\[
M_\sigma = \begin{pmatrix}
  hP^- \nu^- & (1 - h)P^+ \nu^- \\
  (1 - h)P^- \nu^+ & hP^+ \nu^+
\end{pmatrix}.
\]
To get the upper left entry, a low-type neighbor has \( \nu_\sigma^- \) additional neighbors in expectation, a portion \( h \) of whom are low-type players and a portion \( P^-_\sigma \) of whom are potential adopters. The other entries are similarly easy to check.

**Theorem 8.** Let \( r_\sigma \) denote the spectral radius of \( M_\sigma \). If \( r_\sigma > 1 \), then \( \alpha > 0 \) with positive probability. If \( r_\sigma \leq 1 \), then \( \alpha = 0 \) with probability one.

**Proof.** This is immediate from Theorem 7 and standard results on branching processes. 

Just like the quantity \( P_\sigma \nu_\sigma \) in the last section, the spectral radius \( r_\sigma \) serves as a metric for the level of diffusion here. Since the matrix \( M_\sigma \) is small, we can write an explicit formula for the spectral radius

\[
r_\sigma = \frac{h}{2}(P^-_\sigma \nu^-_\sigma + P^+_\sigma \nu^+_\sigma) + \frac{1}{2} \sqrt{h^2(P^-_\sigma \nu^-_\sigma + P^+_\sigma \nu^+_\sigma)^2 - 4(2h - 1)P^-_\sigma P^+_\sigma \nu^-_\sigma \nu^+_\sigma}.
\]

One way we can explore the effect of homophily on diffusion is to look at the partial derivative of \( r_\sigma \) with respect to the homophily parameter \( h \):

\[
2 \cdot \frac{\partial r_\sigma}{\partial h} = P^-_\sigma \nu^-_\sigma + P^+_\sigma \nu^+_\sigma + \frac{2h(P^-_\sigma \nu^-_\sigma + P^+_\sigma \nu^+_\sigma)^2 - 8P^-_\sigma P^+_\sigma \nu^-_\sigma \nu^+_\sigma}{\sqrt{h^2(P^-_\sigma \nu^-_\sigma + P^+_\sigma \nu^+_\sigma)^2 - 4(2h - 1)P^-_\sigma P^+_\sigma \nu^-_\sigma \nu^+_\sigma}}.
\]

This derivative gives us an idea of the marginal impact from adding homophily, holding individual behavior fixed. A short exercise in algebra reveals that whenever \( P^-_\sigma \nu^-_\sigma = P^+_\sigma \nu^+_\sigma \), this derivative is negative. If the ratio of these two quantities is sufficiently large, with either in the numerator, then the derivative is positive. In essence, if one type is far more likely to adopt than the other, homophily helps diffusion because it concentrates the connections among these individuals. If differences in the propensity to adopt are small, homophily impedes diffusion because awareness is less likely to cross over between the two types.

### 4.2.2 Clustering

One way to include clustering in the configuration model is to randomly add connected triangles in addition to our randomly matched pairs. I describe local beliefs first before specifying the corresponding model of the network. As before, a player observes only her valuation type and her degree, and she believes that neighbors have independent valuation types drawn according to \( \nu \). Also as before, we write \( D \) for the degree distribution in the network. To simplify the exposition, I suppose that all players have an even number of neighbors, but this is straightforward to relax. When nature realizes a player’s degree, it pairs the player’s neighbors at random and creates an edge between each pair with independent probability \( \gamma \in [0, 1] \). For example, if a player has four neighbors, exactly two are linked to each other with probability \( 2\gamma(1 - \gamma) \), and there are two linked pairs with probability \( \gamma^2 \).

The reason to specify local beliefs this way is that the degree distribution for each neighbor is independent, despite the effect of clustering. Since degrees are even for all players, a neighbor always has a second link on top of the link to our player; the only question is
whether it links to a new player or whether it links to another neighbor. Consequently, our player believes that each neighbor has degree distributed independently according to the size-biased distribution $D'$, and equilibrium behavior is exactly the same as in the previous section.\footnote{Allowing more general beliefs would create correlations between neighbors’ degrees, but our equilibrium analysis would remain essentially unchanged. Galeotti et al. (2010) permit correlations in neighbors’ degrees, and their results would still apply.}

To obtain the corresponding configuration model, first realize a sequence of stubs according to $D$. Each node has an even number of stubs, so we can pair them. For each pair of stubs, with probability $1 - \gamma$ the two remain as normal stubs, and with probability $\gamma$, we merge them into a single triangle stub. Once the stubs are labeled, we pair normal stubs uniformly at random as before, and we choose triples of triangle stubs uniformly at random to create triples of linked nodes. This variant of the configuration model does not immediately map to a multitype configuration model, but we can construct a corresponding branching process approximation with two types that has the same distribution of connected components, and this allows us to apply Theorem 7 to study the extent of diffusion. The key idea is to fix the root of the graph and consider the distribution of new connections that each neighbor has. A neighbor who is linked with a normal link yields new connections according to the forward distribution $D''$, while a neighbor who is linked as part of a triangle yields new connections according to $D'' - 1$.

**Theorem 9.** Let $\sigma$ be an equilibrium strategy profile with $P_\sigma$ the corresponding neighbor adoption probability, and define $\nu_\sigma$ as in equation (4.5). If

$$\frac{P_\sigma}{2} \left( \nu_\sigma + \sqrt{\nu_\sigma^2 - 4\gamma(\nu_\sigma - 1)^2} \right) > 1$$

then $\alpha > 0$ with positive probability. If

$$\frac{P_\sigma}{2} \left( \nu_\sigma + \sqrt{\nu_\sigma^2 - 4\gamma(\nu_\sigma - 1)^2} \right) \leq 1,$$

then $\alpha = 0$ with probability one.

**Proof.** The first step is to derive a multitype branching process approximation for the configuration model with clustering. We fix a realized network and choose a root from which to explore the network, pruning triangles in the process. We conduct a breadth first search from the root, successively adding all nodes at distance 1, 2, and so on, from the root. I refer to all nodes at a given distance from the root as a cohort. Each time we add a new cohort to the graph, we remove all links between members of the same cohort. After fully exploring one component in this way, select an unvisited node and repeat the process until we have visited all nodes in the graph. This process results in a graph with the same connected components as the original, but the pruned graph is locally tree-like. The argument proving Theorem 7 in the appendix allows us to use a branching process approximation to understand the connected components.
In the network of potential adopters, a random player has degree distributed according to $D_\sigma$ as defined in equation (4.4), and each neighbor is a potential adopter with independent probability $P_\sigma$. Exploring from a fixed root in the network of potential adopters, the first stage in our branching process $T_\sigma$ has offspring distribution $D_{\sigma,P_\sigma}$, and subsequently there are two types of potential adopters, depending on whether they were reached via normal links or triangle links. A player reached through a normal link has the forward distribution $D'_{\sigma,P_\sigma}$, while a player reached through a triangle link has the forward distribution $(D'_{\sigma} - 1)P_\sigma$.

To determine whether the survival probability of $T_\sigma$ is positive, we look at the mean-offspring matrix

\[
\begin{pmatrix}
P_\sigma + P_\sigma(1 - \gamma)(\nu_\sigma - 1) & P_\sigma \gamma(\nu_\sigma - 1) \\
P_\sigma(1 - \gamma)(\nu_\sigma - 1) & P_\sigma \gamma(\nu_\sigma - 1)
\end{pmatrix}.
\]

I leave it as an exercise to check the entries of this matrix; the extra term in the upper left occurs because players reached through a normal link always have a single additional neighbor who is not connected in a triangle. Whether adoption can spread to a positive fraction of the network depends on the spectral radius of this matrix, which we compute as

\[
\frac{P_\sigma}{2} \left( \nu_\sigma + \sqrt{\nu_\sigma^2 - 4\gamma(\nu_\sigma - 1)^2} \right).
\]

In the absence of clustering, the condition for a large cascade reduces to $P_\sigma \nu_\sigma > 1$ as before. As we increase $\gamma$, creating more clustering in the network, the likelihood of a large cascade is reduced. Intuitively, this is clear because there are more redundant edges in a clustered network.

Adding random triangles to a network is not the only way we can introduce clustering in the network. I could just as easily add random $k$-cliques to this model for some fixed $k \geq 4$, and we would similarly construct a branching process approximation with an additional type. An alternative approach is to use a configuration model with household structure, in which we first construct small groups of players with a particular link structure in each group, and then link the groups according to a configuration model. In either case, the techniques used in the proof of Theorem 7 allow us to establish a direct correspondence between connected component sizes and a multitype branching process. Since the behavior of branching processes is well-understood, this provides a powerful tool to study networks with a wide range of structural features.

### 4.2.3 Local Information

Return to the original model with independent valuations and degrees, and suppose $V$ is symmetric around $\frac{1}{2}$ with no atoms. For each neighbor $j$, a player observes a binary signal $s_j \in \{0, 1\}$. If neighbor $j$ has valuation $v_j \geq \frac{1}{2}$, then $s_j = 1$ with probability $g$ and 0 with probability $1 - g$; if $v_j < \frac{1}{2}$, then $s_j = 1$ with probability $1 - g$ and 0 with probability $g$. Assume that $g > \frac{1}{2}$ and signals are independent across neighbors. We have expanded
the information set of a player to a triple \((v, d, s)\), where \(s\) denotes the number of 1 signals observed. A strategy profile \(\sigma\) now maps these triples \((v, d, s)\) to adoption decisions.

As in the model of homophily, I refer to players with valuations below \(\frac{1}{2}\) as low types and the others as high types, with \(V^-\) and \(V^+\) their respective valuation distributions. Let \(B(n, p)\) denote a binomial random variable with \(n\) trials and success probability \(p\). For a random player with degree \(d\), the number \(s\) of high signals she receives is distributed according to \(B(d, \frac{1}{2})\). Her neighbors’ signals will depend on her type. If our player is low-type, a neighbor with degree \(d\) has \(s\) distributed according to \(S^-_d = B(d - 1, \frac{1}{2}) + B(1, 1 - g)\), where the two binomial distributions are independent. If our player is high-type, a neighbor with degree \(d\) has \(s\) distributed as \(S^+_d = B(d - 1, \frac{1}{2}) + B(1, g)\).

Fix a strategy profile \(\sigma\). A random low-type neighbor of a low-type player is willing to adopt with probability

\[
P^- = \mathbb{E}[\sigma(V^-, D', S^-_d)],
\]

and a high-type neighbor of a low-type player is willing to adopt with probability

\[
P^+ = \mathbb{E}[\sigma(V^+, D', S^-_d)].
\]

We can similarly define adoption probabilities for a low and high-type neighbor of a high-type player

\[
\begin{align*}
P^+_\sigma &= \mathbb{E}[\sigma(V^-, D', S^+_d)], & P^+_\sigma &= \mathbb{E}[\sigma(V^+, D', S^+_d)].
\end{align*}
\]

Consequently, a low-type player with degree \(d\) and \(s\) high signals expects her number of adopting neighbors to follow the distribution

\[
A^-_{\sigma, d, s} = B\left(s, gP^-_\sigma + (1 - g)P^-_\sigma\right) + B\left(d - s, (1 - g)P^+_\sigma + gP^-_\sigma\right),
\]

and a high-type player with degree \(d\) and \(s\) high signals expects her number of adopting neighbors to follow the distribution

\[
A^+_{\sigma, d, s} = B\left(s, gP^+_\sigma + (1 - g)P^-_\sigma\right) + B\left(d - s, (1 - g)P^+_\sigma + gP^-_\sigma\right),
\]

where again all of the binomial distributions are independent.

**Proposition 7.** There exists a local Bayesian equilibrium in which both \(P^- \leq P^+_\sigma\) and \(P^- \leq P^+_\sigma\).

In any such equilibrium, strategies are non-decreasing in both \(v\) and \(s\).

**Proof.** Consider the set of strategy profiles \(\sigma\) such that \(P^- \leq P^+_\sigma\), and \(P^- \leq P^+_\sigma\). It should be clear that any best reply to such a strategy profile shares this property and is non-decreasing in \(v\) and \(s\). Since best replies are also monotone in adoption probabilities, Tarski’s fixed point theorem gives us existence. \(\Box\)
Saying much about how more or less informative signals change the equilibrium is difficult without further assumptions. As $g$ increases, there is a tradeoff between players with many high signals becoming more likely to adopt, and those with many low signals becoming less likely to adopt. Which effect dominates will depend on the other parameters in the model. For this example, I leave as an exercise the calculation of the offspring distribution for the branching process $T_\sigma$. Note that in this case, we cannot write the branching process with only two types as in the previous examples. The offspring distribution depends on both the node’s type and the parent’s type, so there will be four types: a low-type with a low-type parent, a low-type with a high-type parent, a high-type with a low-type parent, and a high-type with a high-type parent. Unlike in the previous two examples, the parameter of interest—in this case $g$—does not appear explicitly in the mean offspring matrix. Information will affect diffusion only through the equilibrium, not through any change to the network structure.

### 4.3 Choosing When to Adopt

We do not always buy a new product, or adopt a new technology, the moment we become aware of it. Often we wait, sometimes for a significant amount of time; we might have more pressing concerns, or we have an old version that is good enough until it wears out. Waiting can also be a strategic choice. Many technologies require a large upfront investment, and the payoffs are realized over a long time horizon. If there are payoff externalities, there is option value in waiting for more neighbors to adopt. We now extend our framework to better understand how the option to wait affects diffusion.

We still describe the network using a configuration model with degree distribution $D$. Time is discrete, and a single random player is endowed with a new technology prior to the first period. In each period, players who are aware of the opportunity to adopt, but have not yet done so, decide whether to irreversibly adopt the technology or wait. A player becomes aware the period after her first neighbor adopts. As before, we characterize players using a valuation type and a degree, distributed according to $V$ and $D$ respectively. Unlike the previous sections, players need not make the decision to adopt immediately upon becoming aware. A player observes the number of her neighbors who have already adopted, and she can condition her adoption decision on this information. If she declines to adopt in the current period, she can change her mind in the future. A full description of a player in any particular period is a triple $(v, d, a)$ including a valuation type $v$, a degree $d$, and a number of adopting neighbors $a$.

Players pay the cost of adoption $c$ upfront, and payoffs are realized over time. If $A_t$ neighbors have adopted by the end of period $t$, then adoption yields a payoff $f(v, d, A_t)$ in period $t$, with $f$ non-decreasing in $v$ and $A_t$. Future payoffs are discounted at a common rate $\delta$. A player of type $(v, d)$ who adopts in period $t^*$ earns a total payoff

$$\sum_{t=t^*}^{\infty} \delta^t f(v, d, A_t) - \delta^{t^*} c.$$
To optimize, a player must form expectations about the timing of neighbors’ adoption decisions, conditional on the timing of her own.

Using a local model imposes constraints on the information that a player can use to inform her decision. Sadler (2015a) points out a positive relationship between the length of a player’s memory and the depth of the neighborhood she implicitly models. Since we restrict players to model only immediate neighborhoods, we necessarily restrict their memory. Our players retain no knowledge of the timing or history of other players’ decisions; they observe in the present moment only how many neighbors have adopted. Based on this, they form beliefs about what their neighbors are observing and how their neighbors will act. A strategy profile $\sigma$ for the players is a function mapping triples $(v, d, a)$ to binary adoption decisions.

Each player believes that her neighbors have valuations drawn independently according to $V$ and degrees drawn independently according to $D'$. If a player has $d$ neighbors and $a$ of them have adopted, she uses the fraction $r = \frac{a}{d}$ as an estimate of the probability that a random player in the network has adopted. She supposes that each neighbor of her neighbors, other than herself, has adopted independently with probability $r$. This induces a distribution $F_r$ over triples $(v, d, a)$ that represent neighbors’ valuations, degrees, and number of adopting neighbors. A player supposes that each neighbor has a type drawn independently according to $F_r$. We should expect that the distribution of triples $(v, d, a)$ for a neighbor who has adopted is different than for a neighbor who has not. A strategy profile together with a distribution over triples $F_r$ allows us to define conditional distributions $F^0_{r,\sigma}$ and $F^1_{r,\sigma}$ over triples that do not adopt under the profile $\sigma$ and those that do respectively. If other players follow the profile $\sigma$, a player supposes that each non-adopting neighbor has a type drawn independently according to $F^0_{r,\sigma}$.

To finish our description of a player’s decision problem, we need to specify expectations over future neighbor adoptions conditional on current period observations. If a player has degree $d$ and $a$ currently adopting neighbors, she has in mind a probability $p_{d,a}$ that a non-adopting player will adopt in any given period. Given a strategy profile $\sigma$, we impose a consistency requirement on these adoption probabilities. Recall the notation $B(n, p)$ for a binomial distribution with $n$ trials and success probability $p$. The adoption probabilities must satisfy

$$p_{d,a} = \mathbb{E}_{F^0_{r,\sigma}} [\sigma(V, D, A + B(D - A, p_{d,a}))].$$

This condition states that the probability that a non-adopting neighbor adopts is equal to the probability that enough neighbors of this neighbor adopt to induce adoption under the profile $\sigma$. There may be multiple sets of adoption probabilities that satisfy the condition, but note that $p_{d,a} = 0$ for all $(d, a)$ always satisfies it; throughout this section we will focus on these adoption probabilities. The assumption that $p_{d,a} = 0$ for all $(d, a)$ corresponds to the most pessimistic forecast in which a player expects no new neighbors to adopt unless she does.

We can now write down a player’s optimization problem. If she chooses not to adopt, her expected payoff is zero as she never expects to adopt later. Define $p_{d,a}^* = \mathbb{E}_{F^0_{r,\sigma}} [\sigma(V, D, A + 1)]$. 

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as the expected probability that a non-adopting neighbor will adopt the period after our player does. If our player chooses to adopt, her expected payoff is

$$\pi(v, d, a) - c \equiv f(v, d, a) + \frac{\delta}{1 - \delta} \mathbb{E} \left[ f \left( d, v, a + B(p^*_d, d - a) \right) \right] - c.$$ 

The optimal response is to adopt or wait depending on whether $\pi(v, d, a)$ is larger or smaller than the cost $c$. A strategy profile $\sigma$ is a local Bayesian equilibrium if for all triples $(v, d, a)$ we have $\sigma(v, d, a) = 1$ whenever $\pi(v, d, a) > c$ and $\sigma(v, d, a) = 0$ whenever $\pi(v, d, a) < c$.

The possibility of waiting substantially changes individual behavior in equilibrium because of informational effects. Even if utility from adoption is strictly increasing in the number of adopting neighbors, players can use non-monotonic strategies because additional adopting neighbors provide information about the types of non-adopting neighbors. If having more adopting neighbors suggests that those not yet adopting are unlikely to ever do so, then a player with more adopting neighbors may be less inclined to adopt herself.

We can see this effect in a simple example. Suppose all players have 4 neighbors, and there are three valuation types, which we denote $v_1$, $v_2$, and $v_3$. The distribution $V$ is such that $v_1$ and $v_2$ each occur with probability $\frac{1}{6}$ and $v_3$ occurs with probability $\frac{2}{3}$. Let the cost be $c = 28.5$ and the discount rate $\delta = 0.9$. Players of type $v_1$ earn nothing from adoption, players of type $v_2$ earn $a$ units of utility in each period after adoption, and players of type $v_3$ earn $3a$ units of utility in each period after adoption, where $a$ is the number of adopting neighbors. We show that it is an equilibrium if players of type $v_1$ never adopt, players of type $v_3$ adopt as soon as any neighbor does, and players of type $v_2$ adopt if 1, 3, or 4 neighbors are adopting, but not 2.

The equilibrium strategies of types $v_1$ and $v_3$ are clear since the former can never obtain positive value from adoption, and the latter can justify adoption if only one neighbor ever adopts. For type $v_2$, it is clear that adoption is optimal if 3 or 4 neighbors are adopting; we analyze the decision of these players when only 1 or 2 neighbors have adopted.

Suppose that a player with valuation type $v_2$ observes that one neighbor has adopted. She then believes that each neighbor has 0, 1, 2, or 3 adopting neighbors with probabilities $\frac{27}{64}$, $\frac{27}{64}$, $\frac{9}{64}$, and $\frac{1}{64}$ respectively. Conditional on not adopting, a neighbor is of type $v_1$ with probability $\frac{8}{15}$, but will adopt after our player adopts with probability $\frac{7}{15}$. The expected utility from adoption is

$$1 + \frac{\delta}{1 - \delta} \frac{40}{13} \approx 28.7 > c,$$

so adoption is optimal.

Alternatively, suppose that a player with valuation type $v_2$ observes that two neighbors have adopted. She then believes that each neighbor has 0, 1, 2, or 3 adopting neighbors with probabilities $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$ respectively. Conditional on not adopting, a neighbor is of type $v_1$ with probability $\frac{8}{15}$, but will adopt after our player adopts with probability $\frac{7}{15}$. The lower bound on the payoff for type $v_3$ is $30 > 28.5$. 

\footnote{A lower bound on the payoff for type $v_3$ is $30 > 28.5$}

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expected utility from adoption is then

\[ 2 + \frac{\delta}{1 - \delta} \cdot \frac{44}{15} = 28.4 < 1, \]

so adoption is strictly suboptimal.

This example is remarkable because we obtain non-monotonic equilibrium strategies with linear payoff functions. This means we cannot rule out unintuitive strategies through a simple condition like concave payoffs. Having one more adopting neighbor can strongly affect beliefs about what non-adopting neighbors will do, and this can overwhelm the value from having the additional adopting neighbor.

Given an equilibrium, we can obtain results on the extent and rate of diffusion just as before, but with one added detail: the branching process approximation is based on only those players who adopt after only one neighbor adopts. Since rooted subgraphs in the configuration model are trees, no player will have multiple adopting neighbors until a significant fraction of the population has already adopted, so diffusion is highly dependent on the number of “early adopters” who do not wait to invest. Besides slowing down the diffusion process, this suggests a different approach to encouraging adoption. There may be a benefit to concentrating seeds in a small part of the network, and clustering in the network could also prove helpful since players could have multiple adopting neighbors earlier in the diffusion process.

### 4.4 Remarks

This analysis of diffusion games makes two key contributions, one economic and one technical. The model gives a microfounded account of why we see such varied and stochastic adoption patterns that is roughly consistent with the distribution of outcomes observed in empirical work. In contrast with many papers that rely on steady state analysis, we can capture short-run dynamics in cases when adoption decisions are not immediately reversible. The network structure impacts the outcome distribution apart from individual adoption behavior, and in particular the first two moments of the degree distribution play a special role. Players with many neighbors help facilitate diffusion regardless of their other characteristics, which partly explains the observed influence of central individuals. Several additional patterns emerge from the model, which call for empirical investigation.

To summarize the technical contribution, I can simply say that random graphs are branching processes. Making this idea rigorous takes some work, but once the correspondence is achieved, analysis of random graphs becomes much easier. The idea of a branching process approximation is relatively old in the mathematics literature, but extending it to multitype graphs allows us to use this insight in a wide range of economic applications. Through a careful selection of types, we can capture important network features like homophily and even clustering within a tractable framework. Though beyond the scope of this paper, I am

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7See Lemma 4 in the appendix.
hopeful that these results may facilitate a closer relationship between the theoretical study of diffusions and econometric analysis of networks. The branching process approximation allows us to capture the essential features of a diffusion process using a small number of parameters that should be relatively straightforward to estimate from data. Despite its flexibility, the model of diffusion based on the multitype configuration model has important limitations. We can face difficulty representing some types of clustering and correlations between neighborhoods. Expanding the set of types can help, but doing so is costly in terms of model complexity. In my examples, I am able to minimize the number of types in the branching process through careful modeling choices, but in general, a network with \( n \) types requires a branching process with \( n^2 \) types. This somewhat overstates the scaling problem because most entries in the mean offspring matrix are zero (e.g. a type \( a \) node with a type \( b \) parent cannot have a type \( c \) child with a type \( d \) parent), but a model with many types will still prove cumbersome.
Chapter 5

Innovation Adoption and Collective Experimentation

Mounting evidence suggests that information transmission through social ties plays a central role in the diffusion and acceptance of innovations. Many studies of agricultural technologies emphasize the role of social learning in adoption decisions (Munshi, 2004; Conley and Udriy, 2010), and other work finds learning effects in domains ranging from health products (Dupas, 2014) and health plans (Sorensen, 2006) to investments (Duflo and Saez, 2003) and microfinance (Banerjee et al., 2013). This research provides an understanding not only of the aggregate effects of learning on technology adoption, but the strategic choices individuals make when they can learn from others (Foster and Rosenzweig, 1995; Kremer and Miguel, 2007).

One reason that information from our friends and neighbors can influence our adoption decisions is uncertainty about the merits of an innovation. Experience using a new high yielding seed variety, or the latest smart phone, provides information regarding its value, but we face an opportunity cost to acquire this experience. Learning from friends and acquaintances complicates the choice of whether to experiment, and broader patterns of information sharing influence who experiments, how much, and ultimately whether society adopts the innovation.

We shall study a game that captures important aspects of social experimentation. Players in a fixed social network jointly face a two-armed bandit problem, and the players share with their neighbors any information that they gather. Our goal is to develop an intuitive understanding of how information sharing patterns affect both individual incentives to experiment and the collective choice to explore or exploit. I argue that network structure can have important, but nuanced effects on innovation adoption. In particular, without some knowledge of the distribution of initial beliefs, this structure has ambiguous implications for long run adoption. Individual beliefs determine who finds it worthwhile to experiment, and these beliefs interact with the network to shape outcomes. Nevertheless, the model suggests several regularities, offering insights on how the internal structure of groups or organizations may regulate collective behavior.

On a macro level, the network governs a fundamental tradeoff between disseminating
information efficiently and encouraging more total experimentation. When information is quickly and broadly shared, people gather less information, and bad early experiences may cause the group as a whole to abandon a new technology. These groups are faster to eliminate bad innovations, but more likely to abandon good ones. This suggests that dense or centralized networks will experience more volatile outcomes—either shortly adopting the innovation throughout or abandoning it altogether—though this effect depends crucially on who the initial adopters are.

I provide several examples to highlight important effects. One shows clearly how dense connectivity discourages experimentation. Beliefs tend to move together in a dense network since people learn from the same set of experiences. This leads to early consensus and less double checking, which results in more errors. A second example demonstrates how central individuals can inhibit adoption in the long run. Counterintuitively, making a central player more optimistic about the innovation can make it less likely that the group adopts: the central player experiments more, and her experiences have a disproportionate influence on others. If the central player has a negative experience and quits experimenting, the rest of the group might abandon the innovation too. Finally, I consider how to “seed” an innovation in a network—that is, how to select key individuals to carry out experiments so that broader adoption becomes more likely. Clustering the seeds in one part of a network is self-defeating; the seeds have correlated beliefs and they will quickly adopt or abandon together. Isolating early adopters from one another ensures that more independent information is gathered.

Two strategic effects are important for individual decisions: the free-rider effect and the encouragement effect. The ability to free-ride on others’ experimentation discourages people from gathering information themselves, but good results today could encourage more experimentation by others tomorrow. Those with many neighbors experience both effects more strongly, and the interaction between the two effects leads to complicated behavior among the most well-connected players. All players experiment with sufficiently optimistic beliefs. At moderate belief levels, the free-rider effect dominates for the high degree players. Those with many neighbors refrain from experimenting, allowing low degree players to gather information instead. At more pessimistic belief levels, the encouragement effect takes over. The highest degree players are the ones who experiment, while low degree players wait for some encouraging results.

Taken together, our findings can help us understand when adoption is more or less likely and when adoption is more or less difficult to predict. We have a framework to think about the tradeoffs that different group structures entail with regard to experimentation. The structure of the information sharing network affects individual incentives to gather information in predictable ways, and ultimately the total amount of information gathered determines the long-term behavior of the group.

A broader contribution of the paper is a novel approach to studying games on networks. In an attempt to both capture realistic features of human reasoning and simplify the analysis, I introduce elements of bounded rationality into players’ decision making. The players apply Bayes’ rule to update their beliefs about the innovation, but a player’s expectations about others’ behavior are based on an incomplete or “local” model of the world that only includes
her immediate neighbors. This way of forming expectations reflects the difficulty of reasoning about unknown individuals who are distant in the network. We can also see this as a middle road between fully rational expectations and the simpler heuristics that many authors use to study behavior in networks. Instead of choosing one extreme, we balance tractability against our desire to understand incentives and strategies. The example of this paper can offer a template for other researchers to apply.

This model serves as a bridge between the growing literatures on social learning and strategic experimentation. A significant branch of the social learning literature has recently focused on the role network structure plays in disseminating and aggregating dispersed information, exploring the long-run efficiency of learning in large networks. These papers typically eliminate strategic aspects of learning, either assuming a sequential game in which all players act only once (Acemoglu et al., 2011; Lobel and Sadler, 2014a) or employing a heuristic decision rule (Golub and Jackson, 2010, 2012; Jadbabaie et al., 2012). In contrast, the strategic experimentation literature centers on full observation settings in which all players observe the experimentation of all others, allowing a sharper focus on the strategic behavior of individuals (Bolton and Harris, 1999; Keller et al., 2005). I unify several aspects of these models, studying a setting in which strategic decisions to gather information interact with the network structure to determine learning outcomes.

I build most directly on the work of Bolton and Harris (1999) and Bala and Goyal (1998). I adopt the same continuous-time formulation of a two-armed bandit as Bolton and Harris (1999) with a more general information sharing structure; their strategic experimentation game becomes a special case of ours with a complete network and common initial beliefs about the innovation. As in this paper, Bala and Goyal (1998) study a model with local information acquisition and sharing in a general network, but players are myopic, always choosing the action with the highest current expected payoff. Players ignore the value of information and effects on other players, and the principal focus is on long run outcomes in very large networks. We go a step further, considering strategic players who assess the future implications of current decisions. Moreover, the results of section 4 enrich our understanding of aggregate outcomes in relatively small networks.

Less immediately, this work contributes to the study of information processing and organizational structure. One way to think about a firm is as a collection of information processors that takes inputs from the environment and produces decisions. Taking this view, we can see that the costs of communication, information processing, and delays in decision making have implications for how firms are organized and who makes which decisions (Radner, 1993; Bolton and Dewatripont, 1994; Garicano, 2000). We build on this work, showing that incentives to gather information also have implications for the structure of organizations. If members have significant autonomy to decide how to carry out their work, then the way these individuals communicate with one another will partly determine how best practices diffuse within the organization. A firm that efficiently shares information should be quick to either eliminate or adopt new practices, while a firm organized into relatively isolated teams might appear more patient. An environment favoring one side of this tradeoff creates pressure to adopt particular internal structures.
For expositional clarity I first describe a discrete time version of the game before passing to the continuous time formulation that is the focus of our analysis. After describing the game, I introduce the notion of a local Bayesian equilibrium, giving some discussion of the assumptions and choices this approach entails. Section 3 analyzes individual behavior in equilibrium, and section 4 presents the main results on network structure and long-run outcomes. I conclude with a brief discussion.

5.1 The Experimentation Game

We begin in discrete time. We have \( N \) players, and in every period \( t \), each of them faces a choice between two competing technologies: technology 0 is the “standard” technology, and technology 1 is the “innovation.” Assume players can continuously allocate use between the two technologies in any period; that is, they can split their efforts between the two technologies in any way they choose. Imagine for instance a new crop variety that we can plant in some fraction of available land, using the remaining land for the old crop. For player \( i \), let \( \alpha_i(t) \in [0,1] \) denote the proportion of period \( t \) devoted to the innovation.

Player \( i \)'s payoff in a period is the sum of two independent normally distributed random variables \( \pi_0^i(t) \) and \( \pi_1^i(t) \), representing the payoffs from the standard technology and the innovation respectively. If player \( i \) allocates a portion \( \alpha_i \) of effort to the innovation, her payoff from the standard technology has mean \( (1 - \alpha_i) \mu_0 \) and variance \( (1 - \alpha_i) \sigma^2 \), and her payoff from the innovation has mean \( \alpha_i \mu_1 \) and variance \( \alpha_i \sigma^2 \). The mean payoff from the standard technology \( \mu_0 \) and the variance parameter \( \sigma \) are fixed and commonly known, while the innovation has unknown mean payoff \( \mu_1 \in \{L,H\} \) with \( L < \mu_0 < H \). Players discount the future at a common rate \( \frac{1}{1+r} \in (0,1) \), and their goal is to maximize the discounted sum of present and future payoffs.

At the beginning of period \( t \), player \( i \) believes that \( \mu_1 = H \) with probability \( p_i(t) \). Realized payoffs convey some information about \( \mu_1 \), with a higher \( \alpha_i(t) \) producing a more informative signal. In addition to her own effort allocation and payoffs, each player observes those of her neighbors in a social network. We represent this network as a directed graph \( G \), using \( G_{ij} \in \{0,1\} \) to denote the corresponding entry in the adjacency matrix. If \( G_{ij} = 1 \) we say that \( j \) is a neighbor of \( i \), and at the end of each period player \( i \) observes the values \( \alpha_j(t) \) and \( \pi_j^1(t) \). We further assume that \( G_{ii} = 1 \) for all \( i \), and we let \( G_i = \{ j \mid G_{ij} = 1 \} \) denote the set of player \( i \)'s neighbors. The value \( d_i = |G_i| - 1 \) is player \( i \)'s degree, and \( F \) denotes the distribution of player degrees, which is common knowledge.\(^1\) At the end of period \( t \), player \( i \) observes \( \{ \alpha_j(t), \pi_j^1(t), \forall j \in G_i \} \) and applies Bayes’ rule to update her belief \( p_i(t) \) to \( p_i(t+1) \).

To choose the best course of action, players must form expectations about how much their neighbors will experiment, and this is where cognitive limitations enter our model. In choosing these particular assumptions about the way players think, I aim to capture, as simply as possible, four intuitive features of the problem:

\(^1\)Within this paper, we shall always assume that players beliefs about the degree distribution coincide with the empirical distribution, but we can just as easily accommodate subjective beliefs about player degrees.
1. Players form expectations about neighbors’ experimentation decisions, and they assume their neighbors do the same;

2. Players find it easier to form expectations about immediate neighbors than those more distant in the network;

3. Neighbors’ beliefs are likely higher (lower) when a player’s beliefs are high (low);\(^2\)

4. Direct experimental observations are more salient than the factors that led neighbors to choose a particular level of experimentation, particularly in light of (b).

Limiting the scope of reasoning about the network entails limiting the information players use to inform their strategies. If players condition their actions on fine details of a long history, then they must at least implicitly hold beliefs about what led their neighbors to choose the actions they did. That is, beliefs about neighbors of neighbors, neighbors of those neighbors, and so on. To avoid this, we shall restrict a player \(i\) to strategies that are functions only of her current belief \(p_i(t)\), ignoring the history that led to this belief. We can imagine someone who finds it hard to remember the past precisely or finds it too costly to incorporate this information into a decision at each moment. This assumption means that the system as a whole behaves as a Markov chain with a vector of state variables \(\{p_i(t)\}_{i \leq N}\).

In predicting the beliefs of her neighbors, a player uses her own beliefs as a guide. We make a particularly simple assumption: a player supposes that her neighbors always share her belief. That is, at any time player \(i\) thinks that \(\mu_1 = H\) with probability \(p_i\), she will act as if all of her neighbors \(j \in G_i\) hold the belief \(p_j = p_i\). In addition to capturing our intuition from point (c) above, these expectations could reflect the false consensus effect that psychological studies document (Ross et al., 1977).

Finally, given the players’ memory limitations, it seems especially difficult to infer any information from a neighbor’s chosen level of experimentation, apart from that the realized payoffs provide. Although it could provide an additional way for players to learn, we assume that players ignore this source of information and update their beliefs using only the information that comes directly from experimental results. They do not infer anything more from neighbors’ decisions to experiment.\(^3\)

To formalize the preceding discussion, we define a strategy for a player as a function \(s : [0, 1] \to [0, 1]\), giving an experimental effort level \(\alpha(p)\) for each possible belief \(p\). We consider symmetric strategy profiles \(s(d, p) : N \times [0, 1] \to [0, 1]\), which specify an action for each player as a function of her degree and her belief. Given a symmetric strategy profile \(s\), define the average level of experimentation for a random player with belief \(p\) as \(\alpha_s(p) = E_{D \sim F}[s(D, p)]\). At any moment player \(i\) has belief \(p_i(t) = p\), she expects each neighbor \(j \in G_i\) to choose \(\alpha_j(t) = \alpha_s(p)\).

\(^2\)This is particularly evident when the underlying network is highly clustered.

\(^3\)Equivalently, players treat the actions of their neighbors as independent of the underlying state. Eyster and Rabin (2005) provide an alternative justification for this bias and explore its implications through the solution concept of “cursed equilibrium.”
Consequently, player \( i \) expects to observe total experimentation \( d_i \alpha_s(p) + \alpha_i(p) \) in any period in which her initial beliefs are \( p_i(t) = p \). For any strategy she chooses, she expects her future beliefs to evolve as a Markov process, and the function \( \alpha_s(p) \) completely determines the transition probabilities if other players adopt the strategy profile \( s \). We can therefore define player \( i \)'s expected payoff from choosing the strategy \( s_i \) as

\[
u_{s_i,s}(d_i, p) = \sum_{0}^{\infty} \left( \frac{1}{1 + r} \right)^t \mathbb{E}_{s_i,s} \left[ \pi_i^0(t) + \pi_i^1(t) \right].\]

The strategy profile \( s \) is a local Bayesian equilibrium if for each player \( i \) (or equivalently each possible degree \( d_i \)) we have

\[
u_{s,s}(d_i, p) \geq \nu_{s_i,s}(d_i, p)
\]

for all strategies \( s_i \).

The discrete time model is easier to describe, but a continuous time version is easier to analyze. All formal results pertain to a continuous time model that we can view as an appropriate limit of discrete time models.\(^4\) A player \( i \) must now continuously allocate effort \( \alpha_i(t) \in [0, 1] \) at each instant \( t \) to using the innovation, earning instantaneous payoffs

\[
d\pi_i^0(t) = (1 - \alpha_i)\mu_0 dt + \sigma\sqrt{1 - \alpha_i}dZ_i^0(t), \quad \text{and} \quad d\pi_i^1(t) = \alpha_i\mu_1 dt + \sigma\sqrt{\alpha_i}dZ_i^1(t),
\]

where \( \{Z_i^j\} \) is a collection of mutually independent standard Brownian motions. Her beliefs \( p_i(t) \) are continuously updated on the basis of observed experimental outcomes. Given a strategy profile for the other players \( s \), player \( i \) expects to observe total experimentation \( d_i \alpha_s(p) + \alpha_i(p) \) at any instant she holds beliefs \( p_i(t) = p \). Given these expectations, player \( i \) chooses the strategy \( s_i \) to maximize discounted expected utility

\[
u_{s_i,s}(d_i, p) = \mathbb{E}_{s_i,s} \left[ \int_{0}^{\infty} re^{-rt} (d\pi_i^0(t) + d\pi_i^1(t)) \right].
\]

A strategy profile \( s(d, p) \) is a local Bayesian equilibrium if for each player \( i \) and each strategy \( s_i \) we have

\[
u_{s,s}(d_i, p) \geq \nu_{s_i,s}(d_i, p).
\]

### 5.1.1 Discussion of Local Equilibrium

The solution concept in this paper expresses a notion of bounded rationality. Players use an imperfect procedure to form expectations about others’ behavior. We restrict players to condition their actions on a limited set of variables—namely, their degrees and belief levels—and the players assume a distribution over these variables for their neighbors, given their own beliefs. We treat these distributions as given and immutable: they are part of the basic data of the game. In this sense, a players’ beliefs about her neighbors at each instant are similar to prior beliefs, even though they change dynamically over time.

\(^4\)See Bolton and Harris (1999) for a detailed derivation of an analogous continuous time model as a limit of discrete models.
Unlike other approaches to bounded or costly rationality, an advantage here is that analysis of players’ decisions becomes simpler, rather than more complex. A player’s reasoning about the network is constrained to her local neighborhood, so we can represent her beliefs using no details beyond the degree distribution. Importantly, other aspects of the network structure will still impact aggregate outcomes. A comparison to other boundedly rational solution concepts, such as $k$-level reasoning (e.g. Rubinstein, 1989) or analogy based expectations Jehiel (2005), may prove illuminating. Our players exhibit a bounded depth of reasoning regarding the network, as opposed to a bounded depth of reasoning regarding the belief hierarchy. The notions are related but distinct. Our players implicitly hold infinitely many orders of beliefs, but the hierarchy is incomplete: a player has beliefs about her neighbors and beliefs about her neighbors’ beliefs about their neighbors, but she does not hold her own distinct beliefs about the neighbors of her neighbors. There is also an element of coarse reasoning as players always treat each neighbor as having an average degree, despite any observations over the course of the game.

There are two ways to view our bounded rationality assumptions. One is to take these assumptions as descriptive of real human behavior, or at least as a better description than fully rational expectations. There is some empirical support for this view (Chandrasekhar et al., 2012; Corazzini et al., 2012), finding that simple heuristics based on local updating rules can better account for observed behavior in social networks. Any similarity between reasoning to a limited depth in a network and limited depth reasoning in a belief hierarchy means that the literature studying the latter (e.g. Stahl and Wilson, 1995) bolsters this perspective.

A second view is that these assumptions represent a reasonable approach to dealing with the real constraints, both cognitive and environmental, that players face. Updating beliefs over a large network of players is a significant challenge, rendering the costs of memory and computation highly non-trivial. Moreover, such a complex problem necessitates that players restrict their thinking to a relatively limited set of the most salient variables. Though player reasoning is imperfect, note that if $G$ is a complete network and players share the same initial beliefs, our model coincides exactly with that of Bolton and Harris (1999). This suggests that players do not stray far from full rationality, and our slight relaxation allows fruitful analysis of a novel problem.

Our experimentation game offers an instantiation of a more general framework for modeling games on networks in which we constrain players to condition behavior on a limited set of locally observable variables. In the present paper, players condition on degrees and the belief levels $\{p_i(t)\}$, but we could model a more general space of player types. A player considers only the behavior of a limited set of neighbors who directly impact her payoff. She forms beliefs about each neighbor’s type, and a strategy profile giving actions as a function of type induces expectations on each neighbor’s action. We can then define a local Bayesian equilibrium as above. This framework entails a number of modeling choices regarding what variables are salient and what players’ local beliefs are, necessitating some caution in any application. We can think of this as a relatively sophisticated heuristic decision process that captures some important aspects of rational behavior. I examine the general framework in
more detail in a forthcoming paper.

5.2 Individual Decisions and Local Equilibrium

I first characterize how beliefs evolve in response to experimentation and derive a Bellman equation expressing players’ expected payoffs as a function of their beliefs. These are analogous to results of Bolton and Harris (1999), and I relegate the proofs to an appendix.

Define $\Phi(p) = p(1-p){H-L \over \sigma}$, let $\nu_p \equiv (1-p)L + pH$ denote the expected payoff from the risky action under belief $p$, and let $b \equiv \mu_0L/H-L$ denote the value of $p$ for which $\nu_p = \mu_0$.

The following lemma describes belief evolution of player $i$ as a function of the action profile $\{\alpha_j\}_{j=1}^N$.

**Lemma 1.** The random process $p_i(t)$ evolves as

$$dp_i(t) = \Phi(p_i(t)) \sum_{j \in G_i} \sqrt{\alpha_j(t)} dZ_j^1(t).$$

This lemma implies that, conditional on players’ actions, the instantaneous change in beliefs $dp_i(t)$ is normally distributed with mean zero and variance $\Phi(p_i(t)) \sum_{j \in G_i} \alpha_j(t) dt$. Given a strategy profile $s$ for the other players, player $i$ anticipates a normally distributed belief change with mean zero and variance $\Phi(p_i(t)) (\alpha_i(t) + d_i \alpha_s(p)) dt$. This leads us to a Bellman equation describing the expected payoff from playing a best reply values to a given strategy profile.

**Lemma 2.** Suppose the players $j \neq i$ adopt the strategy profile $s$. The value function for player $i$ is the unique solution of

$$u(d_i, p) = \max_{\alpha \in [0,1]} \left( (1-\alpha)\mu_0 + \alpha \nu_p + \frac{1}{r} (\alpha + d \alpha_s(p)) \frac{\Phi(p) \partial^2 u(d_i, p)}{2 \partial p^2} \right).$$

Lemma 2 also implicitly defines the best reply strategy profile $s^*$. From the Bellman equation for the value function $u$, we can see that a strategy profile $s^*$ is a best reply to $s$ if and only if

$$s^*(d, p) = \begin{cases} 0 & \text{if } \frac{\Phi(p) \partial^2 u(d, p)}{2r} < \mu_0 - \nu_p, \\ 1 & \text{if } \frac{\Phi(p) \partial^2 u(d, p)}{2r} > \mu_0 - \nu_p. \end{cases} \quad (5.1)$$

This expression neatly captures the factors that influence the decision to experiment. The threshold $\mu_0 - \nu_p$ is the opportunity cost of experimentation, while the term $\frac{\Phi(p) \partial^2 u(d, p)}{2r}$ represents the value of experimentation. This value comprises the discount factor $1/r$, the informativeness of experimentation $\Phi(p)$, and the shadow price of information $1/2 \partial^2 u/dp^2$. An optimal strategy experiments whenever the value of experimentation exceeds the opportunity cost and refrains whenever this value is lower than the cost.
5.2.1 Properties of Equilibrium Strategies

The Bellman equation in Lemma 2 directly implies several useful properties of the value function.

**Lemma 3.** Let \( s \) and \( \hat{s} \) denote two strategy profiles, and let \( u \) and \( \hat{u} \) denote the value functions associated with playing best replies to \( s \) and \( \hat{s} \) respectively. We have

1. \( \max(\mu_0, \nu_p) \leq u \leq (1 - p)\mu_0 + pH \);
2. \( \frac{\partial^2 u}{\partial p^2} \geq 0 \);
3. If \( s \geq \hat{s} \), then \( u \geq \hat{u} \);
4. \( u(d, p) \geq u(\hat{d}, p) \) for all \( d \geq \hat{d} \).

**Proof.** Property (a) is immediate since the upper bound is the complete information payoff, and the lower bound is attained using a myopic strategy. For property (b), observe that the Bellman equation defining \( u \) implies

\[
u \mu_0 + (1 + d\alpha_s(p)) \frac{\Phi(p)}{2r} \frac{\partial^2 u}{\partial p^2} \geq u \]

with at least one equality. Suppose the first is an equality. If \( \alpha_s(p) = 0 \), then \( u(d, p) = \mu_0 \); a minimum is attained, so we must have \( \frac{\partial^2 u}{\partial p^2}(d, p) \geq 0 \). Otherwise,

\[
\frac{\Phi(p)}{2r} \frac{\partial^2 u}{\partial p^2} = \frac{u - \mu_0}{d\alpha_s(p)} \geq 0.
\]

Now suppose the second holds with equality. This implies

\[
\frac{\Phi(p)}{2r} \frac{\partial^2 u}{\partial p^2} = \frac{u - \nu_p}{1 + d\alpha_s(p)} \geq 0.
\]

Hence, property (b) holds everywhere.

For property (c), the function \( u \) solves

\[
u \frac{\Phi(p)}{2r} \frac{\partial^2 u}{\partial p^2} = \max_{\alpha \in [0,1]} \left( (1 - \alpha)\mu_0 + \alpha\nu_p + \frac{1}{r} (\alpha + d\alpha_s(p)) \frac{\Phi(p)}{2} \frac{\partial^2 u(d, p)}{\partial p^2} \right),
\]

which by property (b) implies

\[
u \frac{\Phi(p)}{2r} \frac{\partial^2 u}{\partial p^2} = \max_{\alpha \in [0,1]} \left( (1 - \alpha)\mu_0 + \alpha\nu_p + \frac{1}{r} (\alpha + d\alpha_s(p)) \frac{\Phi(p)}{2} \frac{\partial^2 u(d, p)}{\partial p^2} \right).
\]

Comparing with the Bellman equation defining \( \hat{u} \) shows that \( u \geq \hat{u} \). Property (d) follows analogously. \( \square \)
The final three properties all constitute different ways of saying that information has value. Payoffs are increasing in neighbors’ experimentation, so a player benefits from either having more neighbors or having neighbors that engage in more experimentation. Figure 5.1 shows the value function for a typical player; the upper line is the full information payoff $(1 - p)\mu_0 + pH$, and the lower line is the myopic payoff $\max(\mu_0, \nu_p)$. Lemma 3 implies that the value function is convex and increasing in $p$ as shown in the figure. The slope of the myopic payoff line provides an upper bound $H - L$ on the derivative of $u$. Furthermore, since $\Phi(p)$ converges to zero as $p$ approaches zero, the value of experimenting does as well, and no player experiments below some positive belief level.

![Figure 5.1: A typical value function](image)

Using Lemma 3 together with the best reply characterization, we can infer that the highest degree players begin experimenting at more pessimistic belief levels than all other players. In this sense, high degree players are the “early adopters” when people are initially skeptical of the innovation. These players are more willing to take risks at low belief levels because they have the most to gain from promising results. This shows the dominance of the encouragement effect at low belief levels: high degree players are willing to experiment because it may induce their neighbors to gather even more information.
Proposition 8. Let $\bar{d}$ denote the highest player degree in the network, and let $s$ denote an equilibrium strategy profile. The profile $s$ satisfies
\[ \inf\{p : s(\bar{d}, p) > 0\} \leq \inf\{p : s(d, p) > 0\} \]
for all player degrees $d$.

Proof. Define $p = \inf\{p : \alpha_s(p) > 0\}$. For all $p \leq p$ we have $u(d, p) = \mu_0$ and \( \frac{\partial^2 u(d, p)}{\partial p^2} = 0 \). Since $u(\bar{d}, p) \geq u(d, p)$ for all player degrees $d$, there is an interval $(p, p + \epsilon)$ in which \( \frac{\partial^2 u(d, p)}{\partial p^2} \geq \frac{\partial^2 u(d, p)}{\partial p^2} \), with strict inequality for $d < \bar{d}$. This together with equation (5.1) implies the statement. \( \square \)

I next provide an alternative characterization of best reply strategies as a function of neighbors’ expected experimentation, allowing us to clearly distinguish the free-rider effect and the encouragement effect.

Proposition 9. Let $u(d, p)$ denote the best reply value function for the profile $s$. Define
\[ \beta(d, p) = \frac{u(d, p) - \mu_0}{d(\mu_0 - \nu_p)}. \]
A strategy profile $s^*$ is a best reply to $s$ if and only if
\[ s^*(d, p) = \begin{cases} 0 & \text{if } \beta(d, p) < \alpha_s(p) \text{ and } p < b \\ 1 & \text{if } \beta(d, p) > \alpha_s(p) \text{ or } p \geq b. \end{cases} \quad (5.2) \]

Proof. Assume $p < b$, and consider three cases. First, suppose $\gamma \equiv \Phi(p) \frac{\partial^2 u(d, p)}{\partial p^2} < \mu_0 - \nu_p$. From the Bellman equation, this means $s^*(d, p) = 0$, and we have
\[ u(d, p) = \mu_0 + \gamma d \alpha_s(p), \]
which implies $\beta(d, p) < \alpha_s(p)$. Next, suppose $\gamma = \mu_0 - \nu_p$. In this case, any effort allocation is optimal, and a similar calculation shows $\beta(d, p) = \alpha_s(p)$. Finally, if $\gamma > \mu_0 - \nu_p$, then $s^*(d, p) = 1$, and we have
\[ u(d, p) = \nu_p + \gamma (1 + d \alpha_s(p)) \implies u(d, p) > \mu_0 + \gamma d \alpha_s(p), \]
which immediately implies $\beta > \alpha_s(p)$. Since we have exhausted all possibilities when $p < b$, the first line of equation (5.2) follows. The second line is immediate from the third case and the Bellman equation. \( \square \)

There are two ways to see the free-rider effect and the encouragement effect in this characterization. First, we can understand the effects via the expected neighbor experimentation $\alpha_s(p)$. An increase in $\alpha_s(p)$ directly reduces the best reply profile $s^*$ by increasing the decision threshold; this is the free-rider effect. However, Lemma 3 implies that utility increases,
so there is a corresponding increase in $\beta$ that shifts $s^*$ in the opposite direction. We can also see the two effects at work across different player degrees. Increases in $d$ raise the denominator of $\beta(d, p)$, expressing a stronger free-rider effect that discourages experimentation. The encouragement effect appears in the numerator of $\beta(d, p)$ as $u(d, p)$ increases.

Although for any particular player the relative importance of the two effects is ambiguous, for any fixed $p$ the free-rider effect clearly dominates for sufficiently high player degrees because $u$ is bounded above by $(1 - p)\mu_0 + pH$. In an extreme case, we could consider an infinite network in which $F$ has full support on $\mathbb{N}$. Proposition 9 then implies that for any $p < b$, all players with sufficiently high degree must refrain from experimenting. In light of Proposition 8, this highlights the complex interplay between the encouragement and free-rider effects in a network: equilibrium strategies are non-monotone in beliefs. High degree players have the most to gain both from free-riding and from encouraging; they are the first to start experimenting, but they may drop out when others join.

5.2.2 Existence of Equilibrium

I now use the characterization of best replies to show that an equilibrium exists.

Theorem 10. A local Bayesian equilibrium exists.

Proof. Let $U$ denote the set of Lipschitz continuous functions $v : [0, 1] \to [\mu_0, H]$ such that $0 \leq u' \leq H - L$ almost everywhere, and let $V$ denote the set of Borel measurable functions $\alpha : [0, 1] \to [0, 1]$. I define a map from $U^N \to U^N$. We can interpret an element $u \in U^N$ as a vector of single variable value functions for every possible degree $d$, with $u(d, p)$ the $d$th component.

Define the function $\phi(u) = \sup \{ x : \mathbb{P}_F(\beta(D, p) \geq x) \geq x \}$, mapping elements of $U^N$ to $V$. Let $\psi$ denote a function from $V$ to $U^N$ with $\psi(\alpha)$ returning the value function corresponding to best replies when $\mathbb{E}_F[s(D, p)] = \alpha(p)$. Proposition 9 implies $u$ represents an equilibrium value function if and only if it is a fixed point of $\psi \circ \phi$. The function $\phi$ is clearly non-decreasing, and $\psi$ is non-decreasing by Lemma 3. Tarski’s fixed point theorem implies the existence of minimal and maximal fixed points, and any fixed point can be supported as an equilibrium outcome via the strategies defined in Proposition 9.

5.3 Long Run Collective Behavior

While beliefs about immediate neighbors determine individual decisions, the broader context may have implications for aggregate welfare and the long run success or failure of innovations. I focus here on asymptotic outcomes of the learning process and how the network $G$ impacts these outcomes. In particular, I analyze the likelihood that players discard a good innovation and the total amount of experimentation they expend on bad innovations.

Definition 5. If $\lim_{t \to \infty} \alpha_i(t) = 1 (0)$, we say player $i$ adopts (abandons) the innovation. If all players adopt (abandon) the innovation, we say that society adopts (abandons) the
innovation. Let $A_0$ denote the event that society abandons the innovation and $A_1$ the event that society adopts the innovation.

The total experimentation of player $i$ through time $t$ is

$$\eta_i(t) = \int_0^t \alpha_i(s) \, ds.$$  

The total experimentation in society through time $t$ is

$$\eta(t) = \sum_{i=1}^N \eta_i(t).$$

Long-run behavioral conformity is a robust feature of this model.

**Theorem 11.** If $G$ is connected, then with probability one either society adopts the innovation or society abandons the innovation. That is,

$$\mathbb{P}(A_0) + \mathbb{P}(A_1) = 1.$$  

**Proof.** From Lemma 1, the beliefs of each player evolve according to a martingale, so they must converge almost surely. Since $\Phi(p)$ is positive on $(0, 1)$, this means for each player $i$ either $\lim_{t \to \infty} p_i(t) \in \{0, 1\}$ or $\sum_{j \in G_i} \alpha_j(t)$ converges to zero. The former implies that player $i$ learns the true state, while the latter implies that the player and all neighbors abandon the innovation. Since a player who learns the true state either abandons or adopts the innovation, according to whether the limit belief is 0 or 1, each individual player must with probability one either abandon or adopt the innovation.

Suppose one player adopts the innovation. Since this player continues experimenting indefinitely, she learns the true state, and continuing to experiment is optimal only if the true state is $H$. All players observing this experimentation must also learn that the true state is $H$, and will therefore also adopt the innovation. Iterating the argument over a connected network implies that all players adopt the innovation. The only other possibility is that all abandon the innovation. 

Theorem 11 is similar in spirit to payoff equalization results in the social learning literature using both Bayesian and Non-Bayesian approaches. In this boundedly rational framework, players still consistently estimate the underlying state, and the same intuition applies. Since continuing to experiment means learning the true value of the innovation, one adopting player means the innovation must be good, and all players eventually learn this in a connected network. In case of adoption, players reach a belief consensus, but this need not occur if the innovation is abandoned. Some players will quit using the innovation before others, those with different degrees will have different belief thresholds for abandonment, and some may observe additional negative information their neighbors gather. When a player quits using the innovation, this occurs after observing a finite amount of experimentation, implying an asymmetry between outcomes in the two states.
Corollary 4. If the innovation is bad, society abandons it with probability one. If the innovation is good, society abandons it with positive probability. We have

$$\mathbb{P}(A_0 | \mu_1 = L) = 1, \quad \text{and} \quad \mathbb{P}(A_0 | \mu_1 = H) > 0.$$  

This asymmetry motivates two long-run metrics that I study. First, I consider how the network structure affects the probability of abandoning a good innovation $$\mathbb{P}(A_0 | \mu_1 = H)$$. This is a first order concern for a patient social planner, and a variation of this question is the focus in a large segment of the social learning literature. Although bad innovations are always abandoned eventually, we may also care about how much total experimentation $$\eta(\infty)$$ occurs before a bad innovation is rejected. In general, there is a direct tradeoff between minimizing the probability of abandoning a good innovation and minimizing the experimentation required to stop using a bad one.

Suppose $$\mathbf{s}$$ is an equilibrium strategy profile, and define the threshold

$$p_d = \sup \{ p : \mathbf{s}(d, p) = 0 \}.$$  

If player $$i$$ abandons the innovation, we must have $$p_i \leq p_{d_i}$$. Now define

$$\bar{Y}_i = \ln \left( \frac{1 - p_{d_i}}{p_{d_i}} \right) - \ln \left( \frac{1 - p_i(0)}{p_i(0)} \right),$$

and let $$\bar{Y}$$ denote the corresponding vector of thresholds. We can think of $$\bar{Y}_i$$ as a measure of how much bad news player $$i$$ can tolerate before stopping experimentation. Consider the linear program

$$\begin{align*}
\min_{\bar{Y}} & \quad \sum_{i \leq N} Y_i \\
\text{s.t.} & \quad G \bar{Y} \geq \bar{Y} \\
& \quad \bar{Y} \geq 0,
\end{align*}$$

We obtain the following bounds.

**Theorem 12.** Let $$y^*$$ denote the minimal objective value for the problem (5.3). We have

$$\mathbb{P}(A_0 | \mu_1 = H) \leq e^{-y^*} \quad \text{and} \quad \mathbb{E}[\eta(\infty) | \mu_1 = L] \geq \frac{2\sigma^2}{(H - L)^2} y^*.$$  

**Proof.** I introduce an alternative representation of belief evolution in the network. The experimentation of player $$i$$ at time $$t$$ generates a normally distributed signal $$X_i(\eta_i(t))$$ with mean $$\eta_i(t)\mu_1$$ and variance $$\eta_i(t)\sigma^2$$. Given a realization of $$X_i$$, the associated likelihood ratio is

$$l_i(t) \equiv \frac{d\mathbb{P}(X_i | \mu_1 = L)}{d\mathbb{P}(X_i | \mu_1 = H)} = e^{\frac{1}{2\sigma^2}(2(L-H)X_i + \eta_i(t)(H^2-L^2))}.$$  

We shall focus on the logarithm of the likelihood ratio process $$Y_i(t) = \ln l_i(t)$$. Conditional on the realization of $$\mu_1$$, this process is a time-changed Brownian motion with drift. The
total experimentation $\eta_i$ is the “clock” of the Brownian motion, and hereafter I shall write $Y_i$ as a function of $\eta_i$. Conditional on $\mu_1 = L$ we have
\[
dY_i(\eta_i) = \frac{1}{2\sigma^2}(H - L)^2d\eta_i + \frac{H - L}{\sigma}dB_i(\eta_i),
\]
where $B_i$ is a standard Brownian motion. Similarly, conditional on $\mu_1 = H$ we have
\[
dY_i(\eta_i) = -\frac{1}{2\sigma^2}(H - L)^2d\eta_i + \frac{H - L}{\sigma}dB_i(\eta_i).
\]
Note the processes $\{Y_i\}_{i \leq N}$ are mutually independent conditional on the experimentation levels. All dependencies stem from correlated experimentation rates.

I note two well-known facts about Brownian motions that will prove useful. First, let $X(t) = \sigma B(t) + \mu t$ be a Brownian motion with drift $\mu < 0$ and variance $\sigma^2 t$. Suppose $X(0) = 0$, and let $M = \max_{t \geq 0} X(t)$ denote the maximum the process attains. The probability that $M$ is above some threshold $x > 0$ is
\[
P(M > x) = e^{-\frac{\sigma^2}{2\mu}x}.
\]
This will allow us to bound the probability of abandonment. If we suppose instead that the drift $\mu$ is positive, then the expected hitting time of $x > 0$ is
\[
E(T_x) = \frac{x}{\mu}.
\]
This will allow us to bound the expected total experimentation.

The belief of agent $i$ at time $t$ is
\[
p_i(t) = \left(1 + \frac{1 - p_i(0)}{p_i(0)} e^{\sum_{j \in G_i} Y_j(\eta_j)}\right)^{-1},
\]
and note that
\[
\sum_{j \in G_i} Y_j \geq \ln \left(\frac{1 - \bar{p}_d_i}{\bar{p}_d_i}\right) - \ln \left(\frac{1 - p_i(0)}{p_i(0)}\right) \equiv Y_i
\]
is necessary for player $i$ to abandon the innovation. Suppose $\mu_1 = H$, and let $x$ be an $N$-dimensional real vector. The probability that we ever have $Y \geq x$ is no more than
\[
\prod_{i \leq N} P \left(\max_{\eta_i \geq 0} Y_i(\eta_i) \geq x_i\right) = e^{-\sum_{i \leq N} \max(0, x_i)}.
\]
The linear program 5.3 exactly maximizes this probability subject to the necessary condition for abandonment. Similarly, conditional on $\mu_1 = L$, the expected total experimentation that player $i$ observes must satisfy
\[
\sum_{j \in G_i} E[Y_j(\infty) \mid \mu_1 = L] \geq \frac{2\sigma^2}{(H - L)^2} Y_i.
\]
The same linear program describes minimal expected experimentation in the network.
Theorem 12 directly ties the probability of abandonment and the expected total experimentation to the network structure and the distribution of initial beliefs. The proof offers as much insight as the statement. We can interpret the objective vector $Y$ as a scaled allocation of experimental effort that leads to abandonment. The constraint $GY \geq Y$ represents a bound on the amount of experimentation each player must observe before abandoning the innovation. As a general rule, we minimize total experimentation when we allocate effort to those who are most widely observed. Such allocations induce the greatest shifts in societal beliefs for a given amount of experimentation. Some examples will illustrate the broader implications of these findings.

5.3.1 Network Density

Define $G'_i = \{j \mid G_{ji} = 1\}$ as the set of players who observe player $i$, and consider a network in which all players have the same visibility: we have $|G'_i| = k$ for all $i$ and some integer $k$. If we further suppose that each player $i$ begins the game with beliefs $p_i(0) \geq p_d$, the optimization problem (5.3) now admits a particularly simple solution: if $y = \frac{1}{N} \sum_{i \leq N} Y_i$, then the minimum is simply $\frac{p}{k}$. Theorem 12 implies the following scaling result.

**Corollary 5.** Suppose $|G'_i| = k$ for each $i$, and the average abandonment threshold is $\bar{y}$. We have

$$
\mathbb{P}(A_0 \mid \mu_1 = H) \leq e^{-\frac{N}{k} \bar{y}} \quad \text{and} \quad \mathbb{E}[(\eta(\infty)) \mid \mu_1 = L] \geq \frac{2\sigma^2}{(H-L)^2} \frac{N}{k} \bar{y}.
$$

Fixing the distribution of degrees and initial beliefs, the probability of suboptimal abandonment declines exponentially with the size of the network, and the total experimentation increases linearly. The exponent and the slope in the respective bounds are smaller when the network is more dense. This indicates that on average dense networks experiment less, making more long-run mistakes.

A comparison with the complete network is instructive. Assuming that $p_i(0) = p_0$ for each $i$, the results of Bolton and Harris (1999) imply that

$$
\mathbb{P}(A_0 \mid \mu_1 = H) \geq \left[ \frac{1}{1-p_0} \frac{H - \mu_0}{\mu_0 - L} \frac{4r \sigma^2}{N} (1 + \zeta) \right]^{-1},
$$

and

$$
\mathbb{E}[(\eta(\infty)) \mid \mu_1 = L] \leq \frac{2\sigma^2}{(H-L)^2} \left[ \ln \left( \frac{p_0(H - \mu_0)}{(1-p_0)(\mu_0 - L)} \right) + \ln \left( 1 + \frac{N(H-L)^2}{4r \sigma^2} (1 + \zeta) \right) \right],
$$

where $\zeta = \sqrt{1 + \frac{8r \sigma^2}{N(H-L)^2}}$. This means the probability of abandonment declines no faster than $\frac{1}{N}$, and the expected total experimentation scales at most logarithmically. The different scaling rate reflects that for large $N$, networks with bounded degrees become increasingly sparse relative to the complete network, requiring relatively more information before abandoning an innovation.
5.3.2 Centralized Experimentation

Consider the two networks in Figure 5.2. In network (a), players each observe a single neighbor, forming a ring structure, while network (b) has a central player whom all others observe. To simplify the example, imagine that the central player in network (b) observes and is observed by a single other player with initial belief zero; hence, all players have the same degree in both networks, and all face the same individual decision problem. Suppose all players in network (a) and all peripheral players in network (b) have the same initial beliefs $p_0 > \overline{p}$, where $\overline{p}$ is the threshold at which players cease experimenting. Define

$$\bar{y} = \ln \left( \frac{1 - \overline{p}}{\overline{p}} \right) - \ln \left( \frac{1 - p_0}{p_0} \right).$$

Corollary 5 implies that in network (a), we have

$$P(A_0 \mid \mu_1 = H) \leq e^{-\frac{\bar{y} N}{2}}$$

where $N$ is the number of players in the ring.

![Figure 5.2: Centralized Experimentation](image)

Compare this with network (b) as we vary the initial beliefs $p_0^c$ of the central player. If $p_0^c < \overline{p}$, the central player will never experiment, and the likelihood of total abandonment $e^{-\overline{p} N}$ is less than in network (a) because all of the peripheral players experiment independently. Now consider what happens if the central player has a much higher initial belief. Let $A$ denote the event that the central player abandons a good innovation, and let $P_a$ denote the probability that a peripheral player abandons a good innovation conditional on $A$. The probability that network (b) abandons a good innovation is then at least

$$P(A)P_a^N,$$

where $N$ is the number of peripheral players. As $p_0^c$ approaches 1, event $A$ implies more negative experimental results; fixing $p_0$, we have $P_a$ approaching 1. In particular, for sufficiently high $p_0^c$, we have $P_a > e^{-\frac{\bar{y}}{2}}$, which means that for sufficiently large $N$, the probability of abandonment in network (b) is greater than that in network (a).
Central players can create more correlation in the abandonment decisions of others. In this example, if the central player carries out a lot of experimentation with negative results, all peripheral players observe this, and all of them are far more likely to abandon the innovation.\(^5\) Since beliefs determine experimentation, whether the central player introduces these correlations depends crucially on her initial beliefs. Perhaps counterintuitively, increasing one player’s perceived value of an innovation may end up reducing its ultimate likelihood of adoption.

### 5.3.3 Seeding

Suppose that all players in a network begin with extremely pessimistic beliefs \(p_0 = \epsilon > 0\). Now imagine that we can “seed” the network by exogenously increasing the beliefs of a subset of players. This type of seeding could represent an intervention via an educational program or subsidies. How should we select individuals for the intervention if our goal is to maximize the likelihood of long-run adoption?

![Figure 5.3: Seeding Strategies](image)

Figure 5.3 illustrates two possible seedings in a particularly simple network. Let \(p\) denote the belief threshold below which individuals stop experimenting, suppose the red nodes are seeded with the belief level \(p' > p\), and define

\[
y' = \ln\left(\frac{1 - \frac{p}{p'}}{p}\right) - \ln\left(\frac{1 - p'_0}{p'_0}\right).
\]

Since the unseeded nodes have such low belief levels, we can essentially ignore them in estimating the probability of abandonment: if one of the seeds adopts, the network will eventually adopt, otherwise the innovation is abandoned.

In seeding (a), there are two independent chances to adopt, and the probability of abandoning a good innovation is approximately \(e^{-2y'}\). In seeding (b), the two seeds share identical beliefs unless and until the unseeded players join in using the innovation. Since their beliefs

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\(^5\)The role of the central player here is similar to that of a “Royal Family” in the model of Bala and Goyal (1998).
move together, we in essence have only one seed, and the probability of abandoning a good innovation is approximately $e^{-y'}$. Placing seeds adjacent to each other in this network eliminates independent experimentation and therefore reduces the amount of information the network will gather. More generally, when seeds share information with one another, their decisions to adopt or abandon are positively correlated, and this can reduce the long-term likelihood of adoption.

### 5.4 Remarks

Information sharing networks are important drivers of innovation diffusion in firms, communities, and other organizations. When individuals must engage in costly experimentation to learn about an innovation, the network structure has complex effects on incentives to gather information and on long term adoption patterns. A key tradeoff occurs between gathering information and efficiently sharing information. The network structure and individual beliefs jointly determine how the group as a whole conducts this tradeoff. When individuals who are separated from one another gather most of the information, the group is less likely to reject useful innovations, but it takes longer to eliminate inferior ones.

These findings have implications for seeding innovations within skeptical communities. In contexts requiring individuals to experiment and learn about a new technology, seeding individuals in relative isolation, rather than in clusters, may render long-run acceptance more likely. This recommendation contrasts with our intuition for seeding strategies when local payoff externalities influence adoption decisions. In these cases, the decision to adopt is part of a local coordination game, and a behavior or technology will quickly die out if early adopters are isolated from one another. This suggests that identifying the mechanisms of peer influence in different contexts is important for designing appropriate interventions.

Information sharing patterns partly determine the extent to which a group gathers information before its members collectively accept or reject an innovation. In this sense, we can interpret network structure as an expression of collective time or risk preference. Sparse information sharing networks correspond to groups that are relatively patient or less risk averse regarding new technologies or practices. When jobs within a firm or other organization require risky experimentation by individuals, the structure of the organization may play a role in aligning individual incentives with organizational objectives.

The notion of a local Bayesian equilibrium is a novel contribution in this paper. This solution concept requires us to think carefully about how we specify players’ beliefs; in this game, we sought simple and transparent assumptions that reflect intuitive features of the problem. The power of this approach lies in simplifying the analysis of individual choices while preserving key aspects of rational decision making. Players follow a relatively sophisticated heuristic that we can use to better understand strategic behavior in complex settings.

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6 This occurs in the model of Morris (2000) for instance.
Chapter 6

Final Remarks

A major contribution of this dissertation is a new solution concept, local Bayesian equilibrium. This concept fills a gap in our theoretical tool kit, providing a decision framework that is neither fully rational nor entirely mechanistic. My approach to simplifying individual decisions centers on the idea of locality: players model local neighborhoods while ignoring details about the size and structure of the larger network. Like a number of other approaches to dealing with cognitive limitations, local models represent a version of coarse reasoning. Empirical researchers have only just begun exploring how individuals make choices in a network, but recent results suggest that locality is a realistic feature of human decision making. Recent laboratory studies show that even when more information about the network is available, subjects follow degree-based heuristics (Chandrasekhar et al., 2012; Gallo and Yan, 2014). The notion of locality also suggests an explanation for disagreement among individuals in a large network: the structure of the network can induce arbitrary sets of local beliefs.

I study two mechanisms through which people may influence the choices of their neighbors, and each mechanism has different implications for how behaviors diffuse in a network. When decisions are made quickly, and there is little uncertainty about the value of the new product in question, we should expect dense and centralized networks to facilitate faster and more widespread diffusion. When people learn about an innovation over time through use, these network features have the additional effect of reducing incentives to gather information and may produce more uncertainty in the outcome. This suggests different approaches to encouraging widespread adoption, depending on what mechanisms are most important for individual decisions. Both mechanisms allow for volatility in the adoption path of any particular item while accounting for overall trends that depend on network structure.

Many of my results depend on a key technical innovation, which characterizes the component structure of a random graph model. I establish a correspondence between the components in a multitype configuration model and a branching process approximation. This allows us to leverage extensive mathematical results for branching processes to understand the structure of random graphs and their diffusion properties. By extending this characterization to multitype graphs, we are able to capture important properties of real networks—homophily, clustering—in a tractable framework, and we can study the effects of local information.
With this set of tools available, the multitype configuration model becomes a viable option to serve as a standard way to represent social networks in economic models. Moreover, since the branching process approximation characterizes the network structure using a relatively small number of local parameters, this model has potential empirical applications as well.

While the contributions of this dissertation, both substantive and methodological, should improve our understanding of behavior in networks and facilitate more extensive work in this area, many significant challenges remain. Local Bayesian equilibrium may prove a double-edged sword. I believe it offers a valuable tool, but it also necessitates great care in our modeling choices. As with many efforts to model bounded rationality, there is a risk that our conclusions result from our particular arbitrary choices rather than more fundamental features of the situations we study. Fortunately, the tractability that this approach affords should allow us to explore sensitivity to our assumptions more completely, which at least partly mitigates this concern. Both of the models analyzed yield qualitative predictions about diffusion patterns that call for empirical verification. The empirical networks literature is now quite extensive, but I believe still in its infancy. Much of the current research still deals with the very difficult issues around identification and disentangling causal effects from homophily. A few recent efforts have begun testing theoretical predictions, and I am hopeful that the work I have presented can facilitate more such efforts.
Appendix A

Diffusion Games

This appendix is dedicated to the proof of Theorem 7. Although the argument is substantively identical to the proof of Bollobás and Riordan (2015) for the single type case, I choose to present the proof with the necessary modifications because it is fairly involved and likely unfamiliar to my audience. The results here allow us to use a branching process approximation to do much more than characterize the component structure for the configuration model. We can completely capture any “local” property of the random graph—I make this notion precise later—using a branching process to approximate.

I prove a stronger result than Theorem 7, which immediately implies it. To state the result, I must define a few terms and notation. I use \( \mathbf{d} = \{ (d_1, d_2, ..., d_\Theta) \}_{i=1}^N \) to denote the sequence of realized types and degree tuples. Fixing \( \mathbf{d} \), I write \( m_{\theta, \theta'}(\mathbf{d}) \) for the number of edges connecting type \( \theta \) nodes and type \( \theta' \) nodes, and I write \( n_{\theta, \mathbf{d}}(\mathbf{d}) \) for the number of type \( \theta \) nodes with degree tuple \( \mathbf{d} \). Finally, I define the configuration distance as

\[
l(\mathbf{d}, D) = \max \left\{ \frac{1}{N} \sum_{\theta=1}^{\Theta} \sum_{d} \left| \frac{n_{\theta, \mathbf{d}}(\mathbf{d})}{N} - \overline{a}_{\theta} \mathbb{P}(D_\theta = d) \right| \right\},
\]

where \( \overline{d} = \sum_{\theta=1}^{\Theta} d_\theta \) is the total degree associated with \( \mathbf{d} \). This is a measure of how much the realized degree sequence deviates from that prescribed by the distribution \( D \). It is straightforward to check that \( l(\mathbf{d}, D) \) converges to zero as \( N \) grows if and only if the realized degree sequence converges to \( D \) both in distribution and in expectation. Hence, the law of large numbers implies that the configuration distance converges to zero almost surely as \( N \) goes to infinity. The following theorem is our main result.

**Theorem 13.** For any \( \epsilon > 0 \) and any \( k \geq 1 \), there exists \( \delta > 0 \) such that if \( l(\mathbf{d}, D) < \delta \) we have

\[
\mathbb{P} \left( |N_k(G) - N \mathbb{P}(|T| = k)| \geq \epsilon N \right) \leq e^{-\delta N}.
\]

If additionally the degree distribution \( D \) assigns positive probability to some type having 3 or more neighbors, then there exists \( \delta > 0 \) such that if \( l(\mathbf{d}, D) < \delta \) we have

\[
\mathbb{P} \left( |L_1(G) - N \mathbb{P}(|T| = \infty)| \geq \epsilon N \right) \leq e^{-\delta N}, \quad \text{and} \quad \mathbb{P}(L_2(G) \geq \epsilon N) < e^{-\delta N}.
\]

(A.1) (A.2)
I first prove Theorem 13 for a configuration model that allows multigraphs. That is, we generate the network as described, but we do not condition on realizing a simple graph: we permit self-links and multiple links between the same pair of nodes. This version of the configuration model is easier to work with because we can allow the stubs to be paired uniformly at random without any further conditioning. After proving the result for the multigraph case, transitioning to simple graphs is fairly easy.

I will use $G^*$ to denote a graph realized according to the multigraph configuration model. Depending on the context, I use $G$ to denote either a graph realized according to the configuration model or a generic graph. I use $v$ to denote a generic node in $G$, and $G_v$ to denote the graph $G$ rooted on $v$. I use $T$ to refer both to the branching process defined at the beginning of section 4 and the corresponding tree, viewed as a rooted graph. For a positive integer $r$, I write $G_{v,r}$ for the subgraph of radius $r$ rooted on $v$ (i.e. the graph comprising only nodes at distance $r$ or less from $v$), and similarly I write $T_r$ for the tree $T$ truncated after the $r$th generation. I will occasionally abuse notation, writing $G_{v,r} = T_r$ to indicate that $T_r$, viewed as a rooted graph, is isomorphic to $G_{v,r}$.

There are two major steps in the proof. The first step is to show that the distribution of component sizes converges to the distribution of tree sizes generated by the branching process $T$. We can prove this for the multigraph case using a straightforward coupling argument, matching the branching process with a breadth first search process starting at a random node. However, passing to simple graphs later requires a more powerful concentration result, giving exponential bounds on the rate of convergence. We obtain these bounds by applying the Azuma-Hoeffding inequality to a martingale that arises through a process that explores possible link stub pairings.

Once we establish the correspondence between component sizes and tree sizes, we show that essentially all “large” components are connected in one “giant” component. This relies on a coloring and sprinkling argument in which we first retain links independently with some probability $p \in (0, 1)$, and then sprinkle the remaining links back in, taking advantage of conditional independence between the retained links and the sprinkled links. Large components that exist in the thinned graph are likely to be connected by the additional links. The assumption that at least one type has three or more neighbors with positive probability is necessary for this step. It ensures that the survival probability of the thinned tree converges to that of $T$ as $p$ approaches 1. An argument showing that the results carry over if we condition on realizing a simple graph completes the proof.

The Branching Process Approximation

The first part of the proof establishes a coupling between rooted graphs of finite size $G^*_{v,r}$ and truncated trees $T_r$. This in turn implies that any property of the rooted graph $G^*_v$, which depends only on those vertices within distance $r$ of $v$, is asymptotically characterized by the branching process $T$. This is the sense in which $T$ captures the “local” properties of $G^*$. The bulk of this section is devoted to proving bounds on the probability of deviations.

**Lemma 4.** Let $v$ be a vertex of $G^*$ chosen uniformly at random, and suppose $\{d_N\}_{N \in \mathbb{N}}$ is a sequence for which $l(d_N, D)$ converges to zero. For any finite $r$, we can couple the
random graphs $G^*_{v,r}$ and $T_r$ so that they are isomorphic with probability approaching 1 as $N$ approaches infinity.

**Proof.** Begin with a realized sequence $d$, and suppose that $l(d, D) < \epsilon$ for some $\epsilon > 0$. We will reveal the rooted graph $G^*_{v,r}$ one node at a time, following a breadth first search procedure, coupling it with $T_r$ at each step and bounding the probability that the coupling fails. Given our assumption on the configuration distance, we can couple the degree $d$ of the root $v$ with the offspring distribution of the root of $T_r$ with probability at least $1 - \epsilon$.

At each subsequent step, we start with a node of some type $\theta$ and reveal a partner for a link of another type $\theta'$. At the $j$th step, the probability that this is an unvisited node with degree tuple $d$ is precisely $d_{\theta'}(n_{\theta',d}(d) - u_{\theta',d,j})$ $m_{\theta,\theta'}(d) - u_{\theta,\theta',j}$, where $u_{\theta',d,j}$ is the number of type $\theta'$ nodes with degree vector $d$ that have been visited before the $j$th step, and $u_{\theta,\theta',j}$ is the number of completed edges between type $\theta$ and type $\theta'$ nodes before the $j$th step. Note that $u_{\theta',d,j} \leq j$ and $u_{\theta,\theta',j} \leq j$. This implies that for any fixed $j$, the difference between this quantity and $\mathbb{P}(D_{\theta',\theta'} = d)$ is no more than $\epsilon + o(1)$, so the coupling succeeds with probability $1 - \epsilon - o(1)$.

To complete the proof, note that for any $\epsilon > 0$, there is a constant $M$ such that $|T_r| \leq M$ with probability at least $1 - \epsilon$, and for sufficiently large $N$, we have $l(d, D) < \epsilon$ with probability at least $1 - \epsilon$. For $N$ larger than this, the probability that the coupling fails is no more than $2\epsilon + M(\epsilon + o(1))$, and the conclusion follows.

One immediate consequence of this lemma is that the rooted graphs $G^*_{v,r}$ are trees with probability approaching 1. More generally, the branching process $T$ characterizes any “local” property of the graph $G^*$. Let $\mathcal{P}$ be a property of rooted graphs, meaning a set of rooted graphs that is closed under isomorphisms. We can also think of $\mathcal{P}$ as a property of vertices, taking the root of the graph as the relevant vertex. We write $G_v \in \mathcal{P}$ to say that the graph $G$ rooted on $v$ has the property $\mathcal{P}$ and we write $N_\mathcal{P}(G)$ for the number of vertices with property $\mathcal{P}$. For any positive integer $r$, we say that $\mathcal{P}$ is $r$-local if whether $G_v \in \mathcal{P}$ depends only on $G_{v,r}$. The following corollary is immediate from Lemma 4

**Corollary 6.** Let $\mathcal{P}$ be a $r$-local property of rooted graphs, let $v$ be a vertex of $G^*$ chosen uniformly at random, and suppose $\{d_N\}_{N \in \mathbb{N}}$ is a sequence for which $l(d_N, D)$ converges to zero. Then,

$$\lim_{N \to \infty} \mathbb{P}(G^*_v \in \mathcal{P}) = \mathbb{P}(\mathcal{T} \in \mathcal{P}).$$

Equivalently, for any $\epsilon > 0$, there exists $N_\epsilon$ such that if $N \geq N_\epsilon$ we have

$$|\mathbb{E}[N_\mathcal{P}(G^*)] - N\mathbb{P}(\mathcal{T} \in \mathcal{P})| \leq \epsilon N.$$

We require a slightly modified version of this result, which follows from the previous corollary.
Corollary 7. Let $\mathcal{P}$ be a $r$-local property of rooted graphs, and let $v$ be a vertex of $G^*$ chosen uniformly at random. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $l(d, D) < \delta$, then conditional on the degree sequence $d$ we have

$$|\mathbb{E}[N_{\mathcal{P}}(G^*)] - N\mathbb{P}(T \in \mathcal{P})| \leq \epsilon N.$$ 

We focus on the $k$-local property $\mathcal{P}_k$ that a vertex is in a graph component with exactly $k$ nodes, meaning

$$N_{\mathcal{P}_k}(G) = N_k(G), \quad \text{and} \quad \mathbb{P}(T \in \mathcal{P}_k) = \mathbb{P}(|T| = k).$$

Corollary 6 of course implies convergence of $\frac{N_k(G^*)}{N}$ to $\mathbb{P}(|T| = k)$, but we require a stronger bound on the rate of convergence. We make repeated use of the following concentration result.

Proposition 10. Let $\mathcal{P}$ be a $r$-local property of rooted graphs. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $l(d, D) < \delta$ then

$$\mathbb{P} \left( |N_{\mathcal{P}}(G^*) - N\mathbb{P}(T \in \mathcal{P})| \geq \epsilon N \right) \leq e^{-\delta N}.$$ 

The first step to obtain this bound is a lemma using the Azuma-Hoeffding inequality. Fixing a degree sequence $d$, we can consider different pairings of stubs. We say that two pairings $\pi_1$ and $\pi_2$ are related by a switching if we can obtain $\pi_2$ from $\pi_1$ by deleting two pairs of the same type $\{a, b\}$ and $\{c, d\}$ and replacing them with the pairs $\{a, d\}$ and $\{c, b\}$. Let $f$ be a real-valued function defined on pairings of $d$. We say that $f$ is $C$-Lipschitz if for any $\pi_1$ and $\pi_2$ related by a switching, we have $|f(\pi_1) - f(\pi_2)| \leq C$.

Lemma 5. Let $f$ be a $C$-Lipschitz function of pairings of some degree sequence $d$, let $M$ denote the total number of pairs. If $\pi$ is chosen uniformly at random from all pairings of $d$, then for any $r \geq 0$ we have

$$\mathbb{P} \left( |f(\pi) - \mathbb{E}[f(\pi)]| \geq r \right) \leq 2e^{-\frac{r^2}{2C^2M}}.$$ 

Proof. Let $S_\theta^\prime = \{s_1, s_2, ..., s_m\}$ denote the set of stubs leading from type $\theta$ nodes to type $\theta'$ nodes, with $S_\theta^\prime = \{s_1', s_2', ..., s_m'\}$ the set of potential partners. We consider a random process in which we sequentially reveal the pairing. Conditional on the partners of $s_1, ..., s_i$, let $\Omega$ denote the set of pairings between $S_\theta^\prime$ and $S_\theta^\prime$ that are consistent with the information revealed so far. For any possible partner $b$ of $s_{i+1}$, let $\Omega_b$ denote the subset of $\Omega$ containing all possible pairings in which $s_{i+1}$ is matched to $b$. For any two potential partners $b$ and $c$, there is a bijection between $\Omega_b$ and $\Omega_c$ in which each $\pi_1 \in \Omega_b$ is related by a switching to its image $\pi_2 \in \Omega_c$: just switch the pairs $\{s_{i+1}, b\}$ and $\{s_j, c\}$ to $\{s_{i+1}, c\}$ and $\{s_j, b\}$.

Iterate the revelation process over each type of link, and let $\mathcal{F}_i$ be the sigma-field generated by the sequential revelation process up to $s_i$. The process $X_i = \mathbb{E}[f(\pi) \mid \mathcal{F}_i]$ is clearly a martingale. The bijection together with the Lipschitz property implies that

$$\left| \mathbb{E}[f(\pi) \mid \mathcal{F}_i] - \mathbb{E}[f(\pi) \mid \mathcal{F}_{i+1}] \right| \leq C.$$ 

The sequence $\{X_i\}_{i=0}^M$, with $X_0 = \mathbb{E}[f(\pi)]$ and $X_M = f(\pi)$, is a martingale with differences bounded by $C$, and the result follows from the Azuma-Hoeffding inequality.
This lemma is sufficient to prove our concentration result for a local property $\mathcal{P}$ if $N_{\mathcal{P}}(G)$ is $C$-Lipschitz for some $C$, but this is not universally true for all local properties. However, if we modify the property to avoid high-degree vertices, we can obtain a $C$-Lipschitz function of the graph and use it to prove the concentration bounds. For $\Delta \geq 2$ and $r \geq 0$, let $\mathcal{M}_{\Delta,r}$ be the property of rooted graphs that every node within distance $r$ of the root has degree at most $\Delta$. This is a $r+1$-local property.

**Lemma 6.** Let $\mathcal{P}$ be a $r$-local property, and let $\mathcal{Q} = \mathcal{P} \cap \mathcal{M}_{\Delta,r}$. The number of vertices $N_{\mathcal{Q}}(G)$ with property $\mathcal{Q}$ is $16\Delta^r$-Lipschitz.

**Proof.** Suppose $v$ is a vertex of $G$ such that exactly one of $G_v$ and $(G + e)_v$ has property $\mathcal{Q}$, for some edge $e$. This implies that $G_v$ has property $\mathcal{M}_{\Delta,r}$ since removing an edge can only reduce the degree of a vertex. Suppose $x$ and $y$ are the endpoints of $e$. Since only one of $G_v$ and $(G + e)_v$ has property $\mathcal{Q}$, one of $x$ and $y$ is connected through a path of length at most $r$ to $v$ in $G$, and each vertex along this path has degree at most $\Delta$. For each endpoint of $e$, there can be at most $(1 + \Delta + \ldots + \Delta^r) \leq 2\Delta^r$ such paths, so adding or removing an edge can change the number of vertices with property $\mathcal{Q}$ by at most $4\Delta^r$. Since a switching corresponds to removing two edges and adding two edges, the result follows.

The next lemma formalizes the idea that we can safely ignore high-degree vertices.

**Lemma 7.** For any $r \geq 0$ and $\epsilon > 0$, there exist $\delta > 0$ and an integer $\Delta$ such that whenever $l(d, D) < \delta$ we have

$$\Pr(\mathcal{T} \in \mathcal{M}_{\Delta,r}) \geq 1 - \frac{\epsilon}{4}, \text{ and}$$

$$\Pr\left( N_{\mathcal{M}_{\Delta,r}}(G^*) \leq N(1 - \epsilon) \right) \leq e^{-\delta N}.$$ 

**Proof.** The first part is immediate since the total number of offspring in $\mathcal{T}$ within distance $r$ of the root is finite with probability one. For $\delta$ sufficiently small, Corollary 7 then implies that $\E\left[ N_{\mathcal{M}_{\Delta,r}}(G^*) \right] \geq N\left(1 - \frac{\epsilon}{2}\right)$. Apply Lemma 6 to the trivial $r$-local property (i.e. the property that always holds), which shows that $N_{\mathcal{M}_{\Delta,r}}(G^*)$ is a $C$-Lipschitz function for some $C$. The second part now follows from Lemma 5.

We can now complete the proof of Proposition 10. Choose $\Delta$ sufficiently large so that

$$\left| \Pr(\mathcal{T} \in \mathcal{P}) - \Pr(\mathcal{T} \in \mathcal{P} \cap \mathcal{M}_{\Delta,r}) \right| \leq \Pr(\mathcal{T} \notin \mathcal{M}_{\Delta,r}) \leq \frac{\epsilon}{4}. \quad \text{(A.3)}$$

Let $B = N - N_{\mathcal{M}_{\Delta,r}}(G^*)$ denote the number of high-degree vertices in the graph $G^*$. Since $\left| N_{\mathcal{P}}(G^*) - N_{\mathcal{P} \cap \mathcal{M}_{\Delta,r}}(G^*) \right| \leq B$, Lemma 7 implies that for some $\delta_1 > 0$, whenever $l(d, D) < \delta_1$ we have

$$\Pr\left( \left| N_{\mathcal{P}}(G^*) - N_{\mathcal{P} \cap \mathcal{M}_{\Delta,r}}(G^*) \right| \geq \frac{\epsilon N}{2} \right) \leq e^{-\delta_1 N}. \quad \text{(A.4)}$$

Lemma 6 implies that $N_{\mathcal{P} \cap \mathcal{M}_{\Delta,r}}(G^*)$ is $C$-Lipschitz for some $C$, so Corollary 7 and Lemma 5 together imply that for another $\delta_2 > 0$, whenever $l(d, D) < \delta_2$ we have

$$\Pr\left( \left| N_{\mathcal{P} \cap \mathcal{M}_{\Delta,r}}(G^*) - N_{\mathcal{P}}(\mathcal{T} \in \mathcal{P} \cap \mathcal{M}_{\Delta,r}) \right| \geq \frac{\epsilon N}{4} \right) \leq e^{-\delta_2 N}. \quad \text{(A.5)}$$
The inequalities (A.3), (A.4), and (A.5), with an application of the triangle inequality, now imply that for some \( \delta < \min(\delta_1, \delta_2) \), whenever \( l(d, D) < \delta \) we have
\[
\mathbb{P}
\left(
\left| N\mathcal{P}(G^*) - N\mathbb{P}(T \in \mathcal{P}) \right| \leq \epsilon N
\right)
\geq 1 - e^{-\delta N},
\]
proving the result. \( \square \)

Proposition 10 immediately proves (A.1) for the multigraph configuration model. Summing over component sizes above some lower bound, we also find that the number of vertices in “large” components concentrates around \( N\mathbb{P}(|T| = \infty) \).

**Corollary 8.** Fix \( \epsilon > 0 \). For all sufficiently large \( K \), there exists \( \delta > 0 \) such that if \( l(d, D) < \delta \) we have
\[
\mathbb{P}\left(\left| \sum_{k \geq K} N_k(G^*) - N\mathbb{P}(|T| = \infty) \right| \geq \epsilon N \right) \leq e^{-\delta N}. \tag{A.6}
\]

**Proof.** For sufficiently large \( K \), we have \( \sum_{k=1}^{K} \mathbb{P}(|T| = k) \geq 1 - \frac{\epsilon}{2} - \mathbb{P}(|T| = \infty) \). The result follows from (A.1), replacing \( \epsilon \) with \( \frac{\epsilon}{2K} \). \( \square \)

Corollary 8 implies (A.2) if \( \mathbb{P}(|T| = \infty) = 0 \), and it will also play a key role in the next section as we address the case in which \( \mathbb{P}(|T| = \infty) > 0 \).

**Coloring and Sprinkling**

Having established branching process approximation results for component sizes, we now show that essentially all “large” components are connected. I assume throughout this section that \( \mathbb{P}(|T| = \infty) > 0 \). The basic idea of the argument is to thin the graph \( G^* \) by retaining edges with some probability \( p \). For \( p \) close to 1, the component structure of the thinned graph is similar to that of \( G^* \). When we “sprinkle” back in the remaining edges, any large components are very likely joined together.

I choose only one type of edge to thin. By assumption there exists a type \( \theta_1 \) which has three or more neighbors with positive probability. Since the graph is irreducible and \( T \) survives with positive probability, there exists a type \( \theta_2 \) that connects to type \( \theta_1 \) nodes with positive probability and has 2 or more neighbors with positive probability. These conditions ensure that in the branching process \( T \), with positive probability we will encounter both type \( \theta_1 \) parents with type \( \theta_2 \) offspring and type \( \theta_2 \) parents with type \( \theta_1 \) offspring. Let \( G'' \) denote the subgraph of \( G^* \) that we obtain by deleting edges between type \( \theta_1 \) and type \( \theta_2 \) nodes independently with some probability \( p \in (0, 1) \), and let \( G''' \) denote the subgraph formed by the deleted edges. We can also view \( G^* \) as a colored graph, in which the edges of \( G' \) are red and those of \( G'' \) are blue. I will sometimes write \( G^*(p) \) to emphasize that I am talking about the colored version of \( G^* \). Let \( d' \) denote the degree sequence of \( G' \), and let \( d'' \) denote the degree sequence of \( G''' \). The sprinkling argument relies on the following lemma.

**Lemma 8.** For any \( d \) and any \( 0 < p < 1 \), the random graphs \( G' \) and \( G'' \) are conditionally independent given \( d' \).
Proof. This follows from the definition of the configuration model. The graph \( G^* \) is a uniform random pairing of the stubs determined by \( d \). Color each pair red, except color edges between type \( \theta_1 \) and type \( \theta_2 \) nodes blue with independent probability \( 1 - p \). Given the set of stubs in red pairs, which determines \( d' \) and \( d'' \), the pairing of these stubs is uniformly random, and similarly the blue stubs are paired uniformly at random.

The method used to prove Proposition 10 allows us to state similar concentration results for the colored subgraphs. Let \( T(\theta) \) denote the branching process \( T \) in which we color edges between type \( \theta_1 \) and type \( \theta_2 \) nodes blue with independent probability \( 1 - p \). Let \( T'(\theta) \) denote the red subtree containing the root, and let \( D_{\theta} \) denote the thinned degree distribution. Note that \( D_{\theta} \) is the asymptotic degree distribution of \( G' \), and \( T'(\theta) \) is the corresponding branching process that approximates rooted graphs in \( G' \). I omit the proof of the following result as it is essentially identical to that of Proposition 10.

**Proposition 11.** Let \( P \) be a \( r \)-local property of colored rooted graphs, and fix \( \epsilon > 0 \) and \( p \in (0, 1) \). There exists \( \delta > 0 \) such that if \( l(d, D) < \delta \) then

\[
\mathbb{P} \left( \sum_{k \geq K} N_k(G^* \theta) - N \mathbb{P}(T(\theta) \in P) \geq \epsilon N \right) \leq e^{-\delta N}.
\]

We also require a simple lemma bounding the probability that no links are formed between sets of stubs. Recall that \( m_{\theta_1, \theta_2}(d) \) is the number of edges connecting type \( \theta_1 \) and type \( \theta_2 \) nodes, given the degree sequence \( d \).

**Lemma 9.** Let \( \{A_i\}_{i=1}^2 \) and \( \{B_i\}_{i=1}^2 \) be disjoint sets of stubs, with \( A_1 \) and \( B_1 \) containing stubs attached to type \( \theta_1 \) nodes leading to type \( \theta_2 \) nodes, and vice versa for \( A_2 \) and \( B_2 \). The probability that no stubs in \( A_1 \cup A_2 \) are paired to stubs in \( B_1 \cup B_2 \) is no more than

\[
e^{-\frac{|A_1| |B_1| + |A_2| |B_2|}{2m_{\theta_1, \theta_2}(d)}}.
\]

**Proof.** Without loss of generality, assume \( |A_1| \leq |B_1| \), and conduct the following exercise. One at a time, select a random unpaired stub in \( A_1 \) and reveal its partner. Conditional on having no matches in \( B_1 \) yet, the probability of finding a partner in \( B_1 \) is at least \( \frac{|B_1|}{m_{\theta_1, \theta_2}(d)} \). Hence, the probability that we have no matches in \( B_1 \) is at most

\[
\left(1 - \frac{|B_1|}{m_{\theta_1, \theta_2}(d)}\right)^{|A_1|} \leq e^{-\frac{|A_1||B_1|}{2m_{\theta_1, \theta_2}(d)}}.
\]

Repeat the argument for \( A_2 \) and \( B_2 \), and the result follows.

We are now ready to prove (A.2) for the multigraph configuration model. Let \( L_i = L_i(G^*) \) denote the number of vertices in the \( i \)th largest component of \( G^* \), and fix \( \epsilon > 0 \). By Corollary 8, there are constants \( K \) and \( \delta > 0 \) such that if \( l(d, D) < \delta \), then

\[
\mathbb{P} \left( \sum_{k \geq K} N_k(G^*) \geq N \left( \mathbb{P}(|T| = \infty) + \frac{\epsilon}{4} \right) \right) \leq e^{-\delta N}.
\]

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Trivially, we know $L_1 + L_2 \leq 2K + \sum_{k \geq K} N_k(G^*)$. For sufficiently large $N$, we have $K \leq \frac{cN}{8}$, implying
\[
\begin{align*}
P\left(L_1 + L_2 \geq N \left( P(|T| = \infty) + \frac{\epsilon}{2} \right) \right) & \leq P\left(\sum_{k \geq K} N_k(G^*) \geq N \left( P(|T| = \infty) + \frac{\epsilon}{4} \right) \right) \\
& \leq e^{-\delta N}.
\end{align*}
\]

(A.7)

To complete the proof for the multigraph $G^*$, it suffices to show for some $\delta'$, with $0 < \delta' \leq \delta$, that if $l(d, D) < \delta'$ we have
\[
P\left(L_1 \geq N \left( P(|T| = \infty) - \frac{\epsilon}{2} \right) \right) \geq 1 - e^{-\delta' N}.
\]

(A.8)

As $p$ approaches 1, the distribution $D_p$ converges to the distribution $D$. Here we make use of the assumption that at least one type has three or more neighbors with positive probability. This means that in the forward distribution $D''$ for the branching process $T$, there is a positive probability of having two or more offspring. This rules out the case in which a node in $T$ always has one child, and one can check that under this assumption, the survival probability $P(|T'(p)| = \infty)$ converges to $P(|T| = \infty)$. For the rest of the proof, fix a $p$ such that $P(|T'(p)| = \infty) \geq P(|T| = \infty) - \epsilon 8$.

We need a lower bound on the number of stubs in $G''$ that are attached to large components of $G'$. Given $\Delta \geq 2$ and $r \geq 0$, for a vertex $v$, we define the $r$-local property $H_{\Delta, r}$, which is satisfied if two conditions hold. First, no vertex within distance $r$ of $v$ has more than $\Delta$ neighbors in $G'$. Second, at least one of the following statements is true:

1. In the component of $G'$ containing $v$, no vertex lies at distance $r$ or greater from $v$

2. Within distance $r$ of $v$ in $G'$, there exists both a type $\theta_1$ node with a stub in $G''$ and a type $\theta_2$ node with a stub in $G''$.

Lemma 10. Fix $\epsilon > 0$. We can choose $\Delta$ and $r$ for which there exists $\delta_1 > 0$ such that if $l(d, D) < \delta_1$ we have
\[
P\left(N_{H_{\Delta, r}}(G^*(p)) \leq N \left( 1 - \frac{\epsilon}{8} \right) \right) \leq e^{-\delta_1 N}.
\]

Proof. Choosing $r$ sufficiently large ensures that, conditional on surviving until the $r$th generation, the red subtree $T'_r(p)$ has blue stubs of both types with probability at least $1 - \frac{\epsilon}{24}$. By Lemma 7 we can find $\Delta$ so that $P(T'(p) \in M_{\Delta, r}) \geq 1 - \frac{\epsilon}{24}$. Consequently, we have
\[
P\left(T(p) \in H_{\Delta, r} \right) \geq 1 - \frac{\epsilon}{12}.
\]

The result follows from Proposition 11.
Fix the $\Delta$ and $r$ obtained in Lemma 10, let $S_k$ denote the set of vertices in components of $G'$ with at least $k$ vertices, and let $M = N p_0 \mathbb{E}[D_{\theta_1} \cdot e_{\theta_2}]$ be the expected number of links between type $\theta_1$ and type $\theta_2$ nodes. By Corollary 8 (or rather, the analog based on Proposition 11), there exists $k \geq \max\left(K, \frac{M 16\Delta r}{\epsilon^2}\right)$ and $\delta_2 > 0$ such that whenever $l(d, D) < \delta_2$ we have

$$
P \left( |S_k| \leq N \left( \mathbb{P}(|T| = \infty) - \frac{\epsilon}{4} \right) \right) \leq \mathbb{P} \left( |S_k| \leq N \left( \mathbb{P}(|T'(p)| = \infty) - \frac{\epsilon}{8} \right) \right) \leq e^{-\delta_2 N}. \quad (A.9)$$

Call a partition $(X, Y)$ of $S_k$ a potentially bad cut if both $|X| \geq \frac{\epsilon N}{4}$ and $|Y| \geq \frac{\epsilon N}{4}$, and there are no edges of $G'$ connecting $X$ and $Y$. The partition is a bad cut if additionally no edge in $G''$ connects $X$ and $Y$. Each component of $G'$ in $S_k$ must lie entirely in $X$ or in $Y$, so in any realization there are at most

$$2^{\frac{|S_k|}{k}} \leq 2^\frac{N}{8} \leq e^\frac{N}{8}$$

potentially bad cuts.

Fix a realization of $d'$ and $G'$ such that

$$N_{\Delta, r}(G^*(p)) \geq N \left( 1 - \frac{\epsilon}{8} \right).$$

Suppose that $(X, Y)$ is a potentially bad cut. Both $X$ and $Y$ contain at least $\frac{\epsilon N}{4}$ vertices with property $H_{\Delta, r}$. Since $k \geq \Delta^r$, and no vertex in $H_{\Delta, r}$ can reach more than $\Delta^r$ other vertices within $r$ links in $G'$, we know that each of these vertices satisfies condition (b). For any particular stub in $G''$, there are no more than $2\Delta^r$ paths of length $r$ connecting it to a vertex in $H_{\Delta, r}$. Therefore, both $X$ and $Y$ contain at least $\alpha N = \frac{\epsilon N}{16\Delta^r}$ stubs of each type in $G''$.

For small enough $\delta$, the graph $G''$ contains no more than $M$ edges. By Lemma 9, the probability that no edges in $G''$ connect $X$ and $Y$ is no more than

$$e^{-\frac{2N}{M}} \leq e^{-\frac{2N}{8}}.$$

This implies that the expected number of bad cuts, given $d'$ and $G'$, is at most $e^{-\frac{N}{8}}$, and the probability of having any bad cuts is at most $e^{-\frac{N}{8}}$. If there are no bad cuts, then

$$L_1 \geq |S_k| - \frac{\epsilon N}{4} \geq N \left( \mathbb{P}(|T| = \infty) - \frac{\epsilon}{2} \right).$$

Taking $\delta' < \min\left(\delta, \delta_1, \delta_2, -\frac{1}{k}\right)$ completes the proof for the multigraph configuration model.

Simple Graphs

We are almost done. To transfer the result for the multigraph $G^*$ to the simple graph $G$, we use a lower bound on the probability that $G^*$ results in a simple graph. Let $\mathcal{S}$ denote the event that $G^*$ is simple.
Lemma 11. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $l(d, D) < \delta$ we have

$$P(G^* \in S) \geq e^{-\epsilon N}.$$  

Proof. Here I will spare my reader the details and refer to the argument in Lemma 21 of Bollobás and Riordan (2015). The required modifications are slight. 

To complete the proof of Theorem 13, suppose that $\mathcal{P}$ is any property of graphs, and note the distribution of $G^*$ conditional on event $S$ is precisely the configuration model $G$. We have

$$P(G \in \mathcal{P}) = P(G^* \in \mathcal{P} \mid S) \leq \frac{P(G^* \in \mathcal{P})}{P(G^* \in S)}.$$  

Consider the statement (A.1), and take $\mathcal{P}$ as the property that $|N_k(G) - N\mathbb{P}(|T| = k)| \geq \epsilon N$. There exists $\delta_1 > 0$ such that $P(G^* \in \mathcal{P}) \leq e^{-\delta_1 N}$. From Lemma 11, there exists $\delta_2$ such that $P(G^* \in S) \geq e^{-\delta_1 N/2}$. Taking $\delta < \min(\delta_1/2, \delta_2)$, we have $P(G \in \mathcal{P}) \leq e^{-\delta N}$. The statement (A.2) follows analogously. \qed
Appendix B

Experimentation

Proof of Lemma 1

All information is derived from the payoffs $d\pi_i^1(t)$; via a normalization this is equivalent to observing $d\tilde{\pi}_i(t) = \sqrt{\alpha_i(t) \mu^2} dt + dZ_i^1(t)$. The distribution of payoffs in the “period” $[t, t + dt)$ has density

$$
\frac{1}{(\sqrt{2\pi}dt)^{d_i+1}} e^{-\frac{1}{2\alpha(t)\mu^2} \sum_{j \in G_i} (d\tilde{\pi}_j^1(t) - \sqrt{\alpha_j(t)} \mu dt)^2}.
$$

Define $Q_i(\mu) = e^{\sum_{j \in G_i} \frac{1}{2} \sqrt{\alpha_j(t)\mu^2} d\tilde{\pi}_j^1(t) - \frac{1}{2\alpha(t)\mu^2} \alpha_j(t)\mu^2 dt}$. Applying Bayes’ rule we obtain

$$
p_i(t + dt) = \frac{p_i(t) Q_i(H) - p_i(t) Q_i(L)}{p_i(t) Q_i(H) + (1 - p_i(t)) Q_i(L)},
$$

implying that

$$
dp_i(t) = \frac{p_i(t)(1 - p_i(t))(Q_i(H) - Q_i(L))}{p_i(t) Q_i(H) + (1 - p_i(t)) Q_i(L)}.
$$

(B.1)

Now, expand $Q_i(\mu)$ as a Taylor series, using that $(d\tilde{\pi}_j^1)^2 = dt$ and $d\tilde{\pi}_j^1 d\tilde{\pi}_k^1 = 0$ if $j \neq k$, and discard terms of order higher than $dt$ to obtain

$$
Q_i(\mu) \approx 1 + 1\sigma \sum_{j \in G_i} \sqrt{\alpha_j(t)\mu} d\tilde{\pi}_j^1.
$$
Substituting into Eq. (B.1) and simplifying yields

\begin{align*}
dp_i(t) &= \frac{p_i(t) (1 - p_i(t)) (H - L) \sum_{j \in G_i} \sqrt{\alpha_j(t)} d\bar{\pi}_j^1}{\sigma + \sum_{j \in G_i} \sqrt{\alpha_j(t)} \nu_{\pi_i(t)} d\bar{\pi}_j^1} \\
&= \frac{p_i(t) (1 - p_i(t)) (H - L)}{\sigma} \left( \sum_{j \in G_i} \sqrt{\alpha_j(t)} d\bar{\pi}_j^1 \right) \left( 1 - \frac{1}{\sigma} \sum_{j \in G_i} \sqrt{\alpha_j(t)} \nu_{\pi_i(t)} d\bar{\pi}_j^1 \right) \\
&= \frac{p_i(t) (1 - p_i(t)) (H - L)}{\sigma} \left( \sum_{j \in G_i} \sqrt{\alpha_j(t)} d\bar{\pi}_j^1 \right) - \frac{1}{\sigma} \sum_{j \in G_i} \alpha_j(t) \nu_{\pi_i(t)} dt \\
&= \frac{p_i(t) (1 - p_i(t)) (H - L)}{\sigma} \sum_{j \in G_i} \sqrt{\alpha_j(t)} dZ_j^1.
\end{align*}
\hfill \square

Proof of Lemma 2

Given belief and experimentation levels \( p \) and \( \alpha \), the current period payoff is

\[ r \left( (1 - \alpha) \mu_0 + \alpha \nu \right) dt, \]

and the continuation payoff is

\[ e^{-rdt} u(d_i, p + dp). \]

Note \( \mathbb{E}[dp] = 0 \), and Lemma 1 implies

\[ \mathbb{E}[(dp)^2] = (\alpha + d_i \alpha_s(p)) \Phi(p) dt. \]

Discarding terms of order higher than \( dt \), a Taylor expansion gives \( e^{-rdt} \approx 1 - rdt \) and

\[ u_i(d, p + dp) = u_i(d, p) + \frac{\partial u_i}{\partial p}(d, p) dp + \frac{1}{2} \frac{\partial^2 u_i}{\partial p^2}(d, p)(dp)^2. \]

Summing our expressions for the current and continuation payoffs, taking expectations, and dropping higher order terms, gives

\[ u(d_i, p) + rdt \left( (1 - \alpha) \mu_0 + \alpha \nu + \frac{1}{r} (\alpha + d_i \alpha_s(p)) \frac{\Phi(p) \partial^2 u}{\partial p^2}(d_i, p) - u(d_i, p) \right). \]

The unique bounded solution of the Bellman equation, which reduces to the expression in the statement of the lemma, is the value function for agent \( i \). \hfill \square
Bibliography


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