Pre-order Price Guarantee in E-commerce

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With the development of the Internet and E-commerce, retailers often offer pre-orders for new to-be-released products. To encourage pre-orders, retailers such as Amazon offer pre-order price guarantee (PG). That is, if the product price drops before or on the release date, pre-order consumers automatically receive a refund for the difference between the pre-order price and the new price. We find that if pre-order demand uncertainty is high, a firm should adopt PG in advance selling. If pre-order demand uncertainty is low, then a firm should adopt PG if and only if the percentage of high-valuation consumers is high. Furthermore, we find that a firm’s optimal profit under PG monotonically increases in pre-order demand uncertainty while the firm’s optimal profit without PG stays unchanged. That is, PG enables a firm to profit from pre-order demand uncertainty. In addition, we show that price commitment is dominated by dynamic pricing when the retailer can optimally decide whether or not to offer PG under dynamic pricing. We also demonstrate that a retailer should sell in advance if a product’s marginal cost is less than a certain threshold, which is higher than the traditional threshold in advance selling literature without consideration of PG.

Key words: pre-order, price guarantee, advance selling

1. Introduction

With the development of the Internet and E-commerce, advance selling has been adopted by many retailers as an important marketing strategy (Xie and Shugan 2001). A typical form of advance selling is pre-order for new to-be-released products. Under the advance selling strategy, a retailer encourages consumers to pre-order a new product before its release date to receive guaranteed prompt delivery on the release date. For example, consumers can pre-order most new to-be-released video games through retailers such as Amazon and Bestbuy.
There are advantages and disadvantages for pre-order consumers. On one hand, when consumers place pre-orders, they are guaranteed to receive the product on time so that they can avoid stock-out risk in the selling season. Some extremely popular products can stock out immediately after release. For example, Wii units stocked out for a long time right after its release. Consumers cannot obtain any Wii units unless they have pre-ordered the product before it was released. (Martin, 2006).

On the other hand, pre-order consumers have to face fitting risk and price risk. First, consumer valuation about the new product is uncertain in the advance selling period (Xie and Shugan 2001). For example, in the selling season, consumers can try video games in a local store or check online feedback (reviews) provided by actual users. However, when they pre-order a new to-be-released video game, they do not have these opportunities and may realize negative surplus later. So pre-order consumers face fitting risk because they may realize a low valuation for the product.

Second, a retailer may estimate the market too optimistically and charge a high price in the advance selling period. After realizing a soft market response at the end of the advance selling period, the retailer may want to adjust the price and make a significant price cut after the release. For example, the pre-order price for Nokia N900 smart phone started at $649 and then was reduced to $589 when it was close to release (Goldstein, 2009 and King, 2009, Li and Zhang 2013). Walmart charged the pre-order price $199.99 for myTouch 3G Slide and slashed to $129.99 shortly before the release (Tenerowicz 2010). Amazon Kindle 2 was sold at a pre-order price $359 and then dropped to $299 soon after the release (Carnoy, 2009). Taking the price risk and fitting risk into consideration, rational consumers may hesitate to buy early.

In order to counteract rational consumer waiting behavior and induce them to pre-order, a retailer should design the two-period pricing strategy carefully. It is known that credible price commitment may counteract rational consumer waiting behavior (Xie and Shugan 2001). However, acquiring the external credibility itself is very costly. Also retailers lose the flexibility to change prices in the future. So many retailers are not willing to commit their pricing strategies in advance. For example, Amazon states on its website that “Amazon.com’s price for not-yet-released items sometimes changes between the time the item is listed for sale and the time it is released and/or shipped.”
Instead, retailers like Amazon refine two-period dynamic pricing by introducing pre-order price guarantee (PG). The following explanation of pre-order price guarantee is given on the Amazon.com website: “Pre-order Price Guarantee! Order now and if the Amazon.com price decreases between your order time and the end of the day of the release date, you’ll receive the lowest price.” Usually the launching price on the release date is the price for the selling season, especially considering the short life cycle of a new product such as video games and movie DVDs. Although in theory the retailer can change price shortly after the release date, it rarely happens in practice because the retailer tends to maintain the launching price for a certain period before a post selling season starts. For example, for products such as game consoles, the release prices can last for months. Too quick price changes may yield unfavorable consumer responses. For example, in 2011, Nintendo cut Nintendo 3DS’ price significantly within 6 months after its release, which upset those customers who bought early. As a result, Nintendo’s CEO apologized and offered some free games to the early adopters (Reisinger 2011). Similar story happened with Apple when it cut the iphone price two months later after its release (Fox 2012). Therefore, under pre-order price guarantee, if the selling season price is lower than the pre-order price, pre-order consumers will receive a refund for the price difference. So pre-order price guarantee provides a best price insurance for pre-order consumers. Figure 1 shows that Amazon announces “Pre-order Price Guarantee” together with the pre-order price for a to-be-released video game.

![Figure 1 Pre-order price guarantee is announced with the pre-order price for a to-be-released video game](http://www.amazon.com/gp/promotions/details/popup/AWT354OR7BM1U)
Amazon offers PG for all pre-orders of a book, CD, video, DVD, software, and video game. However, we also observe that Amazon does not offer PG for pre-orders of products such as cameras and TVs. Motivated by this real world practice, we study when a retailer should offer PG and when should not. What are the advantages and disadvantages of PG? What are the key driving forces for the profitability of PG? Furthermore, what is the impact of PG on dynamic pricing performance? If a retailer can optimally decide whether to offer PG under dynamic pricing, is dynamic pricing still inferior to price commitment when facing strategic consumer behavior? In addition, what is the impact of PG on advance selling decision? If a retailer can optimally decide whether to offer PG for pre-orders, what is the retailer’s advance selling strategy? i.e., when should a retailer sell in advance?

We find that pre-order demand uncertainty is the key driving force for the profitability of PG. If pre-order demand uncertainty is higher than a threshold, the retailer should always offer PG. Otherwise, the retailer should offer PG if and only if the percentage of high-valuation consumers is high. Furthermore, we find that a retailer’s optimal profit under PG monotonically increases in pre-order demand uncertainty, i.e., PG enables a retailer to profit from pre-order demand uncertainty.

In addition, we show that PG can greatly improve dynamic pricing performance. If a retailer can optimally decide when to offer PG, then dynamic pricing dominates price commitment when facing strategic consumers. It is known that credible price commitment can counteract strategic consumer behavior better than dynamic pricing without PG (Xie and Shugan 2001). However, it is not easy to implement price commitment because it may require external credibility. Our result suggests that dynamic pricing with the option of PG is a better alternative to price commitment for retailers. It is easier to implement and generates more profits than price commitment.

Furthermore, we demonstrate that a retailer is more likely to sell in advance when she considers offering PG for pre-orders. If a retailer can optimally decide when to offer PG, then the retailer should sell in advance when the marginal cost is less than a certain threshold, which is higher than the traditional threshold in advance selling literature without consideration of PG.
2. Literature

This paper is closely related to several streams of literature: advance selling, price guarantee, and price protection. We first discuss the advance selling literature, followed by the price guarantee literature and the price protection literature.

A stream in the operations management literature studies advance selling from a manufacturer to a retailer. Cvsaa and Gilbert (2002) and Gilbert and Cvsaa (2003) examine the manufacturer’s wholesale price commitment decision and issues related to sale timing. Taylor (2006) studies the sale timing for a manufacturer and decides when to sell to a retailer. Boyaci and Ozer (2010) study when to start and stop AS in order to acquire enough demand information from retailers for capacity planning in a manufacturing company. Cho and Tang (2013) examine and compare three selling strategies from a manufacturer to a retailer, including advance selling only, regular selling only, and combined advance and regular selling.

This research focuses on a retailer’s AS rather than a manufacturer’s. One key difference between a retailer’s AS and a manufacturer’s AS is consumer valuation uncertainty. A retailer’s AS to strategic consumers generates consumer valuation uncertainty, which is not considered in a manufacturer’s AS.

A retailer’s advance selling has been studied in both operations and marketing literature (e.g., Ma et al. 2018, Wei and Zhang 2018, Cachon and Feldman 2017, Noparumpa et al. 2015, Li et al. 2014, Nasiry and Popescu 2012). The marketing literature on advance selling focuses on the benefits from consumer valuation uncertainty under AS and answers the question whether or not a retailer should sell in advance. Shugan and Xie (2000, 2004, 2005) and Xie and Shugan (2001) show that during AS, consumer valuation for a new to-be-released product is uncertain. Consumer valuation uncertainty benefits the retailer in several ways such as removing information disadvantages and increasing sales for the retailer. Xie and Shugan (2009) further investigate the benefits of consumer valuation uncertainty in various scenarios and conclude that consumer valuation uncertainty alone can dramatically increase a retailer’s profit under AS. Fay and Xie (2010) show that both AS and probabilistic selling offer consumers a choice involving buyer uncertainty. Such buyer uncertainty homogenizes heterogeneous consumers and thus benefits the retailer. In the marketing literature, consumer valuation uncertainty is identified as a key driver for the profitability of AS.
In the operations management literature, advance selling from a retailer to consumers has been studied. Another benefit of AS is to reduce selling season demand uncertainty for a retailer. Tang et al. (2004) examine the AS benefit of reduced demand uncertainty by utilizing the realized pre-order demand information to update the demand forecast for the selling season. Zhao and Stecke (2010) consider benefits from both demand learning and consumer valuation uncertainty to decide whether or not a retailer should sell in advance to loss averse consumers. Prasad et al. (2011) also consider both benefits to decide when a retailer should sell in advance to risk averse consumers. Li and Zhang (2013) show that reducing selling season demand uncertainty can hurt a retailer. Gao et al. (2012) analyze weather-conditional rebate programs in AS for seasonal products. Huang and Chen (2013) examine the optimal price in AS without revealing the price, which is known as probabilistic pricing. Yu et al. (2015a) show that capacity rationing in advance selling can serve as an effective tool to signal product quality. Yu et al. (2015b) study the impact of interdependence of customer valuation on a seller’s advance selling strategy under limited capacity. Zhao et al. (2016) show that a retailer’s advance selling capability can hurt the retailer’s profit in a decentralized supply chain.

The aforementioned advance selling studies in the marketing and operations management literature answers the question whether or not a retailer should sell in advance under different scenarios. Our paper focuses on the question naturally following the previous one: if a retailer has already decided to sell in advance, should she offer pre-order price guarantee (PG) for pre-order consumers and what is the key driving force for a retailer’s preference of PG over NPG (no pre-order price guarantee)?

PG is similar to posterior price matching where customers can receive a refund equivalent to the price difference if the seller’s future price drops. Along this line, several papers have studied price guarantee in different contexts.

The following two papers consider selling a fixed capacity (inventory) over two periods using a price guarantee option. Png (1991) studies most-favored-customer price protection (i.e., price guarantee) when selling a fixed capacity (inventory) over two periods to strategic consumers. They show that a seller should offer price protection when capacity is sufficiently large. Levin et. al. (2007) also consider a fixed capacity and study a price guarantee option, under which a consumer
can choose to purchase together with one product. They explore the optimal selling prices for the product and for the option. However, they do not consider rational consumer waiting behavior, which we show as an important reason for a retailer to offer PG. In contrary to Png (1991) and Levin et. al. (2007) focusing on service industries with limited capacity such as hotel rooms and airplane seats, we focus on retail industries, where inventory is an endogenous decision rather than an exogenous fixed amount. In our paper, there is no fixed capacity level. Thus, a retailer’s decision whether or not to offer PG does not depend on the capacity level. We show that a retailer should offer price guarantee when pre-order demand uncertainty is high.

The following two papers study price guarantee in a time frame including a selling season and a post selling season (salvage season). Consumption of products is delayed when purchase is delayed. Therefore, valuation decline is significant over two periods. Lai et al. (2009) study whether a retailer should offer posterior price matching considering consumer valuation decline over two periods. They show that a retailer should offer PG if the valuation decline over time is neither too low nor too high. Xu (2011) explores the optimal design, the duration, and refund ratio of a posterior price matching policy, given that consumer valuation of a product declines over time. However, they do not consider demand uncertainty, which we show as a key driving force for a retailer’s profit boost under PG. In contrary to Lai et al. (2009) and Xu (2011), we focus on a time frame including a pre-order season and a regular selling season. Although pre-orders are committed early, they are fulfilled after selling season starts. Consumption of products occurs in the selling season for both pre-order consumers and regular selling season consumers. Therefore, valuation decline is not significant in our problem. Instead, pre-order demand uncertainty, pre-order consumer valuation uncertainty, and stocking out risk in the selling season are significant in our problem, while the problems in Lai et al. (2009) and Xu (2011) do not have these features. Therefore, our results and insights are significantly different from theirs. Most strikingly, our results suggest that even when consumer valuation decline is as low as zero, the retailer should offer PG if pre-order demand uncertainty is high.

The following paper consider price guarantee in a retailer’s advance selling. Li and Zhang (2013) study the value of reducing the selling season demand uncertainty. They find that the reduction in selling season demand uncertainty may hurt the seller’s overall profit with or without PG for
pre-orders. Our focus is different. While Li and Zhang (2013) focus on the selling season demand uncertainty, we focus on pre-order demand uncertainty. We find that pre-order demand uncertainty is a key driving force for PG’s profitability. The retailer’s profit under PG monotonically increases in pre-order demand uncertainty. As far as we know, we are the first one to focus on pre-order demand uncertainty, and reveal the interesting relationship between PG’s profitability and pre-order demand uncertainty.

Our research is also closely related to price protection, usually offered by manufacturers to retailers, where the retailer can get a refund or rebate if the second period price of the product drops below the purchase price of the first period. Lee et al. (2001) show that price protection is an instrument for channel coordination. A properly chosen protection credit alone coordinates the channel when the retailer has a single buying opportunity. When the retailer has two buying opportunities, the price protection credit and the wholesale price together coordinate the channel. Taylor (2001) focuses on the two-purchase-opportunity model and explores the role of various combinations of price protection, midlife returns and end-of-life returns (PME) in channel coordination. He shows that, under certain conditions, the combination of all these policies, PME, guarantees both coordination and a win-win outcome. Employing a one-period model, Taylor (2002) shows that the target-level rebate policy (under which the manufacturer pays the retailer a rebate for each unit sold beyond a specified target level) plus returns can both coordinate the chain and guarantee a win-win situation. Lu et al. (2007) further develops procedures to identify the win-win policy parameters. In addition, for the two-purchase-opportunity model, they show that there may not exist a win-win policy under linear pricing or a quantity discount pricing strategy. However, under an increasing piecewise pricing strategies, PME can guarantee a win-win outcome. While working on a similar concept, our paper studies price protection (price guarantee in our paper) from a different point of view and with different results compared to the previous literature. We study how a newsvendor retailer’s price guarantee strategy depends on consumer characteristics such as consumer valuation uncertainty for products, proportion of high-valuation consumer segment, and pre-order consumer demand uncertainty. That is, we consider strategic consumer choice decisions based on an explicit characterization of their utility functions. Our results suggest that a retailer
should not offer price guarantee if and only if both pre-order demand uncertainty and proportion of high-valuation consumer segment are low.

3. Base Model without Demand Correlation

A retailer sells a single product over two periods, an advance selling period followed by a regular selling period. Consumers are segmented into two groups, informed and uninformed. Informed consumers know about advance selling and make purchase decisions in the advance selling period. Uninformed consumers do not know advance selling so they arrive in the regular selling period. Let \( N_a \) and \( N_r \) be the size of informed and uninformed consumers, respectively. Suppose that \( N_i, i \in \{a, r\} \), are continuous independent random variables. Let \( G_i, g_i, \mu_i, \) and \( \sigma_i \) be the probability distribution function, probability density function, demand mean, and demand standard deviation of \( N_i \), for \( i \in \{a, r\} \).

Each informed consumer arrives in the advance selling period. The informed consumer needs to strategically decide whether to buy in the advance selling period or wait until the regular selling period. On the one hand, if she buys now, she will be guaranteed to get the product; however, if she delays purchase until the regular selling period, she will face the inventory rationing risk. On the other hand, the consumer's valuation, denoted by \( V \), is uncertain in the advance selling period, only to be realized in the regular selling period. We assume \( V \) follows a Bernoulli distribution, with probability \( q \) and \( 1 - q \) to be \( H \) and \( L \), respectively, where \( H > L \). Correspondingly, let \( EV \) be the mean valuation, which satisfies \( EV = qH + (1 - q)L \). If a consumer delays purchase until the regular selling period, she has the benefit of knowing her realized valuation before making the purchase decision.

The sequence of the events can be described as follows. 1) In the advance selling period, the retailer announces the advance selling price \( p_a \) for the product. 2) The informed consumers arrive with their uncertain valuation \( V \) and decide whether to buy in advance or wait until the regular selling period. 3) At the end of the advance selling period, the pre-order demand \( N_a \) is realized. The retailer decides the regular selling price \( p_r \) and the order quantity \( N_a + Q \) at cost \( c \) per unit, where \( Q \) is the inventory prepared for the regular selling period. 4) In the regular selling period, if the retailer uses PG, each pre-order consumer receives the refund of the price difference \( p_a - p_r \) if
$p_r \leq p_u$; The uninformed consumers arrive with their realized valuation $v$ and make their purchase decisions.

In this paper, we assume that the retailer knows the distribution of consumer valuations. That is, the firm knows the values of $L$, $H$, and $q$. Before a firm announces advance selling of a new product, the firm can do some marketing research. One important part of marketing research is customer value analysis (CVA), which has been well studied in the marketing literature and in practice. Firms can use surveys, projective methods, and group interviews to estimate the valuations of their potential customers (Desarbo et al. 2001, Leber et al. 2014, Endeavor 2015, Barker 2017). When the sample size is large enough, the firm can get a big picture of the distribution of consumer valuations.

We focus on the rational expectations (RE) equilibrium of the game. The concept of rational expectations equilibrium has been widely adopted in operations and marketing literatures (e.g., Su and Zhang 2009, Li and Zhang 2013, and references therein). In a rational expectations equilibrium, a strategic consumer forms a belief $\hat{p}_r$ on the selling season price $p_r$ and a belief $\hat{\theta}$ on the selling season availability probability $\theta$. The RE equilibrium requires that the belief must match the true outcome of the game. That is, in the RE equilibrium, $\hat{p}_r = p_r$ and $\hat{\theta} = \theta$.

Following the vast advance selling literature (see, e.g., Xie and Shugan 2001, Tang et al. 2004, Zhao and Stecke 2010, Prasad, Stecke and Zhao 2011), we assume that no returns are allowed. In practice, a consumer cannot return any media products (for example, movie DVDs, music albums, video games, and software copies) once they open the plastic wrapping. Before a new product is released, consumers have uncertain valuation for the product when they pre-order it. On the release date, the pre-orders are delivered to them. They may realize a low valuation after they open the product box and try the product in person. However, once it is open, they cannot return the product. Since pre-order consumers are usually eager to try the new product as soon as possible. It is reasonable to assume that pre-order boxes will be open and thus not eligible for returns, especially when we consider video game products.

We assume that the retailer faces a newsvendor problem in the second period. For example, for media type products such as video games and DVDs, the retail sales volume drops quickly after the first week. If a retailer stocks out in the first a few weeks, he may miss the majority of sales. So the
effective selling season for video games is very short. For another instance, for non-media products such as fireworks, according to usfireworks.biz, about 95% of US fireworks sales occurs between May 15th and July 4th. It takes four to six weeks to ship the fireworks from China, where most of the fireworks are made, to the U.S. (Quint and Shorten 2005). Given the short selling season and the long transportation lead times, fireworks companies face a newsvendor problem. For another example, video game console sales drop exponentially after the first a few days (Orland 2017). For example, the sales number reached 400,000 units in the Wii U’s first week on sale in North America. The number dropped quickly. Right after the first week, the total 5-week sales from the 2nd to the 6th week is 490,000 units. So the effective selling season is relatively short for game consoles and retailers face a newsvendor problem.

3.1. Advance Selling without Price Guarantee

In this section, we examine the retailer’s optimal pricing and ordering decisions under advance selling without price guarantee.

We use the backward analysis. Before the start of the regular selling period, the retailer decides the regular selling price $p_r$ and the order quantity $Q_r$ to maximize its expected profits. Because each customer arriving during the regular selling period knows her realized valuation of the product, she will accept the regular selling price $p_r$ if and only if her realized valuation ($L$ or $H$) is no less than $p_r$. Facing the high-valuation and low-valuation customers, the retailer needs to decide whether to price lower ($p_r \leq L$) to serve both the high-valuation and low-valuation segments or to price higher ($p_r \in (L, H]$) to serve only the high-valuation segment. The retailer’s maximum expected profits from the regular sales, denoted by $\Pi_r^{NPG}$, can be written as

$$\Pi_r^{NPG} = \max \left\{ \max_{Q_r \geq 0, p_r \leq L} E[p_r \min(N_r, Q_r) - cQ_r], \max_{Q_r \geq 0, p_r \in (L, H]} E[p_r \min(qN_r, Q_r) - cQ_r] \right\}. \quad (1)$$

Note that regardless whether to serve both segments or to serve only high-valuation segment, the retailer’s expected profits always increase in the regular selling price $p_r$, implying that the retailer’s optimal regular selling price is equal to $L$ if it intends to serve both segments and is equal to $H$ if it intends to serve only the high-valuation segment.

When the retailer intends to serve only the high-valuation segment, the retailer’s optimal regular selling price is $H$ and the retailer’s optimal profit is $\max_{Q_r \geq 0, p_r \in (L, H]} E[p_r \min(qN_r, Q_r) - cQ_r]$. Let
Q'_r = \frac{Q_r}{q_r}. Then we have the following:

\[
\begin{align*}
\max_{Q_r \geq 0, p_r \in [L,H]} & \quad E[p_r \min(qN_r, Q_r) - cQ]\n= & \quad \max_{Q'_r \geq 0, p_r \in [L,H]} E[p_r \min(qN_r, qQ'_r) - cqQ'_r] \\
= & \quad q \max_{Q'_r \geq 0, p_r \in [L,H]} E[p_r \min(N_r, Q'_r) - cQ'_r] \\
= & \quad q\pi(H)
\end{align*}
\]

Consequently, we can simplify the retailer’s problem to

\[
\Pi^{NPG}_r = \max\{\pi(L), q\pi(H)\}
\]

where \(\pi(p) = \max_{Q \geq 0} E[p \min(N_r, Q) - cQ]\). Hence, the retailer should serve both segments by setting \(p^{NPG}_r = L\) if and only if the fraction of high-valuation customers is sufficiently small

\[
\pi(L) - q\pi(H) \geq 0. \quad (2)
\]

This leads to the following lemma that characterizes the retailer’s optimal price and order quantity decisions, denoted by \(p^{NPG}_r\) and \(Q^{NPG}_r\), for the regular sales. Let \(\bar{q} = \pi(L)/\pi(H)\).

**Lemma 1 (Selling season price without PG).** If the fraction of high-valuation customers is small, i.e., \(q \leq \bar{q}\), then the retailer serves both segments with \(p^{NPG}_r = L\) and \(Q^{NPG}_r = G^{-1}_r((L - c)/L)\). If the fraction of high-valuation customers is large, \(q > \bar{q}\), then the retailer serves only the high-valuation segment with \(p^{NPG}_r = H\) and \(Q^{NPG}_r = qG^{-1}_r((H - c)/H)\).

Lemma 1 suggests that the retailer’s advance selling price has no impact on the retailer’s subsequent decisions and expected profits from the regular sales. Therefore, it suffices to derive the advance selling price that maximizes the retailer’s expected profits from advance selling subject to the constraint that the informed customers are no worse off under advance purchase relative to delaying purchase to the regular selling period.

Let \(p_a\) be the advance selling price. If an informed customer purchases in advance, then her expected utility is equal to \(EV - p_a\). If she delays purchase to the regular selling period, then her expected utility is equal to 0 if \(q > \bar{q}\) because the retailer would set \(p^{NPG}_r = H\), and is equal to \(\theta(EV - L)\) otherwise, where \(\theta = E[\min\{N_r, Q^{NPG}_r\}]/E[N_r]\) is the probability that a consumer believes to find the product available during the regular selling period. Therefore, to ensure that
it is in the best interest of the informed customer to purchase in advance, the advance purchase price \( p_a \) must satisfy the constraint that \( EV - p_a \geq 0 \) if \( q > \bar{q} \) and that \( EV - p_a \geq \theta(EV - L) \) otherwise. This, together with the fact that the retailer’s expected profits from advance selling are equal to \((p_a - c)\mu_a\), results in the following expression for the retailer’s optimal advance selling price, denoted by \( p_{NPG}^a \):

\[
p_{NPG}^a = \begin{cases} 
\arg max_{p_a \leq EV - \bar{q}(EV - L)} \{ (p_a - c)\mu_a \} & \text{if } q \leq \bar{q}; \\
\arg max_{p_a \leq EV} \{ (p_a - c)\mu_a \} & \text{if } q > \bar{q}.
\end{cases}
\]

Note that the retailer’s expected profits from advance selling \((p_a - c)\mu_a\) always increase in the advance selling price \( p_a \), the retailer should set the advance selling price to the highest as long as the informed customers still prefer buying in advance. This leads to the following lemma that characterizes the retailer’s optimal advance selling price.

**Lemma 2 (Pre-order price without PG).** If the fraction of high-valuation customers is small, i.e., \( q \leq \bar{q} \), then \( p_{NPG}^a = EV - \theta(EV - L) \). If the proportion of high-valuation customers is high, i.e., \( q > \bar{q} \), then \( p_{NPG}^a = EV \).

After characterizing the retailer’s optimal pricing decisions during the advance selling period and the regular selling period, we conclude this section by presenting the closed-form expression for the retailer’s optimal expected profits under advance selling without PG, denoted by \( \Pi_{NPG} \). Naturally, \( \Pi_{NPG} \) consists of the profits from advance selling and the profits from regular sales. Let \( \Pi_{a}^{NPG} \) be the former, and \( \Pi_{r}^{NPG} \) be the latter. By definition, \( \Pi_{NPG} = \Pi_{a}^{NPG} + \Pi_{r}^{NPG} \). It follows from Lemmas 1 and 2 that

\[
\Pi_{a}^{NPG} = (p_{a}^{NPG} - c)\mu_a
\]

and

\[
\Pi_{r}^{NPG} = \max\{q\pi(H), \pi(L)\}.
\]

### 3.2. Advance Selling with Price Guarantee

In this section, we examine the retailer’s optimal pricing and ordering decisions under advance selling with price guarantee.

We start with the analysis for the regular selling period. Unlike the scenario without price guarantee where the advance purchase price \( p_a \) and the realized advance purchase quantity \( n_a \) do
not influence the optimal decisions for the regular selling period, the price guarantee creates a linkage among these variables because the retailer’s refunds to pre-order consumers depends on $n_a$ and the difference between $p_a$ and $p_r$. Similar to the scenario without price guarantee, the retailer faces the market targeting decision: serving both segments by setting $p_r \leq L$ or serving only the high-valuation segment by setting $p_r \in (L, H]$. However, in contrast to the scenario without price guarantee, the regular selling price $p_r$ impacts not only the profits from the regular sales but also the price matching costs. Hence, the retailer needs to take into consideration the matching cost in deciding the regular selling price. Specifically, given $p_a$ and $n_a$, the retailer solves the following problem for the optimal pricing and order quantity decisions for the regular sales:

$$\max \{ \max_{Q_r \geq 0, p_r \leq L} E[p_r \min(N_r, Q_r) - cQ_r - n_a (p_a - p_r)^+], \max_{Q_r \geq 0, p_r \in (L, H]} E[p_r \min(q N_r, Q_r) - cQ_r - n_a (p_a - p_r)^+] \}.$$

Note that regardless whether to serve both segments or to serve only the high-valuation segment, the retailer’s expected profits always increase in the regular selling price $p_r$, implying that the retailer’s optimal regular selling price is equal to $L$ if it serves both segments and is equal to $H$ if it serves only the high-valuation segment. Consequently, we can simplify the retailer’s problem to

$$\max \{ \pi(L) - n_a (p_a - L), q \pi(H) \},$$

implying that the retailer’s optimal regular selling price $p_r^{PG} = L$ if and only if

$$\pi(L) - q \pi(H) \geq n_a (p_a - L)$$

(3)

Hence, if $q > \bar{q}$, then the retailer should serve only the high-valuation segment regardless of the value of $n_a$. This result is consistent with that without price guarantee. However, contrast emerges when $q \leq \bar{q}$. Define

$$\tilde{n}_a(p_a) = (\pi(L) - q \pi(H))/(p_a - L).$$

Now the retailer should serve both the two segments if and only if the realized advance purchase quantity $N_a$ is lower than the threshold $\tilde{n}_a(p_a)$. This leads to the following lemma that characterizes the retailer’s optimal price and order quantity decisions, denoted by $p_r^{PG}$ and $Q_r^{PG}$, for the regular sales.
Lemma 3 (Selling season price under PG). If both the fraction of high-valuation customers and the realized pre-order demand are small, i.e., \( q \leq \bar{q} \) and \( n_a \leq \bar{n}_a(p_a) \), then the retailer serves both segments with \( p_r^{PG} = L \) and \( Q_r^{PG} = G_r^{-1}((L - c)/L) \); otherwise, the retailer serves only the high-valuation segment with \( p_r^{PG} = H \) and \( Q_r^{PG} = qG_r^{-1}((H - c)/H) \).

Recall that without price guarantee, the retailer should serve both segments if and only if the fraction of high-valuation customers is sufficiently small \( (q \leq \bar{q}) \). This is because in making the market targeting decision, the retailer has the sole objective of extracting the highest profits from regular sales and the low-valuation segment is too big to ignore. However, this is no longer true with price guarantee. Lemma 3 indicates that the retailer should abandon the low-valuation segment even if its fraction is high \( (q \leq \bar{q}) \), as long as the realized advance sales volume is sufficiently large \( (n_a > \bar{n}_a(p_a)) \). Such a contrast emerges because in making the regular selling price decision, the retailer needs to balance between the profits from regular sales and the matching costs. With sufficiently large number of advance purchasers \( (n_a > \bar{n}_a(p_a)) \), the price matching cost is too high for the retailer to charge a low regular selling price to serve both segments so that the price match cost dominates the regular sales profits, favoring the use of the high selling price to avoid the high matching cost by sacrificing the profits from regular sales. We call such an effect the price match effect.

Next we turn to the advance selling period. Let \( p_a \) be the advance selling price. If \( q > \bar{q} \), then it follows from Lemma 3 that the regular selling price is always \( H \), implying that the retailer’s optimal advance selling price, denoted by \( p_a^{PG} \), is the same as that without price guarantee, i.e., \( p_a^{PG} = EV \). Contrast emerges when \( q \leq \bar{q} \). Recall from Lemma 3 that the regular selling price is equal to \( L \) if \( n_a \leq \bar{n}_a(p_a) \), and \( H \) otherwise. Thus, if an informed customer purchases in advance, then her expected utility is equal to \( EV - p_a + G_a(\bar{n}_a(p_a))(p_a - L) \) where the third term represents the expected price refund due to price guarantee, which will occur if and only if the realized advance purchase quantity \( n_a \) is less than \( \bar{n}_a(p_a) \). If she delays purchase to the regular selling period, then it follows from Lemma 3 that her expected utility is equal to \( \theta G_a(\bar{n}_a(p_a))(EV - L) \). Therefore, to ensure that it is in the best interest of the informed customer to purchase in advance, the advance purchase price \( p_a \) must satisfy the constraint that \( EV - p_a + G_a(\bar{n}_a(p_a))(p_a - L) \geq \theta G_a(\bar{n}_a(p_a))(EV - L) \).
\[ \theta G_a(\bar{n}_a(p_a))(EV - L), \] or equivalently,
\[ p_a \leq L + \frac{q(H - L)(1 - \theta G_a(\bar{n}_a(p_a))))}{1 - G_a(\bar{n}_a(p_a))}. \]

This leads to the following expression for the retailer’s optimal advance selling price:
\[
\begin{align*}
p_a^{PG} & = \arg \max_{p_a \leq L + \frac{q(H - L)(1 - \theta G_a(\bar{n}_a(p_a))))}{(1 - G_a(\bar{n}_a(p_a))))} \left\{ \int_{\bar{n}_a(p_a)}^{\bar{n}_a(p_a)} ((L - c)n_a + \pi(L))dG_a(n_a) + \int_{\bar{n}_a(p_a)}^{\infty} ((p_a - c)n_a + q\pi(H))dG_a(n_a) \right\} \\
& = \arg \max_{p_a \leq L + \frac{q(H - L)(1 - \theta G_a(\bar{n}_a(p_a))))}{(1 - G_a(\bar{n}_a(p_a))))} \left\{ \int_{\bar{n}_a(p_a)}^{\infty} (L - c)n_a dG_a(n_a) + \int_{\bar{n}_a(p_a)}^{\infty} (p_a - c)n_a dG_a(n_a) \right\}
\end{align*}
\]

It is verifiable that the above objective function always increases in \( p_a \), implying that the constraint must bind at the optimal solution. This leads to the following lemma that characterizes the retailer’s optimal advance selling price.

**Lemma 4.** If \( q \leq \bar{q} \), then the retailer’s optimal advance selling price \( p_a^{PG} \) can be determined by the following equation:
\[ p_a^{PG} = \min \{ L + \frac{q(H - L)(1 - \theta G_a(\bar{n}_a(p_a))))}{1 - G_a(\bar{n}_a(p_a)))}, H \}. \]

If \( q > \bar{q} \), then the retailer’s optimal advance selling price is \( p_a^{PG} = EV \).

Now we turn to the retailer’s optimal expected profits, which we denote by \( \Pi^{PG} \). Naturally, \( \Pi^{PG} \) consists of the profits from advance selling and the profits from regular sales. Let \( \Pi_a^{PG} \) be the former, and \( \Pi_r^{PG} \) be the latter. By definition, \( \Pi^{PG} = \Pi_a^{PG} + \Pi_r^{PG} \). It follows from Lemmas 3 and 4 that if \( q > \bar{q} \) then \( \Pi^{PG} = \Pi^{NPG} \); otherwise,
\[
\begin{align*}
\Pi_a^{PG} & = \int_{0}^{\bar{n}_a(p_a^{PG})} (L - c)n_a dG_a(n_a) + \int_{\bar{n}_a(p_a^{PG})}^{\infty} (p_a^{PG} - c)n_a dG_a(n_a) \\
\Pi_r^{PG} & = \int_{0}^{\bar{n}_a(p_a^{PG})} \pi(L)n_a dG_a(n_a) + \int_{\bar{n}_a(p_a^{PG})}^{\infty} q\pi(H)n_a dG_a(n_a).
\end{align*}
\]

The first term in \( \Pi_a^{PG} \) and \( \Pi_r^{PG} \) represents the retailer’s profits from pre-orders and from regular selling season, respectively, in the scenario where pre-order demand is realized to be less than \( \bar{n}_a \), so the retailer charges a low selling season price and gives refunds to pre-order consumers. The second term in \( \Pi_a^{PG} \) and \( \Pi_r^{PG} \) represents the retailer’s profit from pre-orders and from selling season, respectively, in the scenario where pre-order demand is realized to be higher than \( \bar{n}_a \), so the retailer charges a high selling season price and does not refund pre-order consumers.
3.3. When Should a Retailer Offer PG?

In this section, we compare the retailer’s optimal expected profits under advance selling with and without PG, and identify the necessary and sufficient conditions under which the retailer is better off with PG than without PG. Recall that the retailer is equally well off with and without PG when \( q > \bar{q} \). It suffices to focus on the case when \( q \leq \bar{q} \) in the remainder of this section.

We start with the special case where there is no uncertainty in pre-order demand, i.e., \( \sigma_a = 0 \). This allows us to isolate the effect of the fraction of high-value segment \( (q) \) on influencing the retailer’s optimal performance with and without PG. From the perspective of extracting more profits from regular sales, the retailer should serve both segments with regular selling price \( L \) for \( q \leq \bar{q} \). However, with PG, the retailer faces the tradeoff between extracting more profits from regular sales and reducing the price matching cost. When \( q \) starts decreasing from \( \bar{q} \), the loss from regular sales by abandoning the low-valuation segment is insignificant because its size is small, implying that the retailer would always set the regular selling price to be \( H \). The high regular selling price empties the advance purchasers’ incentives to delay, and allows the retailer to fully extract the surplus from the advance purchasers by setting the advance purchase price to be \( EV \). This is in contrast with the lower advance purchase price \( EV - \theta(EV - L) \) without price guarantee. Further, the gap between these two advance purchase prices decreases as \( q \) decreases. Therefore, price guarantee allows the retailer to earn more profits from advance sales, albeit at the loss in profits from regular sales. As \( q \) keeps decreasing, the loss in profits from regular sales increases while the gain in profits from advance sales decreases. Therefore, there exists a threshold \( \bar{q} \) such that when \( q = \bar{q} \), the loss equalizes the gain so that the retailer is equally well off with price guarantee and without price guarantee. As \( q \) keeps decreasing from \( \bar{q} \), the loss from regular sales outweighs the gain from advance sales, implying that the retailer is worse off with price guarantee for \( q \in (0, \bar{q}) \). The following proposition formalizes these intuitive arguments.

**Proposition 1.** Without pre-order demand uncertainty, i.e., \( \sigma_a = 0 \), there exists a threshold \( \bar{q} \equiv \pi(L)/[\theta(H - L)\mu_a + \pi(H)] \), such that a retailer should offer PG if and only if the fraction of high-valuation customers is higher than the threshold. That is, \( \Pi^{NPG} > \Pi^{PG} \) for \( q \in (0, \bar{q}) \) and \( \Pi^{NPG} < \Pi^{PG} \) for \( q \in (\bar{q}, \bar{q}) \).
We next turn to the role of pre-order demand uncertainty in influencing the retailer’s optimal performance with and without PG, respectively. Recall that the pre-order demand uncertainty has no impact on the retailer’s optimal expected profits under advance selling without PG. This result is intuitive because the retailer’s profit margin for every unit of advance sales is always equal to \( p_{a}^{NPG} - c \) regardless of the realized advance purchase quantity.

Contrast emerges with PG. Now the retailer’s profit margin of advance sales depends on the realized advance sales volume. Specifically, recall that due to the price match effect, the retailer’s optimal pricing strategy during the regular selling period is to set \( p_{r}^{PG} = H(L) \) when the realized advance sales volume is sufficiently high (low), i.e., \( n_a > \bar{n}_a(p_{a}^{PG}) \) \( (n_a < \bar{n}_a(p_{a}^{PG})) \), implying that the retailer’s actual profit margin of advance sales (after deducting the price match cost if any) is equal to \( L - c \) when the realized advance sales volume is small \( (n_a \leq \bar{n}_a(p_{a}^{PG})) \), and jumps up to \( p_{a}^{PG} - c \) for \( n_a > \bar{n}_a(p_{a}^{PG}) \) (see Figure 2). In other words, the retailer’s profit margin of advance sales increases in the realized advance sales volume. This result is driven by a key feature of PG. That is, the regular selling price can be made contingent on the realized advance sales volume.

An implication of the result of increasing profit margin is that the retailer benefits from an increase in the pre-order demand uncertainty. Intuitively, the higher the pre-order demand uncertainty, the more likely the realized advance sales quantity \( n_a \) will take the extreme (very small or large) values. Two effects emerge. First, the shift in probability from small to smaller value of \( n_a \) lowers the retailer’s pre-sales profits. Second, the shift in probability from large to larger value of \( n_a \) improves the retailer’s pre-sales profits. Because of the result of increasing profit margin, the latter effect dominates the former effect, implying that the retailer benefits from the increase in the pre-order demand uncertainty. This, together with the result that \( \Pi^{NPG} \) is independent of \( \sigma_a \),
implies that, *ceteris paribus*, there exists a threshold $\bar{\sigma}$ such that the retailer is better off under the price guarantee if and only if $\sigma_a > \bar{\sigma}$.

To formalize the above insights, let $N_a = \mu_a + \sigma_a Z$ where $Z$ is a zero-mean random variable with standard deviation 1. Hence, $\sigma_a$ is the standard deviation of the pre-order demand. The following proposition summarizes how the pre-order demand uncertainty $\sigma_a$ impacts the retailer's optimal expected profits with and without PG, respectively.

**Proposition 2.** As the pre-order demand uncertainty $\sigma_a$ increases, the retailer's optimal expected profit $\Pi^{PG}$ under advance selling with PG increases whereas its optimal expected profit $\Pi^{NPG}$ under advance selling without PG stays unchanged.

Propositions 1 and 2 together provide necessary and sufficient conditions under which the retailer's preference over PG and NPG is definitive. This is summarized in the following proposition, which is the main result of this paper. This result is also illustrated in Figure 3.

**Proposition 3.** A retailer should offer PG unless both pre-order demand uncertainty and the fraction of high-valuation segment are sufficiently small, i.e., $\sigma_a < \bar{\sigma}_a(q)$ and $q < \tilde{q}$.

Proposition 3 provides analytical results that identify the complete set of regimes based on the values of $q$ and $\sigma_a$, under each of which there is a definitive comparison result between $\Pi^{PG}$ and $\Pi^{NPG}$. Figure 3 depicts these regimes.

An implication of Proposition 3 is that the retailer ought not to blindly offer PG in advance selling. PG hurts the retailer when both the fraction of high-valuation consumers and the pre-order demand uncertainty are sufficiently small. Although PG eliminates the informed consumer’s incentive to delay purchase and thus allows the retailer to charge a higher advance selling price, the retailer faces a tradeoff between charging a low price to incur refunds for pre-orders and charging a high price to abandon the low-valuation segment during the regular selling period. In the extreme case with no pre-order demand uncertainty, the retailer’s market targeting decision in the regular selling period is determined by the fraction of high-valuation segment: When the fraction of high-valuation segment is small, it is too costly to abandon the low-valuation segment and thus the retailer will always set a low regular selling price, resulting in the fact that the advance purchasers will end up paying the price that is lower than that without PG. Hence, the retailer is worse off
Figure 3. PG vs. NPG: $H = $20, $L = $15, $N_a$ and $N_r$ follow normal distributions with $\mu_a = \mu_r = 20$, $\sigma_r = 5$, and $\sigma_a$ increases from 1 to 45 in increment 1.

with PG than without PG when both the fraction of high-valuation consumers and the pre-order demand uncertainty are sufficiently small.

Based on Proposition 3, pre-order demand uncertainty plays an important role when $q$ is small. If pre-order demand uncertainty is high, i.e., $\sigma_a \geq \bar{\sigma}_a(q)$, a retailer should adopt PG in advance selling. If pre-order demand uncertainty is low, i.e., $\sigma_a < \bar{\sigma}_a(q)$, then a retailer should adopt PG if and only if the percentage of high-valuation segment is high, i.e., $q \geq \tilde{q}$.

Our analytical result suggests that the retailer should offer PG in advance selling for those products with relatively high pre-order demand uncertainty. This result is consistent with Amazon’s current practice where PG for pre-orders is offered to most creative digital products such as newly-released movie DVDs and video games but not to the relatively mature products such as cameras and TVs. It is plausible that the former group of products have relatively high degree of pre-order demand uncertainty due to the product novelty.

Next, we examine the magnitude of PG benefits. We compare the profits with and without PG in some numerical experiments used for Figure 3. That is, the following parameter values are used in the numerical experiments: $H = $20, $L = $15, $\mu_a = 20$, $\mu_r = 20$, and $\sigma_r = 5$. $N_a$ and $N_r$ follow independent normal distributions. $q$ is increased from 0.1 to 0.8 in 0.01 increments. $\sigma_a$ is increased
from 1 to 20 in 1 increment. The following value is calculated and presented in Table 1 for each combination of \( q \) and \( \sigma_a \).

\[
\frac{\text{expected profit with PG} - \text{expected profit without PG}}{\text{expected profit without PG}} \times 100\%
\]

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<th>( \sigma_a )</th>
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<th>0.35</th>
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</table>

Positive numbers in Table 1 show the percentage of profit improvement by offering PG. For example, if \( q < \tilde{q} = 0.2983 \) and \( \sigma_a \) is large, the retailer benefits from PG, i.e., positive profit improvement percentage in Table 1. Negative numbers in Table 1 show that the retailer's profit is reduced by offering PG. For instance, if \( q < \tilde{q} = 0.2983 \) and \( \sigma_a \) is small, the numbers in Table 1 are negative and they measure profit reduction percentage by offering PG.

Table 1 shows the following two results. 1) the percentage of profit improvement by offering PG increases in \( \sigma_a \). 2) the possible profit improvement percentage by PG can be as high as 24%. For ease of presentation, Table 1 only shows the percentage of profit change by PG for some selective \( q \) values and \( \sigma_a \) values. However, our experiments show that these two results hold for all of 36000
combinations of \( q \) values (increasing from 0.1 to 0.8 in 0.01 increment) and \( \sigma \) values (increasing from 1 to 45 in 1 increment).

4. Extended Model with Correlation

In our base model, we assume the advance demand and regular demand are independent. In this section, we model the correlated demand by assuming that \( N_a \) and \( N_r \) follow a bivariate normal distribution such that \( N_i \sim N(\mu_i, \sigma_i), i = \{a, r\} \) with a correlation coefficient \( \rho \in [-1, 1] \), where \( \sigma_a > 0 \) and \( \sigma_r > 0 \). Under the correlated demand, the realized advance sales volume \( N_a = n_a \) is useful for the retailer to update the distribution of demand during the regular selling period.

It follows from the property of the bivariate normal distribution that \( N_r | N_a = n_a \) is also normally distributed with mean \( \mu_r + \rho \sigma_r (n_a - \mu_a) / \sigma_a \) and standard deviation \( \sqrt{1 - \rho^2} \sigma_r \). It is convenient to write \( N_r | N_a = n_a = \rho \sigma_r (n_a - \mu_a) / \sigma_a + X \), where \( X \sim N(\mu_r, \sqrt{1 - \rho^2} \sigma_r) \).

We start with the analysis of the scenario without price guarantee. Given the realized advance sales volume \( N_a = n_a \), the retailer faces a newvendor pricing problem with demand \( X + \rho \sigma_r (n_a - \mu_a) / \sigma_a \) at the price \( p_r = L \) and with demand \( q(X + \rho \sigma_r (n_a - \mu_a) / \sigma_a) \) at the price \( p_r = H \), where \( X \sim N(\mu_r, \sqrt{1 - \rho^2} \sigma_r) \). Therefore, given the realized advance sales volume \( N_a = n_a \), the retailer’s maximum expected profit from the regular selling period is

\[
\Pi_{r|N_a=n_a}^{NPG} = \max \{ (L - c)\rho \sigma_r (n_a - \mu_a) / \sigma_a + \pi(L), q(H - c)\rho \sigma_r (n_a - \mu_a) / \sigma_a + q\pi(H) \}
\]

where \( \pi(p) = \max_{Q \geq 0} E[p \min(X, Q) - cQ] \), where \( X \sim N(\mu_r, \sqrt{1 - \rho^2} \sigma_r) \). Let \( Z_a = (N_a - \mu_a) / \sigma_a \sim N(0, 1) \). The retailer’s maximum expected profit from the regular sales is

\[
\Pi_r^{NPG} = E_{Z_a} \max \{ (L - c)\rho \sigma_r Z_a + \pi(L), q(H - c)\rho \sigma_r Z_a + q\pi(H) \}
\]

Therefore, the retailer’s optimal regular selling price \( p_r^{NPG} = L \) if and only if

\[
\pi(L) - q\pi(H) \geq [q(H - c) - (L - c)]\rho \sigma_r Z_a.
\]  

Recall that under the base model without correlation, the retailer should set \( p_r = L \) to serve both segments if and only if the fraction of the high-valuation segment is sufficiently small, i.e., \( q \leq \pi(L) / \pi(H) \). Contrast emerges when the realized advance sales volume contains useful information about the demand during the regular selling period. Specifically, upon observing \( N_a \), the
retailer expects that demand during the regular selling period consists of two parts: a deterministic part \( \rho \sigma_r Z_a \) and a stochastic part \( X \sim N(\mu_r, \sqrt{1-\rho^2} \sigma_r) \). The deterministic part certainly impacts the retailer’s decision on the regular selling price. In particular, the retailer’s earning from this deterministic part is \( (L - c) \rho \sigma_r Z_a \) at \( p_r = L \) and \( q(H - c) \rho \sigma_r Z_a \) at \( p_r = H \). Therefore, if \( q(H - c) \rho \sigma_r Z_a > (L - c) \rho \sigma_r Z_a \), the retailer earns more from this deterministic part of demand by setting \( p_r = H \) relative to \( p_r = L \). The opposite is true if \( q(H - c) \rho \sigma_r Z_a < (L - c) \rho \sigma_r Z_a \). The difference \( [q(H - c) - (L - c)] \rho \sigma_r Z_a \), which depends on the realized advance sales volume \( N_a \) via its normalized term \( Z_a \), captures how the presence of correlation impacts the regular selling price decision. Everything else being equal, the larger the value of \( [q(H - c) - (L - c)] \rho \sigma_r Z_a \), the retailer’s preference over \( p_r = H \) relative to \( p_r = L \) is stronger. We call such an effect the correlation effect, because it is brought into the model by the presence of demand correlation. The correlation effect implies that the retailer’s optimal regular selling price depends on the realization of the normalized advance sales volume \( Z_a \) according to the condition given in (4).

Let \( p_a \) be the advance selling price. If an informed customer purchases in advance, then her expected utility is equal to \( EV - p_a \). If she delays purchase to the regular selling period, then her expected utility is equal to 0 if \( p_{NPG} = H \), and is equal to \( \theta (EV - L) \) if \( p_{NPG} = L \), where \( \theta \) is the probability that a consumer believes to find the product available during the regular selling period. Therefore, to ensure that it is in the best interest of the informed customer to purchase in advance, the advance purchase price \( p_a \) must satisfy the constraint that

\[
EV - p_a \geq \theta (EV - L) \mathbf{P}([q(H - c) - (L - c)] \rho \sigma_r Z_a \leq \pi(L) - q \pi(H)).
\]

This, together with the fact that the retailer’s expected profit from advance selling is equal to \( (p_a - c) \mu_a \), results in the following expression for the retailer’s optimal advance selling price, denoted by \( p_{NPG}^a \),

\[
p_{NPG}^a = EV - \theta (EV - L) \mathbf{P}([q(H - c) - (L - c)] \rho \sigma_r Z_a \leq \pi(L) - q \pi(H)).
\]

Consequently, the retailer’s optimal expected profit under advance selling without PG, denoted by \( \Pi_{NPG} \), is

\[
\Pi_{NPG} = (p_{NPG}^a - c) \mu_a + E_{Z_a} \max\{ (L - c) \rho \sigma_r Z_a + \pi(L), q(H - c) \rho \sigma_r Z_a + q \pi(H) \}
\]
where $Z_a \sim N(0, 1)$.

**Proposition 4.** $\Pi^{NPG}$ is independent of $\sigma_a$ for $\sigma_a > 0$.

Under correlated demand, the realized advance sales volume indeed influences the retailer’s regular selling price and thus her profit from the regular selling period. Therefore, one may conjecture that the retailer’s total expected profit $\Pi^{NPG}$ should depend on the uncertainty of the advance demand. However, Proposition 4 states that $\Pi^{NPG}$ is independent of $\sigma_a$, consistent with the result under the base model without demand correlation. To see the intuition, we note that the presence of demand correlation brings the correlation effect into the retailer’s problem of deciding the regular selling price. However, from the definition of the correlation effect, the retailer’s optimal regular selling price $p_{r}^{NPG}$ depends on the realized advance sales volume $N_a$ only via its normalized term $Z_a = (N_a - \mu_a) / \sigma_a$ which is independent of $\sigma_a$. Such an independence result also implies that the informed customer’s expected utility from delaying purchase depends on the realized advance sales volume only via its normalized term. Therefore, the retailer’s optimal advance selling price $p_{a}^{NPG}$ that always makes the informed customers indifferent between buying early and later is also independent of $\sigma_a$. This, together with the fact that the retailer operates under make-to-order for advance sales, implies that the retailer’s expected profit from advance sales is also independent of $\sigma_a$. This explains Proposition 4.

Next we consider the scenario with price guarantee. Given the realized advance sales volume $N_a = n_a$ and the advance selling price $p_a$, the retailer’s maximum expected profit from the regular selling period (including the cost of refund to advance purchases due to the price guarantee) is

$$
\Pi^{PG}_{r}|_{N_a=n_a} = \max\{(L-c)\rho\sigma_r(n_a - \mu_a)/\sigma_a + \pi(L) - (p_a - L)n_a, q(H-c)\rho\sigma_r(n_a - \mu_a)/\sigma_a + q\pi(H)\}
$$

Let $Z_a = (N_a - \mu_a)/\sigma_a \sim N(0, 1)$. The retailer’s maximum expected profit from the regular selling period is

$$
\Pi^{PG}_r = E_{Z_a, N_a} \max\{(L - c)\rho\sigma_r Z_a + \pi(L) - (p_a - L)N_a, q(H - c)\rho\sigma_r Z_a + q\pi(H)\}.
$$

Therefore, the retailer’s optimal regular selling price $p_{r}^{PG} = L$ if and only if

$$
\pi(L) - q\pi(H) \geq [q(H - c) - (L - c)]\rho\sigma_r Z_a + (p_a - L)N_a
$$

(5)
Comparing the condition for \( p^r_{PG} = L \) under the base model without correlation, i.e., (2), with the condition for \( p^r_{PG} = L \) under the extended model with correlation, i.e., (5), we see that the extra term \( [q(H - c) - (L - c)]\rho \sigma_r Z_a \) shows up in the latter condition influencing the retailer’s optimal selling price decision. This is the correlation effect, as explained under the former case with NPG. Whether or not the retailer should serve only the high-value segment or both segments is now driven by two effects: the correlation effect via the term \( [q(H - c) - (L - c)]\rho \sigma_r Z_a \) and the price match effect via the term \((p_a - L)N_a\). The former depends on the realized advance sales volume via its normalized term \( Z_a \) whereas the latter depends on the full value of \( N_a \). To see how the price match effect works, note that the higher the value of \( N_a \), the larger the price match cost \((p_a - L)N_a\), implying that everything else being equal, a larger volume of advance sales makes the strategy of serving only the high-value segment more preferable. However, the same cannot be said for the correlation effect. Specifically, it depends on the sign of \([q(H - c) - (L - c)]\rho \sigma_r \). If \([q(H - c) - (L - c)]\rho \sigma_r \geq 0\), then similar to the price match effect, the higher the value of \( Z_a \), the more preferable the strategy of serving only the high-value segment. This is because the condition \([q(H - c) - (L - c)]\rho \sigma_r \geq 0\) implies that it is more profitable to serve only the high-value segment if we simply focus on the deterministic part of the demand during the regular selling period due to correlation. When the size of the deterministic demand grows (i.e. \( Z_a \) increases), serving only the high-value segment is more preferable. Contrast emerges when \([q(H - c) - (L - c)]\rho \sigma_r < 0\), which implies that it is more profitable to serve both segments if we simply focus on the deterministic part. Therefore, when the size of such a deterministic part grows (i.e. \( Z_a \) increases), serving both segments is more preferable. Under the former case \([q(H - c) - (L - c)]\rho \sigma_r \geq 0\), the correlation effect works in the same direction as the price match effect in the sense that as \( N_a \) increases, both effects would imply that serving only the high-value segment is more preferable. Under the latter case \([q(H - c) - (L - c)]\rho \sigma_r < 0\), the correlation effect works in the opposite direction as the price match effect because as \( N_a \) increases, the correlation effect would favor serving both segments whereas the price match effect would favor serving only the high-value segment. This observation is the key driving force for the subsequent analytical result on how the pre-order demand uncertainty \( \sigma_a \) impacts the retailer’s optimal performance under PG with demand correlation.
Let $p_a$ be the advance selling price. If an informed customer purchases in advance, then her expected utility is equal to $EV - p_a$. If she delays purchase to the regular selling period, then her expected utility is equal to $0$ if $p_{r}^{PG} = H$, and is equal to $\theta(EV - L)$ if $p_{r}^{PG} = L$, where $\theta$ is the probability that a consumer believes to find the product available during the regular selling period. Therefore, to ensure that it is in the best interest of the informed customer to purchase in advance, the advance purchase price $p_a$ must satisfy the constraint that

$$EV - p_a \geq \theta(EV - L)\mathbf{P}(\pi(L) - q\pi(H)) \geq [q(H - c) - (L - c)]\rho\sigma_r Z_a + (p_a - L)N_a.$$ 

This, together with the fact that the retailer’s expected profit increases in $p_a$, implies that $p_{a}^{PG} = \max\{p_a | EV - p_a \geq \theta(EV - L)\mathbf{P}(\pi(L) - q\pi(H)) \geq [q(H - c) - (L - c)]\rho\sigma_r Z_a + (p_a - L)N_a\}$.

Consequently, the retailer’s optimal expected profit under PG, denoted by $\Pi^{PG}$, is

$$\Pi^{PG} = (p_{a}^{PG} - c)\mu_a + E_{Z_a,N_a} \max\{(L - c)\rho\sigma_r Z_a + \pi(L) - (p_{a}^{PG} - L)N_a, q(H - c)\rho\sigma_r Z_a + q\pi(H)\}.$$ 

**Proposition 5.** If $[q(H - c) - (L - c)]\rho\sigma_r \geq 0$, then $\Pi^{PG}$ increases in $\sigma_a$ for $\sigma_a > 0$. If $[q(H - c) - (L - c)]\rho\sigma_r < 0$, then $\Pi^{PG}$ first decreases and then increases in $\sigma_a$ for $\sigma_a > 0$.

Proposition 5 demonstrates that the sign of $\rho$ and $q(H - c) - (L - c)$ plays an important role in determining the impact of $\sigma_a$ on the seller’s profit under PG. If demands are positively (negatively) correlated and $q$ is high (low), then $\Pi^{PG}$ increases in $\sigma_a$. Otherwise, if demands are positively (negatively) correlated and $q$ is low (high), $\Pi^{PG}$ first decreases and then increases in $\sigma_a$.

In contrast to the result that $\Pi^{NPG}$ is independent of $\sigma_a$, Proposition 5 states that $\Pi^{PG}$ depends on $\sigma_a$. This is because the retailer’s optimal regular selling price depends on the realized advance sales volume $N_a$ not just via the normalized term $Z_a$ but also the full value of $N_a$ (see (5)), due to the fact that the price match cost depends on the full value of realized advance sales volume $N_a$. Therefore, Proposition 5 describes precisely how the pre-order demand uncertainty influences the retailer’s optimal performance under PG. As alluded by the previous discussions on interpreting (5), the results are presented in two distinct cases. In the first case, $[q(H - c) - (L - c)]\rho\sigma_r \geq 0$, which implies that the correlation effect works in the same direction as the price match effect. Both effects would make serving only the high-value segment (by setting $p_{r}^{PG} = H$) more preferable as
Na increases. This implies that similar to the base model without demand correlation, the retailer’s actual profit margin of advance sales (after deducting the price match cost if any) is equal to \( L - c \) when the realized advance sales volume is small, and jumps up to \( p^P - c \) for large value of \( Na \) (similar to Figure 2). In other words, the retailer’s profit margin of advance sales increases in the realized advance sales volume. The presence of correlation effect only enforces such a pattern of increasing profit margin by lowering the cutoff value for \( Na \) above which the retailer switches from serving both segments to serving only the high-value segment. As explained in the previous section, the result of increasing profit margin in \( Na \) implies that the retailer benefits from an increase in the pre-order demand uncertainty \( \sigma_a \). This explains the first part of Proposition 5.

In the second case where \( [q(H - c) - (L - c)]\rho_\sigma Z_a < 0 \), the correlation effect works in the opposite direction from the price match effect, inducing the retailer to serve both segments for large value of realized advance sales. Naturally, the retailer’s optimal decision on the regular selling price depends on which effect is stronger. Interestingly, the strength of the correlation effect, as captured by the term \( [q(H - c) - (L - c)]\rho_\sigma Z_a \), is intact when \( \sigma_a \) increases, whereas the strength of the price match effect (captured by the term \( (p_a - L)Na \)) becomes more prominent as \( \sigma_a \) increases because a large value of \( \sigma_a \) implies more extremely large values of \( Na \) resulting in more extremely large cost of price match. Therefore, as \( \sigma_a \) increases over the range of sufficiently small values, the correlation effect dominates the price match effect, resulting in a pattern of decreasing profit margin (for advance sales) in \( Na \). This is in contrast with the result in the first case. The presence of correlation effect undermines the price match effect, reversing the retailer’s pricing strategy during the regular selling period contingent on the realized advance sales and resulting in decreasing profit margin for advance sales as \( Na \) increases. Hence, the fact that the profit margin for advance sales decreases in \( Na \) implies that the retailer under PG is hurt from an increase in the pre-order demand uncertainty \( \sigma_a \) over the range of sufficiently small values. When \( \sigma_a \) increases over the range of sufficiently large values, the price match effect dominates the correlation effect, restoring the pattern of increasing profit margin in \( Na \) and implying that the retailer under PG again benefits from an increase in the pre-order demand uncertainty over the range of sufficiently large values. This explains the second part of Proposition 5.
The demand correlation brings a new force, so called the correlation effect, into the retailer’s consideration in deciding the regular selling price. This new feature enriches our analytical results on the comparison between PG and NPG under the base model without correlation. An implication from Propositions 4 and 5 is that the retailer’s strategic choice between PG and NPG depends jointly on the pre-order demand uncertainty $\sigma_a$, the correlation coefficient $\rho$, and the fraction of high-value segment $q$. Under the case with $[q(H - c) - (L - c)]\rho\sigma_r \geq 0$ where the correlation effect works in the same direction as the price match effect, the retailer’s strategic choice between PG and NPG is qualitatively similar to that under the base model without correlation, i.e., choosing PG for large values of $\sigma_a$ and NPG for small values of $\sigma_a$. Contrast emerges under the case with $[q(H - c) - (L - c)]\rho\sigma_r < 0$ where the correlation effect works against the price match effect. The retailer should choose NPG if and only if $\sigma_a$ is of intermediate values.

Proposition 5 has the following implications. When a retailer such as Amazon believes demands are correlated in the two periods, she needs to be more careful. The retailer needs to check whether the correlation effect works in the same direction as the price match effect. If so, i.e., both effects favor charging a high selling season price after realizing a high advance sales volume, then our previous guidance in section 3 still hold. The retailer’s optimal profit under PG increases in pre-order demand uncertainty. She should offer PG if pre-order demand uncertainty is high. Otherwise, the guidance needs to be revised. The retailer’s optimal profit under PG first decreases and then increases in pre-order demand uncertainty. The retailer should not offer PG if and only if pre-order demand uncertainty is medium.

5. Discussions

In this section, we provide further discussions on advance selling and pricing strategies. In Section 5.1, we study price commitment (PC) and compare all three pricing strategies: PG, NPG, PC. In Section 5.2, we examine whether or not a retailer should sell in advance, with the option of adopting PG.

5.1. Price Commitment

Other than PG and NPG, price commitment (PC) can also be used in advance selling. Under PC, the retailer commits both pre-order price and selling season price at the beginning of the advance
selling period. The following lemma shows the optimal prices for the retailer using PC, where the superscript “pc” represents price commitment.

**Lemma 5.** If the fraction of high-valuation customers is small, i.e., $q < \bar{q}$, then $p_a^{PC} = EV - \theta(EV - L)$, $p_r^{PC} = L$, and $\Pi^{PC} = (EV - \theta(EV - L) - c)\mu_a + \pi(L)$; Otherwise if the fraction of high-valuation customers is high, i.e., $q > \bar{q}$, then $p_a^{PC} = EV$, $p_r^{PC} = H$, and $\Pi^{PC} = (EV - c)\mu_a + q\pi(H)$.

Lemma 5 shows that the retailer is more likely to set a higher spot price and hence a higher pre-order price under PC than under NPG. This is because PC can eliminate pre-order consumers’ waiting incentive. Lemma 5 also implies that pre-order demand uncertainty has no effect on the profit under PC. Next, we compare the retailer’s profits under PC, PG, and NPG.

**Proposition 6.** PC is dominated by either PG or NPG. That is, by optimally deciding whether or not to offer PG under dynamic pricing, the retailer can get more profit under dynamic pricing than under price commitment.

1. If $q \in (0, \bar{q})$, $\Pi^{PC} = \Pi^{NPG}$.
2. If $q \in (\bar{q}, \hat{q})$, $\Pi^{PC} < \Pi^{PG}$.
3. If $q \in [\bar{q}, 1)$, $\Pi^{PC} = \Pi^{NPG} = \Pi^{PG}$.

Proposition 6 shows that the retailer does not need to consider price commitment even if he can credibly do so. By optimally deciding whether or not to offer PG under dynamic pricing, the retailer’s profit under dynamic pricing is more than or equal to her profit under price commitment.

### 5.2. Optimality of Advance Selling

We study the retailer’s decision whether or not to sell in advance. If the retailer sells in advance, she can further decide whether to offer PG or not. If the retailer does not sell in advance, then all consumers decide whether to buy in the second period. Let superscript “NAS” represent the retailer’s optimal profit without advance selling. We compare $\Pi^{NAS}$, $\Pi^{PG}$, and $\Pi^{NPG}$ to obtain the following.

**Proposition 7.** There exists a threshold $\bar{c} \in (L, EV)$. If the marginal cost is low, i.e., $c < \bar{c}$, the retailer should sell in advance, i.e., $\max\{\Pi^{PG}, \Pi^{NPG}\} \geq \Pi^{NAS}$. If the marginal cost is high, i.e., $c > \bar{c}$, the retailer should not sell in advance, i.e., $\max\{\Pi^{PG}, \Pi^{NPG}\} \leq \Pi^{NAS}$. 

Proposition 7 suggests that retailers sell in advance for products with relatively small marginal costs, i.e., \( c < \tilde{c} \). This is consistent with Xie and Shugan (2001), which does not consider PG and suggests to sell in advance if \( c < L \). Our threshold \( \tilde{c} \) is higher than \( L \) because advance selling is more profitable with the option of adopting PG.

The above analytical result implies that advance selling is appealing for products with relatively small marginal cost, and the use of price guarantee strengthens the value of advance selling. Our result provides theoretical support for Amazon’s practice of using advance selling together with PG for small to medium cost products such as digital products, not so much for large value items.

5.3. Consumer Waiting Cost

In some scenarios, if an informed consumer delays purchase until the regular selling season, there is a waiting cost and thus her utility in the selling season is discounted by \( (1 - \delta) \), \( \delta \in (0, 1) \). When consumer waiting cost is higher for a product, discount rate \( \delta \) is higher. Different products may have different levels of discount rate. For some products like video games and DVDs, customer waiting cost may be higher and thus \( \delta \) may be higher, compared to other products such as TVs and cameras.

With the consideration of consumer waiting cost, our original analyses carry through with slight changes. The discount rate \( \delta \) shows up in the optimal prices and profits in the same places as the in-stock probability \( \theta \) does. If we define \( \theta' = (1 - \delta)\theta \), then all the expressions are unchanged except replacing the original in-stock probability \( \theta \) by the new parameter \( \theta' \). Our results still hold qualitatively.

The impact of a consumer’s waiting cost on the consumer’s decisions is the same as the impact of out-of-stock probability. They both reduce consumer waiting incentive. When the waiting cost increases, consumer waiting incentive decreases. Thus, the seller is able to increase the optimal pre-order price without driving pre-order consumers away. This increases the value of PG and thus enlarges the region where PG dominates NPG. For example, as the waiting cost increases, the thresholds \( \tilde{\sigma}_a \) and \( \tilde{q} \) in Figure 3 decreases. Hence, the lower left region in Figure 3 shrinks. That is, the retailer is more likely to prefer PG for products with higher consumer waiting costs.
The above result is also consistent with our observations from practice. For example, video games and DVDs may have higher waiting costs than TVs and cameras. Amazon offers PG for pre-orders of the former products but no PG for the latter group.

5.4. The Fluid Demand Assumption

In our base model, we adopt the fluid demand assumption that the number of high-value customers is equal to \( qN_\alpha \) for advance sales and \( qN_r \) for regular sales. Now we discretize each individual customer and use a Bernoulli trial to model each customer’s valuation. Such a modeling change only impacts the regular demand when the regular selling price is \( H \). Specifically, the regular demand under the price \( H \) is a binomial variable denoted by \( B(N_r, q) \) (Png 1989) for any realized value of \( N_r \). Hence, the only change we need to make is to replace \( q\pi(H) \) throughout the analysis in our base model by \( \pi(H, q) \), where

\[
\pi(H, q) = \max_{Q \geq 0} \{ E[\min(B(N_r, q), Q)] - cQ \}
\]

It is clear that as \( q \) increases, the regular demand \( B(N_r, q) \) stochastically increases, implying that \( \pi(H, q) \) strictly increases in \( q \). Let \( \bar{q} \) be the threshold such that \( \pi(H, q) \geq \pi(L) \) if and only if \( q \geq \bar{q} \). By repeating the same analysis as in the base model, we can show that all of our results continue to hold qualitatively.

5.5. The Two-Point Valuation Distribution Assumption

In this subsection, we consider a general case with a continuous valuation distribution. Let \( F \) and \( f \) be the cumulative distribution function and probability density function of consumer valuation \( V \) with the support \([0, \infty)\). Both \( F \) and \( f \) are continuous and twice differentiable. Let \( EV \) be the mean of the continuous valuation distribution.

For convenience, define \( \pi(p_r) = \max_{Q \geq 0} \{ p_r E[\min(N, \bar{F}_2(p_r), Q)] - cQ \} \). Under PG, the retailer’s selling season price problem can be formulated as follows.

\[
\Pi^{PG}_r(p^PG_\alpha, n_a) = \max_{p_r} \{ \pi(p_r) - (p_a - p_r)^+ n_a \} . \tag{6}
\]

Recall that under a two-point distribution, the selling season price is either \( L \) or \( H \). So the optimal selling season price can be determined by comparing \( \pi(H) - n_a(p_a - H)^+ \) and \( \pi(L) - n_a(p_a - L)^+ \). In contrast, under a continuous valuation distribution, the above selling season price problem (6)
is a price dependent newsvendor problem. So there is no closed-form solution for the selling season price $p_r$. Further analyses on the pre-order price $p_a$ and profit under PG become intractable as well under a continuous distribution.

In order to verify whether or not our result about pre-order demand uncertainty still hold under a continuous distribution, we perform extensive numerical experiments with the following parameter sets: $c = 5$; $V$ is uniformly distributed within $[\mu_v - \epsilon, \mu_v + \epsilon]$, where $\mu_v \in \{1.5c, 2c, 3c\}$ and $\epsilon = \{1, 1 + \mu_v/10, \ldots, \mu_v\}$; $N_a$ and $N_r$ follow bivariate normal distribution such that $N_i \sim N(\mu_i, \sigma_i)$, $i \in \{a, r\}$, with $\rho \in \{-0.5, 0, 0.5\}$. $\mu_a \in \{10, 20\}$, and for each $\mu_a$, $\mu_r \in \{0.5\mu_a, \mu_a, 2\mu_a\}$. $\sigma_r \in \{1, 4, 10, 20\}$, and $\sigma_a$ evenly takes 20 points between 1 and $10\sigma_r$. In total, we tested 47520 instances and observe two consistent patterns. First, the retailer’s profit under PG increases in pre-order demand uncertainty $\sigma_a$. Second, the retailer is worse off with PG if $\sigma_a$ is small. Figure 4 provides a representative numerical result. The following parameter values are used in Figure 4. The marginal cost $c = $5. Consumer valuation $V$ follows a uniform distribution within $[13, 17]$. The market sizes $N_a$ and $N_r$ follows a bivariate normal distribution with means $\mu_a = 10$, $\mu_r = 10$, $\sigma_r = 3$, and $\rho = 0.5$. $\sigma_a$ is increased from 1 to 10 in 1 increment.

![Figure 4](image-url) The retailer’s profit under PG increases in pre-order demand uncertainty
6. Conclusions

We study the question of whether or not the retailer should offer PG in advance selling. We provide the necessary and sufficient conditions under which the retailer should (or not) use PG. In particular, we find that the retailer is worse off with PG relative to NPG if and only if both the fraction of high-valuation consumers and the pre-order demand uncertainty are sufficiently small. Furthermore, we show that the profit improvement percentage by PG can be as high as 24%, compared to NPG.

In addition to PG and NPG, the retailer may commit both the advance selling price and the regular selling price at the beginning of the advance selling period. We show that price commitment is dominated by either PG or NPG from the retailer’s perspective. Further, the retailer may choose not to sell in advance. We show that the retailer is better off by selling in advance than not selling in advance when marginal cost is higher than a threshold.

Our research provides managerial insights and guidance for companies in practice. For example, for different products, a company’s PG strategy may be different. Depending on a product’s pre-order demand uncertainty level and the fraction of high-valuation consumers in the market for the product, it may be optimal for a company to offer PG for one product but not for another product.

Our result explains Amazon’s current practice where PG for pre-orders is offered to most creative digital products such as newly-released movie dvds and video games but not to the relatively mature products such as cameras and TVs. It is plausible that the former group of products have relatively high degree of pre-order demand uncertainty due to the product novelty.

When a retailer such as Amazon believes demands are correlated in the two periods, she needs to be more careful. The retailer needs to check whether the correlation effect works in the same direction as the price match effect. If so, i.e., both effects favor charging a high selling season price after realizing a high advance sales volume, then our previous guidance still hold. The retailer’s optimal profit under PG increases in pre-order demand uncertainty. She should offer PG if pre-order demand uncertainty is high. Otherwise, the guidance needs to be revised. The retailer’s optimal profit under PG first decreases and then increases in pre-order demand uncertainty. The retailer should not offer PG if and only if pre-order demand uncertainty is medium.
In this paper, we do not consider heterogeneous consumers in the 1st period. As an alternative model that captures customer heterogeneity in pre-order, we can extend our base model by introducing a heterogeneous discount factor $\delta$ into the informed customer’s utility under delayed purchase (see Cachon and Swinney 2009). The discount factor $\delta$ can be interpreted as the customer’s patience for waiting. Customers may differ in their patience level $\delta$. Let $F(\cdot)$ be the distribution function of the patience level $\delta$ with the support interval $[0, 1]$. Each customer knows his own patience level $\delta$ whereas the firm knows that the patience level of each individual customer is randomly drawn from the interval $[0, 1]$ according to the distribution function $F(\cdot)$. Such a model extension leads to the split of informed customers (who are aware of advance purchase) into two groups: those with low patience level purchase early whereas those with high patience level prefer waiting and join the uninformed customers during the regular sales period. Further, the realized pre-order volume reveals the size of those informed customers who have decided to wait and show up during the regular sales period, implying that the pre-order demand is correlated with the regular demand. For such an extended model, while we do not have a full-blown analysis, we are able to prove that our core result that the firm’s expected profits under PG increase in the pre-order demand uncertainty (Proposition 2) continues to hold, and our core comparison result continues to hold, i.e., PG should not be offered when both pre-order demand uncertainty and the fraction of high-valuation segment are sufficiently small.

In this paper, consumers have homogeneous belief on the valuation distribution. A direction for future work is to model the heterogeneous beliefs in the 1st period consumers. For example, in the 1st period, consumers may have different beliefs on the valuation distributions. Some consumers buy early and some do not. Based on consumer behavior in the 1st period, the retailer can infer the information on the belief of the valuation distribution for those consumers who choose to wait and join the pool in the regular season. Thus, based on this information, the retailer can update the valuation distribution of consumers in the regular season. Doing so would enable the analysis of how information advantage through advance selling benefits a retailer.

References


Wei, M., Zhang, F. 2018 Advance Selling to Strategic Consumers: Preorder Contingent Production Strategy with Advance Selling Target *Production and Operations Management* 27(7) 1221-1235.


Appendix A: Proofs of Lemmas and Propositions

**Proof of Lemma 1**

Comparing $\Pi^{NPG}_r(H)$ with $\Pi^{NPG}_r(L)$, we have $\Pi^{NPG}_r(H) - \Pi^{NPG}_r(L) = q\pi(H) - \pi(L)$. If $q \geq \frac{\sigma(L)}{\sigma(H)}$, then $\Pi^{NPG}_r(H) \geq \Pi^{NPG}_r(L)$. So it’s optimal to sell at $H$ if $q \geq \frac{\sigma(L)}{\sigma(H)}$ and to sell at $L$ if $q < \frac{\sigma(L)}{\sigma(H)}$.

**Proof of Lemma 2**

It is easy to observe that the profit function increases in $p_a^{NPG}$. Therefore, the constraints bind at optimality. That is, if $q < \bar{q}$, $p_a^{NPG} = EV - \theta(EV - L)$. If $q \geq \bar{q}$, $p_a^{NPG} = EV$.

**Proof of Lemma 3**

Note that $\Pi^{PG}_r(L) = \pi(L) - n_a(p_a - L)^+$ and $\Pi^{PG}_r(H) = q\pi(H) - n_a(p_a - H)^+$. The retailer compares $\Pi^{PG}_r(L)$ and $\Pi^{PG}_r(H)$ to determine the optimal selling season price.

$$
\Pi^{PG}_r(H) - \Pi^{PG}_r(L) = q\pi(H) - \pi(L) + n_a((p_a - L)^+ - (p_a - H)^+)
$$

If $q > \bar{q}$, by the definition of $\bar{q}$, $\pi(L) < q\pi(H)$. Therefore, $\Pi^{PG}_r(H) > \Pi^{PG}_r(L)$. The retailer’s optimal selling season price $p_r^{PG} = H$. The retailer’s inventory problem given $p_r^{PG} = H$ is a standard newsvendor problem. The optimal order quantity is $Q_r^{PG} = qG_r^{-1}((H - c) / H)$.

If $q \leq \bar{q}$, by the definition of $\bar{q}$, $\pi(L) \geq q\pi(H)$. Considering that $n_a(p_a - L)^+ - n_a(p_a - H)^+ \geq 0$ and increases in $n_a$, we show there exists a threshold $\bar{n}_a$.

$$
\bar{n}_a(p_a) \equiv (\pi(L) - q\pi(H))/[(p_a - L)^+ - (p_a - H)^+].
$$

When $n_a \leq \bar{n}_a(p_a)$, $\Pi^{PG}_r(H) - \Pi^{PG}_r(L) \leq 0$. The retailer’s optimal selling season price $p_r^{PG} = L$. The corresponding optimal newsvendor inventory is $Q_r^{PG} = G_r^{-1}((L - c) / L)$.

Otherwise when $n_a > \bar{n}_a(p_a)$, $\Pi^{PG}_r(H) - \Pi^{PG}_r(L) > 0$. The retailer’s optimal selling season price $p_r^{PG} = H$. The corresponding optimal newsvendor inventory is $Q_r^{PG} = qG_r^{-1}((H - c) / H)$.

**Proof of Lemma 4**

The solution is straightforward when $q \geq \bar{q}$. We now focus on the scenario when $q < \bar{q}$.
Let $\Pi^\text{PG}(p_a)$ be the expected profit under PG for the advance selling price $p_a$. Differentiating $\Pi^\text{PG}(p_a)$ with respect to $p_a$ yields
\[
\frac{\partial}{\partial p_a} \Pi^\text{PG}(p_a) = \mu_a - \int_0^{\bar{n}_a(p_a)} N_a dG_a(N_a) > 0,
\]
which implies that the retailer will choose the highest price possible.

Furthermore, we can show that the right hand of the constraint, i.e., $L + \frac{q(H-L)(1-\theta G_a(\bar{n}_a(p_a)))}{1-G_a(\bar{n}_a(p_a))}$, decreases in $p_a$. This is because $\frac{1-\theta G_a(\bar{n}_a(p_a))}{1-G_a(\bar{n}_a(p_a))}$ increases in $G_a(\bar{n}_a(p_a))$, which increases in $\bar{n}_a(p_a)$. In addition, according to the definition of $\bar{n}_a(p_a)$, we can see that $\bar{n}_a(p_a)$ decreases in $p_a$. Therefore, the right hand of the constraint decreases in $p_a$. Therefore, the constraint $p_a \leq L + \frac{q(H-L)(1-\theta G_a(\bar{n}_a(p_a)))}{1-G_a(\bar{n}_a(p_a))}$ binds at optimality. That is, $p_a^\text{PG} = L + \frac{q(H-L)(1-\theta G_a(\bar{n}_a(p_a)))}{1-G_a(\bar{n}_a(p_a))}$.

Note that we restrict our discussions to $p_a^\text{PG} \leq H$ in this paper. Therefore, the optimal advance selling price is $p_a^\text{PG} = \min\{L + \frac{q(H-L)(1-\theta G_a(\bar{n}_a(p_a)))}{1-G_a(\bar{n}_a(p_a))}, H\}$.

**Proof of Proposition 1**

First, we solve the retailer’s problem under PG. When $\sigma_a = 0$, $n_a = \mu_a$. The retailer’s spot price problem is
\[
\max\{\pi(L) - \mu_a(p_a - L), \pi(H)\}.
\]

The retailer charges a spot price $p_r^\text{PG} = L$ if $\pi(L) - \mu_a(p_a - L) > \pi(H)$, i.e., $p_a \leq L + \frac{\pi(L) - \pi(H)}{\mu_a}$. When $p_r^\text{PG} = L$, the retailer gives refunds to pre-order consumers. Consumer expected utility from advance purchase $(\text{EV} - L)$ is higher than expected utility from waiting $(\theta(\text{EV} - L))$. So all informed consumers buy early. The retailer’s expected profit $\Pi^\text{PG} = (L - c)\mu_a + \pi(L)$.

If $p_a > L + \frac{\pi(L) - \pi(H)}{\mu_a}$, $p_r^\text{PG} = H$. Informed consumers buy early if consumer expected utility from advance purchase $(\text{EV} - p_a)$ is no less than consumer expected utility from waiting $(0)$, i.e., $p_a \leq \text{EV}$. If $\text{EV} \geq L + \frac{\pi(L) - \pi(H)}{\mu_a}$, i.e.,
\[
q \geq \frac{\pi(L)}{(H - L)\mu_a + \pi(H)},
\]
there exists a feasible range for $p_a^\text{PG}$ so that informed consumers buy early and $p_r^\text{PG} = H$. That is, $L + \frac{\pi(L) - \pi(H)}{\mu_a} < p_a \leq \text{EV}$. In order to maximize profit, the retailer should charge $p_a^\text{PG} = \text{EV}$ and $\Pi^\text{PG} = \mu_a(\text{EV} - c) + q\pi(H)$. Otherwise if
\[
q < \frac{\pi(L)}{(H - L)\mu_a + \pi(H)},
\]
then $\text{EV} < L + \frac{\pi(L) - \pi(H)}{\mu_a}$. There does not exist a feasible $p_a^\text{PG}$ that satisfies both $p_a \leq \text{EV}$ and $p_a > L + \frac{\pi(L) - \pi(H)}{\mu_a}$. Therefore, if $q < \frac{\pi(L)}{(H - L)\mu_a + \pi(H)}$, the retailer charges $p_r^\text{PG} = L$ and $p_a^\text{PG}$ can be any number being greater than $L$. The retailer’s expected profit $\Pi^\text{PG} = (L - c)\mu_a + \pi(L)$.

In summary, if $q > \hat{q} \equiv \frac{\pi(L)}{(H - L)\mu_a + \pi(H)}$, then $p_a^\text{PG} = \text{EV}$, $p_r^\text{PG} = H$ and $\Pi^\text{PG} = \mu_a(\text{EV} - c) + q\pi(H)$. If $q \leq \hat{q}$, then $p_a^\text{PG}$ can be any number being greater than $L$, $p_r^\text{PG} = L$, and $\Pi^\text{PG} = (L - c)\mu_a + \pi(L)$. 
Since the left hand side of (10) is independent on parts. Depending on whether
The left hand side of expressions (10) and (11) is independent on $\sigma$
reduced as $\prod$
The transformation from expression (7) to (8) is because $\bar{q}<\sigma$
Note that $\bar{p}$
Second, strategies under NPG is unaffected. When $q \geq \bar{q}$, $\Pi^{NPG} = (EV - c)\mu_a + q\pi(H)$. When $q < \bar{q}$, $\Pi^{NPG} = (EV - \theta(EV - L) - c)\mu_a + \pi(L)$.

If $q \geq \bar{q}$, then $\Pi^{PG} = \Pi^{NPG} = (EV - c)\mu_a + q\pi(H)$. If $q < \bar{q}$, then $\Pi^{PG} = (EV - c)\mu_a + q\pi(H)$, which is higher than $\Pi^{NPG} = (EV - \theta(EV - L) - c)\mu_a + \pi(L)$ based on the definition of $\bar{q}$. If $q < \bar{q}$, then $\Pi^{PG} = (L - c)\mu_a + \pi(L)$, which is lower than $\Pi^{NPG} = (EV - \theta(EV - L) - c)\mu_a + \pi(L)$.

**Proof of Proposition 2**

To show that $\Pi^{PG}$ strictly increases in $\sigma_a$ when $q < \bar{q}$, we start with the expected profit function under PG when $q < \bar{q}$.

$$
\Pi^{PG} = \int_{0}^{\bar{n}_a} [(L - c)n_a + \pi(L)]dG_a(n_a) + \int_{\bar{n}_a}^{\infty} [(p_a^{PG} - c)n_a + q\pi(H)]dG_a(n_a)
$$

$$
= (L - c)\mu_a + \pi(L) + \int_{0}^{\bar{n}_a} (p_a^{PG} - L)n_a dG_a(n_a) - \int_{\bar{n}_a}^{\infty} (\pi(L) - q\pi(H))dG_a(n_a)
$$

$$
= (L - c)\mu_a + \pi(L) + \int_{0}^{\bar{n}_a} \pi(L) - q\pi(H) n_a dG_a(n_a) - \int_{\bar{n}_a}^{\infty} (\pi(L) - q\pi(H))dG_a(n_a)
$$

$$
= (L - c)\mu_a + \pi(L) + \int_{0}^{\bar{n}_a} (n_a - \bar{n}_a) dG_a(n_a)
$$

$$
= (L - c)\mu_a + \pi(L) + \frac{\bar{n}_a \pi(L) - q\pi(H)}{\bar{n}_a} E[N_a - \bar{n}_a]
$$

The transformation from expression (7) to (8) is because $\bar{n}_a = \frac{\pi(L) - q\pi(H)}{\bar{n}_a}$. Thus $p_a^{PG} - L = \frac{\pi(L) - q\pi(H)}{\bar{n}_a}$.

According to Lemma 4, there are two scenarios with different values for $p_a^{PG}$: (1) $p_a^{PG} = L + \frac{1 - \theta G_a(\bar{n}_a)(H - L)}{1 - G_a(\bar{n}_a)}$. (2) $p_a^{PG} = H$. Next, we further analyze $\Pi^{PG}$ under each scenario. For convenience, let $Z \equiv \frac{N_a - \mu_a}{\sigma_a}$. Let $\Phi(\cdot)$ be the distribution function of $Z$. We can normalize $\bar{n}_a$ by defining $z_a = \frac{\bar{n}_a - \mu_a}{\sigma_a}$. Correspondingly, $G_a(\bar{n}_a) = \Phi(z_a)$.

Note that $\bar{n}_a \geq 0$ implies that $z_a \geq -\mu_a/\sigma_a$.

First, if $p_a^{PG} = L + \frac{1 - \theta G_a(\bar{n}_a)(H - L)}{1 - G_a(\bar{n}_a)}$, then considering $\bar{n}_a = \frac{\pi(L) - q\pi(H)}{p_a^{PG} - L}$, we have:

$$
\pi(L) - q\pi(H) \over q(H - L) = \frac{1 - \theta G_a(\bar{n}_a)}{1 - G_a(\bar{n}_a)} \bar{n}_a
$$

$$
= \frac{1 - \theta \Phi(z_a)}{1 - \Phi(z_a)} (\mu_a + \sigma_a z_a)
$$

$$
= \frac{1 - \theta \Phi((\bar{n}_a - \mu_a)/\sigma_a)}{1 - \Phi((\bar{n}_a - \mu_a)/\sigma_a))} \bar{n}_a
$$

The left hand side of expressions (10) and (11) is independent on $\sigma_a$. The right hand sides contain two parts. Depending on whether $\mu_a$ is larger than $\bar{n}_a$, we have two subcases.

Subcase 1: If $\mu_a < \bar{n}_a$, then $z_a > 0$. Thus, the right hand side of (10) increases in $z_a$ and $\sigma_a$, respectively. Since the left hand side of (10) is independent on $\sigma_a$, we know that $z_a$ must strictly decrease in $\sigma_a$.

Considering the profit function $\Pi^{PG} = (L - c)\mu_a + \pi(L) + (\pi(L) - q\pi(H))\frac{E[(Z - z_a)^+]}{\frac{\sigma_a^2}{2} + z_a}$, which can be further simplified as $\Pi^{PG} = (L - c)\mu_a + \pi(L) + (\pi(L) - q\pi(H))\frac{E[(Z - z_a)^+]}{\frac{\sigma_a^2}{2} + z_a}$. It is easy to show that $\frac{E[(Z - z_a)^+]}{\frac{\sigma_a^2}{2} + z_a}$ strictly
decreases in \( z_a \), which has been shown decreasing in \( \sigma_a \). Therefore, we draw the conclusion that \( \Pi^{PG} \) strictly increases in \( \sigma_a \).

Subcase 2: If \( \mu_a > \bar{n}_a \), then \( z_a < 0 \) and \( \bar{n}_a - \mu_a < 0 \). Therefore, the right hand side of (11) strictly increases in \( \bar{n}_a \) and \( \sigma_a \) respectively. Since the left hand side of (11) is independent on \( \sigma_a \), \( \bar{n}_a \) must strictly decreases in \( \sigma_a \).

Considering the profit function \( \Pi^{PG} = (L - c)\mu_a + \pi(L) + \frac{\pi(L) - q\pi(H)}{\bar{n}_a}E[\bar{n}_a - \bar{n}_a] \), which can be further simplified as \( \Pi^{PG} = (L - c)\mu_a + \pi(L) + (\pi(L) - q\pi(H))\frac{E[\bar{n}_a Z - (\bar{n}_a - \mu_a)]}{\bar{n}_a} \). It is easy to show that \( E[(Z\sigma_a - (\bar{n}_a - \mu_a))] \) strictly increases in \( \sigma_a \) given any fixed \( \bar{n}_a > 0 \) and it strictly decreases in \( \bar{n}_a \) given any fixed \( \sigma_a \). Therefore, considering that \( \bar{n}_a \) strictly decreases in \( \sigma_a \), \( E[(Z\sigma_a - (\bar{n}_a - \mu_a))] \) strictly increases in \( \sigma_a \). Thus, \( \Pi^{PG} \) strictly increases in \( \sigma_a \).

Second, if \( p^a_{PG} = H \), then \( \bar{n}_a = \frac{z(H) - q\pi(H)}{E[L]} \), which is independent on \( \sigma_a \). \( \bar{n}_a \equiv \frac{z(H) - q\pi(H)}{E[L]} \). Therefore, we can further simplify the expression of the expected profit function as \( \Pi^{PG} = (L - c)\mu_a + \pi(L) + (H - L)E[\bar{n}_a - \bar{n}_a] \), \( \bar{n}_a \) given any fixed \( \bar{n}_a > 0 \) and it strictly decreases in \( \bar{n}_a \) given any fixed \( \sigma_a \). Therefore, we conclude that \( \Pi^{PG} \) strictly increases in \( \sigma_a \) when \( q < \bar{q} \).

**Proof of Proposition 3**

We show that there exists a threshold for \( \sigma_a \), while the rest of Proposition 3 is obvious based on Propositions 1 and 2. When \( q \leq \bar{q} < \bar{q} \), \( \Pi^{NPG} = (EV - \theta(EV - L - c))\mu_a + \pi(L) = (1 - \theta)q(H - L) + L - c)\mu_a + \pi(L) \), which is independent on \( \sigma_a \). Note that \( \Pi^{PG} = (L - c)\mu_a + \pi(L) + \frac{\pi(L) - q\pi(H)}{\bar{n}_a}E[\bar{n}_a - \bar{n}_a] \). We have

\[
\Pi^{PG} - \Pi^{NPG} = \frac{\pi(L) - q\pi(H)}{\bar{n}_a}E[\bar{n}_a - \bar{n}_a] + (1 - \theta)q(H - L)
\]

If \( p^a_{PG} = H \), then the difference between profits can be further simplified as \( \Pi^{PG} - \Pi^{NPG} = (H - L)(E[\bar{n}_a Z - (\bar{n}_a - \mu_a)] + (1 - \theta)q) = (H - L)(\sigma_a E[Z - (\bar{n}_a - \mu_a)]/\sigma_a) + (1 - \theta)q \), which increases in \( \sigma_a \). Note that \( \bar{n}_a = \frac{z(H) - q\pi(H)}{E[L]} \). When \( \sigma_a = 0 \), \( \Pi^{PG} - \Pi^{NPG} = -(H - L)(1 - \theta)q < 0 \). Since all the terms except \( \sigma_a Z \) are constant, as \( \sigma_a \) is large enough, \( \Pi^{PG} - \Pi^{NPG} \) becomes positive. For example, when \( \sigma_a \to \infty \), \( E[Z - (\bar{n}_a - \mu_a)]/\sigma_a] + \to E[Z^{+}] > 0 \) and thus \( \Pi^{PG} - \Pi^{NPG} > 0 \). Therefore, the monotonicty of \( \Pi^{PG} - \Pi^{NPG} \) implies that there must exist a threshold \( \sigma^*_a \) so that \( \Pi^{PG} - \Pi^{NPG} < 0 \) if \( \sigma_a < \sigma^*_a \) and \( \Pi^{PG} - \Pi^{NPG} \geq 0 \) otherwise.

If \( p^a_{PG} = L + \frac{1 - \theta G_a(\bar{n}_a)}{1 - G_a(\bar{n}_a)} \), the difference between profits can be further simplified as \( \Pi^{PG} - \Pi^{NPG} = (\pi(L) - q\pi(H))\frac{E[\bar{n}_a Z - (\bar{n}_a - \mu_a)]}{\bar{n}_a} + (1 - \theta)q(H - L) \), which strictly increases in \( \sigma_a \). When \( \sigma_a \to 0 \), as shown in the proof for Proposition 1 \( \Pi^{PG} - \Pi^{NPG} < 0 \). When \( \sigma_a \to \infty \), according to equation (10), \( z_a \to 0 \). This is because the left hand side \( \frac{\pi(L) - q\pi(H)}{\bar{n}_a} \) is a non-zero constant and the right hand side strictly increases in \( \sigma_a \). In order to keep both sides of equation (10) balanced, when \( \sigma_a \to \infty \), \( z_a \to 0 \) and thus \( \bar{n}_a \to \mu_a \). Therefore, when \( \sigma_a \to \infty \),

\[
\Pi^{PG} - \Pi^{NPG} = (\pi(L) - q\pi(H))\frac{E[\bar{n}_a Z - (\bar{n}_a - \mu_a)]}{\bar{n}_a} + (1 - \theta)q(H - L) = (\pi(L) - q\pi(H))\frac{E[Z^+] \sigma_a}{\mu_a} + (1 - \theta)q(H - L) >
\]
0. Therefore, there must exist one and only one threshold $\bar{q}_a$ so that $\Pi^P - \Pi^{NP} < 0$ if $\sigma_a < \bar{q}_a$ and $\Pi^P - \Pi^{NP} \geq 0$ otherwise.

**Proof of Lemma 5**

Under PC, the retailer commits the selling season price at the beginning of the advance selling period. Let $p_a^{PC}$ and $p_r^{PC}$ be the optimal advance selling price and selling season price, respectively.

Recall that the optimal selling season price should be either $H$ or $L$. We first consider the case where $p_r^{PC} = L$. Informed consumers buy early if and only if consumer expected utility from advance purchase $(E(V - p_a^{PC}))$ is no less than consumer expected utility from waiting $(\theta(EV - L))$, which implies that $p_a^{PC} \leq EV - \theta(EV - L)$. The total expected profit is $\Pi^{PC} = \max_{p_a \leq EV - \theta(EV - L)} \{(p_a - c)\mu_a + \pi(L)\}$, which implies that the optimal advance selling price $p_a^{PC} = EV - \theta(EV - L)$. Therefore,

$$\Pi^{PC} = (EV - \theta(EV - L) - c)\mu_a + \pi(L), \text{if } p_r^{PC} = L.$$

Next, we consider the case where $p_r^{PC} = H$. Informed consumers buy early if and only if $U_a = EV - p_r^{PC} \geq U_W = 0$, which implies that $p_a^{PC} \leq EV$. The total expected profit is $\Pi^{PC} = \max_{p_a \leq EV} \{(p_a - c)\mu_a + q\pi(H)\}$, which implies that the optimal advance selling price $p_a^{PC} = EV$. Therefore,

$$\Pi^{PC} = (EV - \theta(EV - L) - c)\mu_a + q\pi(H), \text{if } p_r^{PC} = H.$$

Since a rational retailer’s spot price would be either $H$ or $L$, the retailer’s problem is to choose a pair of advance and spot prices between $(EV, H)$ and $(EV - \theta(EV - L), L)$. The optimal expected profit for the retailer is as follows.

$$\Pi^{PC} = \max\{(EV - \theta(EV - L) - c)\mu_a + \pi(L), (EV - c)\mu_a + q\pi(H)\}. \tag{12}$$

Under PC, the seller will choose $(EV, H)$ if and only if

$$(EV - \theta(EV - L) - c)\mu_a + \pi(L) \leq (EV - c)\mu_a + q\pi(H),$$

which implies that $q \leq \bar{q} \equiv \frac{\pi(L)}{\theta(H - L)\mu_a + \pi(H)}$. Otherwise, the seller will choose $(EV - \theta(EV - L), L)$. Note that $\bar{q} < \bar{q}$.

**Proof of Proposition 5**

Let $EUA$ and $EUW$ be the informed customer’s expected utility by purchasing in advance and by delaying purchase to the regular sales period, respectively. By definition, we have

$$EUA = EV - p_a + P\{((p_a - L)\sigma_a - ((L - c) - q(H - c))\rho\sigma_a)z_1 \leq \pi(L) - q\pi(H) - (p_a - L)\mu_a \cdot (p_a - L),$$

$$EUW = P\{((p_a - L)\sigma_a - ((L - c) - q(H - c))\rho\sigma_a)z_1 \leq \pi(L) - q\pi(H) - (p_a - L)\mu_a \cdot (EV - L)\theta.$$
The retailer's expected profit under PG is

$$\Pi^{PG} = \pi(L) + (L - c)\mu_a + E[(p_a - L)\sigma_a - ((L - c) - q(H - c))\rho\sigma_r]z_1 - (\pi(L) - q\pi(H) - (p_a - L)\mu_a).$$

Given any \(\sigma_a\), let \(p_a(\sigma_a)\) be the optimal preorder price. Then we have

$$p_a(\sigma_a) = \max \left\{ p : EUA(p) \geq EUW(p), p \in [L, H] \right\}.$$

Case 1A: \([L - c - q(H - c)]\rho\sigma_r < 0\)

If \([L - c - q(H - c)]\rho\sigma_r < 0\), then for any \(\sigma_a\) and \(p_a \in [L, H]\), \((p_a - L)\sigma_a - [L - c - q(H - c)]\rho\sigma_r > 0\). Hence we have:

$$EUA - EUW = EV - p_a + \Phi(B(p_a, \sigma_a))(p_a - L - \theta(EV - L))$$

where \(B(p_a, \sigma_a) = \frac{\pi(L) + q\pi(H) - (p_a - L)\mu_a}{(p_a - L)\rho_a - ((L - c) - q(H - c))\rho\sigma_r}\), which is both continuous and differentiable in \(p_a\) and \(\sigma_a\). Thus \(p_a(\sigma_a) = H\) or satisfies \(EUA(p_a(\sigma_a)) - EUW(p_a(\sigma_a)) = 0\).

Case 1A: \(\pi(L) - q\pi(H) - (p_a - L)\mu_a \leq 0\)

In this case, \(\frac{\partial B(p_a, \sigma_a)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \geq 0\). This implies \(\frac{\partial(EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \geq 0\).

Given \(\sigma_a\), \(EUA - EUW = 0\) at the optimal \(p_a(\sigma_a)\). If we increase \(\sigma_a\) by a small amount while fixing everything else unchanged, then \(EUA - EUW\) becomes positive. That is, for sufficiently small \(\epsilon > 0\), \(EUA - EUW \geq 0\) at \(p_a(\sigma_a)\) and \(\sigma_a + \epsilon\) because \(\frac{\partial(EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \geq 0\). Hence the retailer is able to charge a higher pre-order price when \(\sigma_a\) is increased by a small amount. In another word, \(p_a(\sigma_a + \epsilon) \geq p_a(\sigma_a)\) for sufficiently small \(\epsilon > 0\). Therefore, according to the definition of total derivative, \(\frac{dp_a(\sigma_a)}{d\sigma_a} \geq 0\).

Furthermore, \(\frac{d\Pi^{PG}}{d\sigma_a} = \frac{d\Pi^{PG}}{dp_a} \cdot \frac{dp_a}{d\sigma_a} + \frac{d\Pi^{PG}}{d\sigma_a} \geq \frac{d\Pi^{PG}}{dp_a} = (p_a - L)\int_E \frac{d\phi(z)}{dz}.\) Therefore, in this subcase 1A, the retailer’s profit \(\Pi^{PG}\) increases in pre-order demand uncertainty \(\sigma_a\).

Case 1B: \(\pi(L) - q\pi(H) - (p_a - L)\mu_a > 0\)

In this case, if we consider EUA-EUW as a function of \(\sigma_a\) and \(p_a\), then we have the following:

$$\frac{\partial(EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} = \phi(B(p_a, \sigma_a))(p_a - L - \theta(EV - L)) \cdot \frac{\partial B(p_a, \sigma_a)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \leq 0$$

If we consider \(B(p_a, \sigma_a)\) as a function of \(p_a\) and \(\sigma_a\), then it is easy to see that \(\frac{\partial B(p_a, \sigma_a)}{\partial p_a} \bigg|_{p_a = p_a(\sigma_a)} \leq 0\).

Therefore, we have the following:

$$\frac{\partial(EUA - EUW)}{\partial p_a} \bigg|_{p_a = p_a(\sigma_a)} = -\phi(B(p_a, \sigma_a)) + \phi(B(p_a, \sigma_a))(p_a - L - \theta(EV - L)) \cdot \frac{\partial B(p_a, \sigma_a)}{\partial p_a} \bigg|_{p_a = p_a(\sigma_a)} \leq 0$$

Therefore,

$$\frac{dp_a(\sigma_a)}{d\sigma_a} = -\frac{\partial(EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \leq 0$$

(13)

On the other hand, we may consider \(EUA - EUW\) as a function of \((p_a, B)\). That is \((EUA - EUW)(p_a, B) = EV - p_a + \Phi(B)(p_a - L - \theta(EV - L))\). Therefore, we have the following:
\[
\frac{dB}{d\sigma_a} = \frac{\partial (EUA - EUW)}{\partial \sigma_a} = -\frac{1 - \Phi(B)}{\sigma(L - \theta(EV - L))} \geq 0,
\]
which gives:
\[
\frac{dB}{d\sigma_a} = \frac{dB}{dp_a} \cdot \frac{dp_a}{d\sigma_a} \leq 0
\] (14)

We note the following:

(i) \(\Pi^{PG} = \pi(L) + (L - c)\mu_a + (\pi(L) - \pi(H) - (p_a - L)\mu_a) \int_{B(p_a,\sigma_a)}^{\infty} \left(\frac{z}{B(p_a,\sigma_a)} - 1\right) \phi(z)dz\).

(ii) By (13), \((\pi(L) - \pi(H) - (p_a - L)\mu_a)\) increases in \(\sigma_a\).

(iii) By (14), \(\int_{B(p_a,\sigma_a)}^{\infty} \left(\frac{z}{\overline{B}(p_a,\sigma_a)} - 1\right) \phi(z)dz\) increases in \(\sigma_a\). Indeed, since \(\pi(L) - \pi(H) - (p_a - L)\mu_a > 0\) we have \(B(p_a,\sigma_a) > 0\). Also Leibniz Rule gives
\[
\frac{d}{d\sigma_a} \left(\int_{B(p_a,\sigma_a)}^{\infty} \left(\frac{z}{B(p_a,\sigma_a)} - 1\right) \phi(z)dz\right) = -\frac{dB}{dp_a} \int_{B(p_a,\sigma_a)}^{\infty} \left(\frac{z}{B(p_a,\sigma_a)} - 1\right) \phi(z)dz \geq 0.
\]

(i), (ii), and (iii) imply that \(\Pi^{PG}\) increases in \(\sigma_a\).

Case 2: \([L - c - q(H - c)]\rho\sigma_z > 0\)

Let \(\tilde{\sigma}_a\) be such that \(\frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a} = \max\{EV - L, \frac{\pi(L) - \pi(H)}{\mu_a}\}\). We further divide case 2 into two subcases. In the first subcase 2I, where \(\sigma_a \geq \tilde{\sigma}_a\), we can show that the coefficient of \(z_1\) is still nonnegative, which is similar to case 1. Therefore, the result in case 2I is also the same as in case 1, i.e., the retailer’s profit increases in pre-order demand uncertainty \(\sigma_a\). In the second subcase 2II, where \(\sigma_a < \tilde{\sigma}_a\), we can show that the coefficient of \(z_1\) is negative, i.e., the denominator of \(B\) is negative. This is contrary to the condition in case 1. We further discuss subcase 2II in two cases depending on whether the numerator of \(B\) is negative.

We show the result in case 2II is contrary to case 1, i.e., the retailer’s profit decreases in pre-order demand uncertainty \(\sigma_a\).

Case 2I: \(\sigma_a \geq \tilde{\sigma}_a\)

Let \(p_a = L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\). By definition of \(\tilde{\sigma}_a\), \(p_a \leq EV\) or \(p_a \leq \frac{\pi(L) - \pi(H)}{\mu_a} + L\). If \(p_a \leq EV\), then \(EUA(p_a) - EUW(p_a) \geq 0\), which implies that \(p_a(\sigma_a) \geq p_a = L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\). If \(p_a \leq \frac{\pi(L) - \pi(H)}{\mu_a} + L\), then \(EUA - EUW = EV - p_a + 1_{\pi(L) - \pi(H) - (p_a - L)\mu_a \geq 0} \cdot \left((p_a - L - \theta(EV - L)) - (EV - L)(1 - \theta) \geq 0\right.\). Therefore, the optimal pre-order price \(p_a(\sigma_a) \geq L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\).

Since \(p_a(\sigma_a) \geq L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\) for any \(\sigma_a \geq \tilde{\sigma}_a\), the coefficient of \(z_1\) is nonnegative. Following the same considerations and expressions as in Cases 1A and 1B, we can get that \(\Pi^{PG}\) increases in \(\sigma_a\) for \(\sigma_a > \tilde{\sigma}_a\).

Case 2II: \(\sigma_a < \tilde{\sigma}_a\)

Note that in this subcase, for any \(p_a \geq L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\), by definition of \(\tilde{\sigma}_a\), we have \(p_a - L > EV - L\) and \(p_a - L > \frac{\pi(L) - \pi(H)}{\mu_a}\). Let \(p = L + \frac{[(L - c) - q(H - c)]\rho\sigma_z + \Delta}{\tilde{\sigma}_a}\) for any \(\Delta \geq 0\). Next we show that \(EUA(p) - EUW(p) < 0\), which implies that the optimal price \(p_a(\sigma_a) < L + \frac{[(L - c) - q(H - c)]\rho\sigma_z}{\tilde{\sigma}_a}\) for \(\sigma_a < \tilde{\sigma}_a\).

\(EUA(p) - EUW(p)\)
\[= EV - p + P \left(\Delta \sigma_a z_1 \leq \pi(L) - q\pi(H) - \frac{[(L - c) - q(H - c)]\rho\sigma_z + \Delta}{\mu_a} \cdot (p - L - \theta(EV - L))\right)\]
where \( B \sigma \pi \)

Case II(B):

On the other hand, we may consider EUA-EUW as a function of the numerator of \( B \), so that

\[
\pi = \frac{\partial EV}{\partial \sigma_a} \quad \text{for } \sigma_a < \bar{\sigma}_a \text{ and } N_a \geq 0.
\]

Therefore, \( p_a(\sigma_a) < L + \frac{[L-c-q(H-c)]\sigma_r}{\sigma_a} \) for \( \sigma_a < \bar{\sigma}_a \).

Because \( p_a(\sigma_a) < L + \frac{[L-c-q(H-c)]\sigma_r}{\sigma_a} \), the coefficient of \( z_1 \) and the denominator of \( B \) are negative. Therefore, we have

\[
EUA - EUW = EV - p_a + (1 - \Phi(B(p_a, \sigma_a)))(p_a - L - \theta(EV - L)),
\]

where \( B(p_a, \sigma_a) = \frac{\pi(L) - q\pi(H) - (L-c-q(H-c))\sigma_r}{(p_a - L)\sigma_a - ((L-c-q(H-c))\sigma_r} \).

Similar to the analyses in case 1, we further discuss case II in two subcases depending on whether or not the numerator of \( B \) is positive.

Case II(A): \( \pi(L) - q\pi(H) \leq (p_a(\sigma_a) - L)\mu_a \)

Note that in this case, the numerator of \( B \) is negative and the denominator of \( B \) is also negative. If we consider EUA-EUW as a function of \( \sigma_a \) and \( p_a(\sigma_a) \), we have the following two properties.

\[
\frac{\partial(EUA-EUW)}{\partial p_a} \mid_{p_a=p_a(\sigma_a)} = -\Phi(B(p_a(\sigma_a), \sigma_a)) - \phi(B(p_a(\sigma_a), \sigma_a))(p_a(\sigma_a) - L - \theta(EV - L)) \cdot \frac{\partial(B(p_a(\sigma_a), \sigma_a))}{\partial p_a} \leq 0,
\]

since \( \frac{\partial(B(p_a(\sigma_a), \sigma_a))}{\partial p_a} \geq 0 \).

\[
\frac{\partial(EUA-EUW)}{\partial \sigma_a} \mid_{p_a=p_a(\sigma_a)} = -\phi(B(p_a(\sigma_a), \sigma_a))(p_a(\sigma_a) - L - \theta(EV - L)) \cdot \frac{\partial(B(p_a(\sigma_a), \sigma_a))}{\partial \sigma_a} \leq 0,
\]

since \( \frac{\partial(B(p_a(\sigma_a), \sigma_a))}{\partial \sigma_a} \geq 0 \).

Using the above two properties, we have

\[
\frac{dp_a(\sigma_a)}{d\sigma_a} = \frac{\partial(EUA-EUW)}{\partial \sigma_a} \frac{\partial(EUA-EUW)}{\partial p_a} \leq 0.
\]

On the other hand, we may consider \( EUA - EUW \) as an implicit function of \( p_a, B \). That is \( EUA - EUW(p_a, B) = EV - p_a + \Phi(B)(p_a - L - \theta(EV - L)) \). Therefore, we have the following:

\[
\frac{dB}{dp_a} = \frac{\partial EUA-EUW}{\partial B} \cdot \frac{\partial(EUA-EUW)}{\partial p_a} = \frac{-\Phi(B)}{-\phi(B)(p_a - L - \theta(EV - L))} \leq 0,
\]

which gives

\[
\frac{dB}{d\sigma_a} = \frac{dB}{dp_a} \cdot \frac{dp_a}{d\sigma_a} \geq 0.
\]

We note the following:

(i) \( \Pi^P = \pi(L) + (L-c)\mu_a + ((p_a - L)\mu_a - (\pi(L) - q\pi(H))) \int_{-\infty}^{B(p_a, \sigma_a)} \left(1 - \frac{z}{B(p_a, \sigma_a)}\right) \phi(z)dz \).

(ii) By Fact [15], \( (p_a - L)\mu_a - (\pi(L) - q\pi(H)) \) decreases in \( \sigma_a \) and is nonnegative.

(iii) By Fact [16], \( \int_{-\infty}^{B(p_a, \sigma_a)} \left(1 - \frac{z}{B(p_a, \sigma_a)}\right) \phi(z)dz \) also decreases in \( \sigma_a \). This is because Leibniz Rule gives

\[
\frac{d}{d\sigma_a} \left(\int_{-\infty}^{B(p_a, \sigma_a)} \left(1 - \frac{z}{B(p_a, \sigma_a)}\right) \phi(z)dz\right) = \frac{dB(p_a, \sigma_a)}{d\sigma_a} \int_{-\infty}^{B(p_a, \sigma_a)} \left(1 - \frac{z}{B(p_a, \sigma_a)}\right) \phi(z)dz \leq 0.
\]

Therefore, (i), (ii), and (iii) together imply that \( \Pi^P \) decreases in \( \sigma_a \).

Case II(B): \( \pi(L) - q\pi(H) > (p_a(\sigma_a) - L)\mu_a \)

Note that the numerator of \( B \) is positive while the denominator of \( B \) is negative in this case. Similarly, if we consider EUA-EUW as a function of \( \sigma_a \) and \( p_a(\sigma_a) \), we have the following two properties.
When \( \bar{q} = q_a \) expected profits under PG, NPG, and PC are the same. This gives:

From Lemmas 3, 4, and 5, we have the following comparison.

\[
\frac{\partial (EUA - EUW)}{\partial p_a} \bigg|_{p_a = p_a(\sigma_a)} = -\Phi(B(p_a(\sigma_a), \sigma_a)) - \Phi(B(p_a(\sigma_a), \sigma_a))(p_a(\sigma_a) - L - \theta(EV - L)) \cdot \frac{\partial (B(p_a(\sigma_a), \sigma_a))}{\partial p_a} \leq 0,
\]

since \( \frac{\partial (B(p_a(\sigma_a), \sigma_a))}{\partial p_a} \geq 0 \).

\[
\frac{\partial (EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} = -\phi(B(p_a(\sigma_a), \sigma_a))(p_a(\sigma_a) - L - \theta(EV - L)) \cdot \frac{\partial (B(p_a(\sigma_a), \sigma_a))}{\partial \sigma_a} \leq 0,
\]

since \( \frac{\partial (B(p_a(\sigma_a), \sigma_a))}{\partial \sigma_a} \leq 0 \).

Using the above two properties, we have

\[
\frac{dp_a(\sigma_a)}{d\sigma_a} = -\frac{\partial (EUA - EUW)}{\partial \sigma_a} \bigg|_{p_a = p_a(\sigma_a)} \geq 0. \tag{17}
\]

On the other hand, we may consider \( EUA - EUW \) as an implicit function of \((p_a, B)\). That is \( (EUA - EUW)(p_a, B) = EV - p_a + \Phi(B) \), \( 0 < \mu_a \). Therefore, we have the following:

\[
\frac{dB}{dp_a} = \frac{\partial (EUA - EUW)}{\partial p_a} \bigg|_{p_a = p_a(\sigma_a), B = B(p_a, \sigma_a)} = -\phi(B)(p_a - L - \theta(EV - L)) \leq 0,
\]

which gives:

\[
\frac{dB}{d\sigma_a} = \frac{dB}{dp_a} \cdot \frac{dp_a}{d\sigma_a} \leq 0. \tag{18}
\]

We note the following:

(i) \( \Pi^{PG} = \pi(L) + (L - c)\mu_a + (\pi(L) - q\pi(H))(p_a - L)\mu_a \int_{-\infty}^{B(p_a, \sigma_a)} \left( \frac{z}{B(p_a, \sigma_a)} - 1 \right) \phi(z) dz. \)

(ii) By Fact (17), \((\pi(L) - q\pi(H))(p_a - L)\mu_a\) decreases in \( \sigma_a \) and is positive.

(iii) By Fact (18), \( \int_{-\infty}^{B(p_a, \sigma_a)} \left( \frac{z}{B(p_a, \sigma_a)} - 1 \right) \phi(z) dz \) also decreases in \( \sigma_a \).

Therefore, (i), (ii), and (iii) together imply that \( \Pi^{PG} \) decreases in \( \sigma_a \).

**Proof of Proposition 6**

In this proof, we check three cases: (a) \( q \geq \bar{q} \), (b) \( \bar{q} < q < \bar{q} \), and (c) \( q \leq \bar{q} \).

(a) When \( q \geq \bar{q} \)

From Lemmas 1, 2, 3, 4 and 5, \( p_a^{PG} = p_a^{NP} = p_a^{PC} = EV \) and \( p_r^{PG} = p_r^{DP} = p_r^{PC} = H \). Therefore, the retailer’s expected profits under PG, NPG, and PC are the same.

(b) When \( \bar{q} < q < \bar{q} \)

From Lemmas 3, 4, and 5, we have the following comparison.

\[
\Pi^{PG} = \int_0^{\bar{n}_a} ((L - c)\mu_a + \pi(L)) g_a(n_a) dn_a + \int_{\bar{n}_a}^{\infty} ((p_a^{PG} - c)n_a + q\pi(H)) g_a(n_a) dn_a
\]

\[
= \int_0^{\bar{n}_a} (\pi(L) - ((p_a^{PG} - c)n_a + q\pi(H))) g_a(n_a) dn_a + \int_{\bar{n}_a}^{\infty} (p_a^{PG} - c)n_a + q\pi(H)) g_a(n_a) dn_a
\]

\[
\geq \int_0^{\infty} ((p_a^{PG} - c)n_a + q\pi(H)) g_a(n_a) dn_a
\]

\[
\geq (EV - c)n_a + q\pi(H)) g_a(n_a) dn_a
\]

\[
= (EV - c)\mu_a + q\pi(H) = \Pi^{PC}
\]

where the first inequality is because \( (L - c)n_a + \pi(L) > (p_a^{PG} - c)n_a + q\pi(H) \) for any \( n_a < \bar{n}_a \), and the second is because \( p_a^{PG} \geq EV \). Therefore, PG dominates PC.
(c) When \( q \leq \tilde{q} \)

It follows from Lemma 12 and Lemma 5 that \( \Pi^{NPG} = \Pi^{PC} \).

**Proof of Proposition 7**

Let \( \tilde{D}_r(p_r) \) be the aggregate demand function at spot price \( p_r \) when there are only spot sales. Then \( \tilde{D}_r(L) = N_a + N_r \) and \( \tilde{D}_r(H) = q(N_a + N_r) \). The total expected profit without advance selling is

\[
\Pi^{NAS} = \max_{Q \geq 0, p_r \in [L, H]} E[p, \min(\tilde{D}_r(p_r), Q) - cQ].
\]

For convenience, let \( \pi^{NAS}(p) = \max_{Q \geq 0} E[p(\min(N_a + N_r, Q)) - cQ] \). Then \( \Pi^{NAS} = \max \{ \pi^{NAS}(L), q\pi^{NAS}(H) \} \).

Note that for any \( Q \geq 0 \),

\[
E[\min(N_a + N_r, Q)] = E_{N_a}[E_{N_r}[\min(N_a + N_r, Q)]] \leq E_{N_r}[\min(E[N_a] + N_r, Q)]
\]

where the inequality is because \( \min(N_a + N_r, Q) \) is concave in \( N_a \) and we apply the Jensen’s inequality such that \( E_{N_a}[\min(N_a + N_r, Q)] \leq \min(E[N_a] + N_r, Q) \). Therefore,

\[
\pi^{NAS}(p) \leq (p - c)E[N_a] + \max_{Q \geq 0} E[p \min(N_r, Q) - cQ] = (p - c)E[N_a] + \pi(p).
\]

First, consider the case where \( L \geq c \). If \( \pi^{NAS}(L) \geq q\pi^{NAS}(H) \), then

\[
\Pi^{NAS} = \pi^{NAS}(L) \leq (L - c)E[N_a] + \pi(L) \leq (EV - \theta(\theta EV - L) - c)E[N_a] + \pi(L).
\]

If \( \pi^{NAS}(L) < q\pi^{NAS}(H) \), then

\[
\Pi^{NAS} = q\pi^{NAS}(H) \leq q(H - c)E[N_a] + q\pi(H) \leq (EV - c)E[N_a] + q\pi(H).
\]

Hence, \( \Pi^{NAS} \leq \Pi^{PC} \). Note that when \( \tilde{q} < q < q^* \) we have \( \Pi^{NPG} < \Pi^{PC} \); otherwise \( \Pi^{NPG} = \Pi^{PC} \). On the other hand, we know that \( \Pi^{PG} \geq \Pi^{PC} \) when \( q \geq \tilde{q} \), which suggests that \( \max\{\Pi^{PG}, \Pi^{NPG}\} \geq \Pi^{NAS} \). In other words, when \( L > c \), advance selling is always more profitable than spot selling only if the retailer can optimally decide when to offer PG and when to offer NPG. Hence, the question of interest is whether to offer PG or NPG.

Next, consider the case where \( L < c \). When \( L < c < H \), we have \( \pi(L) = 0 \) and \( \pi^{NAS}(L) = 0 \), which implies that \( \Pi^{NAS} = q\pi^{NAS}(H) \). If \( \pi(H) > 0 \), then \( \tilde{q} = 0 \) and hence for any \( q > 0 \), \( \Pi^{PG} = \Pi^{NPG} = (EV - c)E[N_a] + q\pi(H) \). Furthermore, if \( EV \leq c < H \), we have

\[
\Pi^{NAS} = q\pi^{NAS}(H) > q\pi(H) \geq (EV - c)E[N_a] + q\pi(H) = \Pi^{PG} = \Pi^{NPG}.
\]

On the other hand, if \( L < c < EV \), the result is different. Suppose \( N_a \) and \( N_r \) are independently and normally distributed with mean \( (\mu_a, \sigma_a) \) and \( (\mu_r, \sigma_r) \), respectively. Here, \( \sigma_a > 0 \).
Then \( \pi(p) = (p-c)\mu_r + \sigma_r[pE[Z \wedge \Phi^{-1}(c/p)] - c\Phi^{-1}(c/p)] \) and \( \pi^{NAS}(p) = (p-c)(\mu_a + \mu_r) + \sqrt{\sigma_a^2 + \sigma_r^2}[pE[Z \wedge \Phi^{-1}(c/p)] - c\Phi^{-1}(c/p)] \), where \( Z \) is the standard normal variable with distribution function \( \Phi(\cdot) \), \( \Phi(\cdot) = 1 - \Phi(\cdot) \). Define \( \Delta(c) = (EV-c)E[N_a] + q\pi(H) - q\pi^{NAS}(H) \). Then

\[
\Delta(c) = (1-q)(L-c)\mu_a - (\sqrt{\sigma_a^2 + \sigma_r^2} - \sigma_r)[HE[Z \wedge \Phi^{-1}(c/H)] - c\Phi^{-1}(c/H)]
\]

Taking derivative yields

\[
\Delta'(c) = (\sqrt{\sigma_a^2 + \sigma_r^2} - \sigma_r)\Phi^{-1}(c/H) - (1-q)\mu_a,
\]

which is strictly decreasing in \( c \) when \( \sigma_a > 0 \). That is, \( \Delta \) is concave. Note that \( \Delta(L) > 0 \) and \( \Delta(EV) < 0 \), which, together with the concavity, implies that \( \Delta \) crosses zero at most once. Hence, there exists a threshold \( \bar{c} \in (L,EV) \) such that \( \Pi^{NAS} \geq \Pi^{PG} = \Pi^{NPG} \) if and only if \( c \geq \bar{c} \).