Equilibrium Term Structure under the Expectations Theory

**PART I**

*Given:* \( r_1 = .02; \) and Expected \( r_{t+1} = .04 \)

*To Prove:* Equilibrium \( r_2 = \left[ (1.02)(1.04) \right]^{1/2} - 1 = .02995 \approx .03 \)

*Approach:* Assume \( r_2 = .02995 = .03 \) and examine investor holding period yields (HPY). If short-term (one-year) investors realize the same HPY by buying \( r_1 \) compared with buying \( r_2 \) and selling after one year, they will be indifferent between investing in \( r_1 \) and investing in \( r_2 \). And if long-term (two-year) investors realize the same HPY by buying \( r_2 \) compared with buying \( r_1 \) and reinvesting at Exp \( r_{t+1} = .04 \) they too will be indifferent between investing in \( r_1 \) compared with \( r_2 \). This would imply no buying or selling pressure in the marketplace, hence prices and yields would be equilibrium.

**Proof:** Uses Pure Discount Bonds

\[
\begin{align*}
\text{NB:} & \quad r_1 = .02 \implies P_1 = \frac{100}{1.02} = 98.0392 \\
r_2 = .03 \implies P_2 = \frac{100}{(1.03)^2} = 94.2596
\end{align*}
\]

**ONE-YEAR INVESTOR ALTERNATIVE STRATEGIES**

Strategy 1: Buy \( r_1 \) at 98.0392 and hold to maturity

\[
\text{HPY} = \frac{100}{98.0392} - 1 = .02
\]
Strategy 2: Buy \( iR_2 \) at 94.2596 and sell after one year. Because \( iR_2 \) becomes a one-year bond after one year and because one-year bonds next year are expected to yield 4%, the expected selling price of \( iR_2 \) next year is:

\[
\frac{100}{1.04} = 96.154
\]

\[
HPY = \frac{96.154}{94.2596} - 1 = 0.02
\]

**TWO-YEAR INVESTOR ALTERNATIVE STRATEGIES**

Strategy 1: Buy \( iR_1 \) and reinvest at \( Exp_{t+1}R_1 \).

\[
HPY = \left[ (1.02)(1.04) \right]^{1/2} - 1 \approx 0.02995 \approx 0.03
\]

Strategy 2: Buy \( iR_2 \) and hold to maturity

\[
HPY = \left( \frac{100}{94.2596} \right)^{1/2} - 1 \approx 0.02999 \approx 0.03
\]

**Conclusion**

Because HPY's for all investors and all strategies are equal, the term structure is in equilibrium with \( iR_2 \) a geometric average of \( iR_1 \) and \( Exp_{t+1}R_1 \).

**PART II**

**Question**

What happens to equilibrium \( iR_2 \) when expectations change? In particular, when \( Exp_{t+1}r_1 = 0.06 \), what is the new equilibrium \( iR_2 \)? How does it come about?

**Given:** \( iR_1 = 0.02 \) and the new \( Exp_{t+1}R_1 = 0.06 \)

**To Prove:** The new equilibrium \( iR_2 = [(1.02)(1.06)]^{1/2} - 1 = 0.0398 \) because portfolio adjustments by market participants will make it so.
**Approach:** Assume for a moment that \( P_2 \) (the price of \( tR_2 \)) remains at 94.2596 so that \( tR_2 \) remains at .03. Examine what portfolio adjustments "two-year" investors will undertake and see what impact that will have on \( P_2 \). After determining the new equilibrium \( P_2 \) that leaves two-year investors indifferent between both of their investment strategies, see if that \( P_2 \) "works" for one-year investors. If so, that is the new equilibrium \( P_2 \) with the associated new equilibrium \( tR_2 \).

**Proof:**

**TWO-YEAR INVESTOR STRATEGIES**

1. Two-year investors prefer \( tR_1 \) and reinvesting in new \( Exp_{t+1}R_1 \)

\[
HPY = \left[(1.02)(1.06)\right]^{1/2} - 1 = .0398
\]

To buying \( tR_2 \) at the old \( P_2 = 94.2596 \)

\[
HPY = \left(\frac{100}{94.2596}\right)^{1/2} - 1 = .092999 \approx .03
\]

2. Therefore, two-year investors want to sell \( tR_2 \) at \( P_2 = 94.2596 \). This selling pressure drives down \( P_2 \) until two-year investors are willing to hold \( tR_2 \). That occurs when its new \( HPY = .0398 \).

Therefore, the new equilibrium \( P_2 \) is

\[
P_2 = \frac{100}{(1.0398)^2} = 92.49
\]

At this new price the yield on \( tR_2 \) is:

\[
HPY = \left(\frac{100}{92.49}\right)^{1/2} - 1 = .0398
\]

Thus two-year investors are indifferent between \( tR_1 \) and \( tR_2 \) when \( P_2 = 92.49 \).

**ONE-YEAR INVESTOR STRATEGIES**

With \( P_2 = 92.49 \) and \( tR_2 = .0398 \) one-year investors are also indifferent between both of their strategies, as shown in the following 2 possibilities:
1. Buy \( iR_1 \) and earn .02

2. Buy \( iR_2 \) and sell after one year. The expected selling price of \( iR_2 \) after one year is now:

\[
\frac{100}{1.06} = 94.3396
\]

Therefore

\[
HPY = \frac{94.3396}{92.49} - 1 = .01999 \approx .02
\]

Therefore,

New equilibrium \( P_2 = 92.49 \)

New equilibrium \( iR_2 = .0398 \)

**Conclusion**

Market forces in the form of portfolio adjustments by investors drive long-term rates into an average of current and expected future short-term rates.