Continuous Compounding: Some Basics

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Because you may encounter continuously compounded growth rates elsewhere, and because you will encounter continuously compounded discount rates when we examine the Black-Scholes option pricing formula, here is a brief introduction to what happens when something grows at \( r \) percent per annum, compounded continuously.

We know that as \( n \to \infty \)

\[
1 + \frac{1}{n}^n = e = 2.71828183\ldots
\]

In our context, this means that if $1 is invested at 100% interest, continuously compounded, for one year, it produces $2.71828 at the end of the year.

It is also true that if the interest rate is \( r \) percent, then $1 produces \( e^r \) dollars after 1 year. For example, if \( r = 0.06 \) we have

\[
1 \cdot e^{0.06} = 1.0618365
\]

After two years, we would have:

\[
e^{0.06} \cdot e^{0.06} = e^{0.06(2)} = 1.127497
\]

More generally, investing \( P \) at \( r \) percent, continuously compounded, over \( t \) years, produces (grows to) the amount \( F \) according to the following formula:

\[
P e^{rt} = F
\]

For example, $100 invested at 6 percent, continuously compounded, for 5 years produces

\[
$100 \cdot e^{0.06(5)} = $134.98588
\]
We can use equation (2) to solve for the present value of $F$ dollars paid after $t$ years, assuming the interest rate is $r$ percent, continuously compounded. In particular,

\[ P = \frac{F}{e^{rt}} \]

Or

\[ P = Fe^{-rt} \]

The term $e^{-rt}$ in expression (4) is nothing more than a discount factor like $\frac{1}{(1 + r)^t}$, except that $r$ is continuously compounded (rather than compounded annually).

For example, suppose $r = .06$ and $t = 1$.

\[ \frac{1}{(1 + r)^t} = \frac{1}{1.06} = .9434 \]

\[ e^{-rt} = e^{-0.06} = .9417 \]

This last result is slightly surprising. Why is the present value of $1 less (.9417) under continuous compounding compared with annual compounding (.9434)?

The answer is: With a fixed dollar amount ($1) at the end of one year, continuous compounding allows you to put away fewer dollars (.9417 rather than .9434) because it grows at a faster (continuously compounded) rate.
A note on EAR: It is quite straightforward to calculate the EAR if you are given a continuously compounded rate. We saw above that $1 compounded continuously at 6% produces 1.061836 at the end of one year:

\[ 1 \times e^{0.06} = 1.061836 \]

Subtracting one from the right hand side of the above shows that a simple annual rate (without compounding) of 6.1836 % would be equivalent to 6% continuously compounded. And that is what we mean by the EAR.

What if you were told that the annual rate without compounding was 6%, could you derive the continuously compounded rate that produces a 6% EAR? The answer is given by solving the following expression for x:

\[ e^x = 1.06 \]

Taking the natural log (ln) of both sides produces:

\[ x = \ln (1.06) = .0582689 \]

Thus, 6 % simple interest is equivalent to 5.82689 % continuously compounded. In general, taking the natural log of ‘one plus’ a simple rate produces the corresponding continuously compounded rate. File away this last point until we discuss options towards the end of the semester.