Annuities and Perpetuities: Present Value

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I. The present value of an annuity, *PV*, can be written as the sum of the present values of each component annual payment, *C*, as follows:

(1) $$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^t}$$

where *r* is the single average interest rate per annum and t is the number of years the annuity is paid.

This can be simplified as follows:

(2) $$PV = C\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots \frac{1}{(1+r)^t}\right].$$

Using a formula for the sum of a geometric progression (as long as $r > 0$), we have:

(3) $$PV = C\left[\frac{1-(1+r)^{-t}}{r}\right],$$

which is the same as:

(4) $$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$$

II. Thus if you have a three-year annuity $(t = 3)$ that pays \$100 per annum $(C = \$100)$ and the average annual interest rate, r, is 6 percent, then from equation (4), we have:

$$PV = \$100.00\left[\frac{1}{.06} - \frac{1}{.06(1.06)^3}\right] = \$267.30$$

You can check that this is correct by calculating:

$$PV = \frac{\$100}{1.06} + \frac{\$100}{(1.06)^2} + \frac{\$100}{(1.06)^3} = \$267.30$$

III. More interesting is what happens to the present value formula when the annual payments, *C*, continue forever. The annuity becomes a perpetuity as $t \rightarrow \infty$ and the formula in (4) becomes:

(5) $$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^{\infty}}\right]$$

(6) $$PV = C\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

Or, finally,

(7) $$PV = \frac{C}{r}$$

IV. Equation (7) is very simple. It says that the present value of an annuity of *C* dollars per annum is *C* divided by r, where r is the average interest rate per annum. This makes considerable sense once you provide a numerical example. Suppose *C*=$10 per annum and the interest rate is .05, or 5 percent. How many dollars, designated by the letter P, would you have to put away today so that it produces $10 in each year forever? The answer is given by solving the following formula for P:

$$P \times .05 = \$10$$

$$P = \frac{\$10}{.05} = \$200.$$

Investing $200 at 5 percent generates $10 in interest per year and continues to do so forever. Thus, if an annuity promises to pay $10 forever and the annual interest rate is 5 percent, the value of that infinite stream of payments is $200. If the annuity were priced in a competitive market its price should be $200