Annuities and Perpetuities: Present Value

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I. The present value of an annuity, \( PV \), can be written as the sum of the present values of each component annual payment, \( C \), as follows:

\[
P V = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \cdots + \frac{C}{(1 + r)^t}
\]

where \( r \) is the single average interest rate per annum and \( t \) is the number of years the annuity is paid.

This can be simplified as follows:

\[
P V = C \left[ \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + \cdots + \frac{1}{(1 + r)^t} \right]
\]

Using a formula for the sum of a geometric progression (as long as \( r > 0 \)), we have:

\[
P V = C \left[ \frac{1 - (1 + r)^{-t}}{r} \right],
\]

which is the same as:

\[
P V = C \left[ \frac{1}{r} - \frac{1}{r (1 + r)^t} \right]
\]

II. Thus if you have a three-year annuity \((t = 3)\) that pays $100 per annum \((C = $100)\) and the average annual interest rate, \( r \), is 6 percent, then from equation (4), we have:
\[ PV = \$100.00 \left[ \frac{1}{.06} - \frac{1}{.06(1.06)^{1}} \right] = \$267.30 \]

You can check that this is correct by calculating:

\[ PV = \frac{\$100}{1.06} + \frac{\$100}{(1.06)^{2}} + \frac{\$100}{(1.06)^{3}} = \$267.30 \]

III. More interesting is what happens to the present value formula when the annual payments, \( C \), continue forever. The annuity becomes a perpetuity as \( t \to \infty \) and the formula in (4) becomes:

\[ PV = \frac{C}{r} - \frac{1}{r(1+r)^{\infty}} \]

(5)

\[ PV = \frac{C}{r} - \frac{1}{r^{\infty}} \]

(6)

Or, finally,

\[ PV = \frac{C}{r} \]

IV. Equation (7) is very simple. It says that the present value of an annuity of \( C \) dollars per annum is \( C \) divided by \( r \), where \( r \) is the average interest rate per annum. This makes considerable sense once you provide a numerical example. Suppose \( C = \$10 \) per annum and the interest rate is .05, or 5 percent. How many dollars, designated by the letter \( P \), would you have to put away today so that it produces \$10 in each year forever? The answer is given by solving the following formula for \( P \):
\[ P \times 0.05 = 10 \]

\[ P = \frac{10}{0.05} = 200. \]

Investing $200 at 5 percent generates $10 in interest per year and continues to do so forever. Thus, if an annuity promises to pay $10 forever and the annual interest rate is 5 percent, the value of that infinite stream of payments is $200. If the annuity were priced in a competitive market its price should be $200.