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Statistics and Data Analysis
B01.1305

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Combinations and Permutations

I. Preliminaries.

How many different ways can 4 kids (Allison,Julie,Betsy,Lesley) get on line for ice cream? (Or is it "in line?" Depends on where in the country the line is.) One possibility is
[Allison,Julie,Betsy,Lesley].
Of course, being first in line matters, and being last, well,… So, with the same 4 kids, the line
[Julie,Betsy,Lesley,Allison]
is a different line. When permutation of the items in the group produces a different group, then "order matters," and the "different ways" are different ***permutations***.

How many different 5 card poker hands can be formed from a deck of 52 cards? Consider, first, this question. With respect to the one particular hand, for an example,

[A♣,5♥,Q♦,7♦,9♠],
if we reorder the cards, say to get
[A♣,7♦,5♥,9♠,Q♦]
then we get a hand that has the same five cards, and it is not a different hand (group). In this case, "order doesn't matter," and the different ways are different ***combinations***.

Before we proceed to the counting rule, you should notice one straightforward point about "how many." Consider the poker hand above. It is what it is. There is one combination of 5 cards that contains those specific 5 cards. But (it will turn out), there are 120 different orderings of the 5 cards, that is, different permutations. This is a general result. In the calculation of "how many (permutations),(combinations) can be made from K objects out of N candidates, there will be more permutations than combinations, because each combination can be rearranged to make many permutations.

II. Permutations.

Now, how many ways can the 4 kids form a line? For convenience, let's call them A, J, B, and L. There are 4 who can be first, so the possible lines are

[A,…], [J,…], [B,…], [L,…].

After the first in line is chosen, then three remain for the second position, so each of the 4 possible lines suggested has 3 possible second place kids, for 12 so far:

[A,J,..], [A,B,..], [A,L,…], [J,A,…], [J,B,…], [J,L,…] and so on.

With the first two positions, each of the 12 can have either of the 2 remaining in the third position, so we are up to 24. Finally, the one remaining child takes the 4^{th} place.

The general pattern is that if we are selecting K items out of N candidates, then the first item can be any of the N, the next can be any of the N-1 remaining, and so on, so that the total is

N(N-1)(N-1)(N-2)…(N-K+1)

Which is denoted $_NP_K$ to mean "Permutations of N objects taken K at a time." There is a useful "shortcut" (actually just the definition) for computing this value. We define the symbol N! (N factorial) to mean (N)(N-1)(N-2)(N-3)…(1) (counting all the way down to 1). The number of permutations is equal to the factorial less the last (N-K) terms. We can compute this as follows, where we illustrate by computing $_{10}P_4$.

$$_{10}P_4 = \frac{10(9)(8)(7)(6)(5)(4)(3)(2)(1)}{(6)(5)(4)(3)(2)(1)}$$

$$= \frac{10!}{(10\text{-}4)!}$$

The general result is $$_NP_K = \frac{N!}{(N-K)!}$$

III. Combinations

How many different 5 card poker hands can be formed from a deck of 52 cards? This is obviously related to the number of permutations. If we start with all the possible permutations of 5 cards out of 52, the answer would be

Number = 52(51)(50)(49)(48)=311,875,200.

However, this overcounts, since, as we noted, the order of the cards doesn't matter. Having chosen the 5 cards, the number above treats all of the permutations of those 5 cards as if they were different. But, since they are the same, we have overcounted by $_5P_5$ (which you can now see is 120). (We need a convention here. If you try to compute $_5P_5$ you will end up computing 0! in the denominator, which equals 1.) The right answer is

$$\text{Combinations of 5 cards} = \frac{_{52}P_5}{_5P_5} = \frac{\left(\frac{52!}{(52-5)!}\right)}{\left(\frac{5!}{(5-5)!}\right)} = \frac{52!}{5!(52-5)!}$$

The general result, which is written as $_NC_K$ (and spoken as "N choose K") is

$$_NC_K = \frac{N!}{K!(N-K)!}$$

There is a more commonly used notation for this computation,

$$\binom{N}{K} = \frac{N!}{K!(N-K)!}$$

The answer to the original question (how many 5 card hands) is

$$\begin{aligned}\binom{52}{5} &= \frac{52!}{5!(52-5)!} \\ &= \frac{52(51)(50)(49)(48)(47)...}{[5(4)(3)(2)(1)][47(46)(54)...]} \\ &= \frac{52(51)(50)(49)(48)}{5(4)(3)(2)(1)} = 2{,}598{,}960\end{aligned}$$

IV. Useful Results

1. Since 0! = 1, it follows that $\binom{N}{0} = \binom{N}{N} = 1$

It makes sense. There is only one way to choose none of the members of a set, and only one way to choose all of them.

2. Since multiplication is commutative, $\binom{N}{K} = \frac{N!}{K!(N-K)!} = \frac{N!}{(N-K)!K!}\binom{N}{N-K}$

V. Large Numbers of Items

There are 435 members of the U.S. House of Representatives. Suppose there are 217 Democrats and 217 Republicans and 1 Independent. How many ways are there to form a Republican vs Democrats softball game, with 10 players on each team and the Independent acting as the umpire. In principle, this is simple using our results. For each party, the number of different teams is

$$_{217}C_{10} = \frac{217!}{10!(217-10)!}$$

In principle, this requires you to compute 217! If you push the factorial number on your calculator it will tell you the computation cannot be done – the largest factorial most calculators and computer programs can compute is about 69. A partial answer is found by taking logs, which turns the product into a sum. Thus, log217! is the sum of the logs of the values from 1 to 217, which is about 954. That doesn't solve the problem, however, since you then have to exponentiate the result, which gives a value with more than 400 digits. But, if you look carefully at the computations, you'll see that terms cancel out in the fractions, so it isn't actually necessary to compute the whole factorials. The result we are looking for above is (217)(216)…(208)/(10!). The result is big, .517 times 10^{17}, but a 17 digit number is manageable while a 400 digit number isn't. (The number of possible two team configurations, since each team can play with any of the other party's teams, is the square, which is .267 times 10^{34}.)

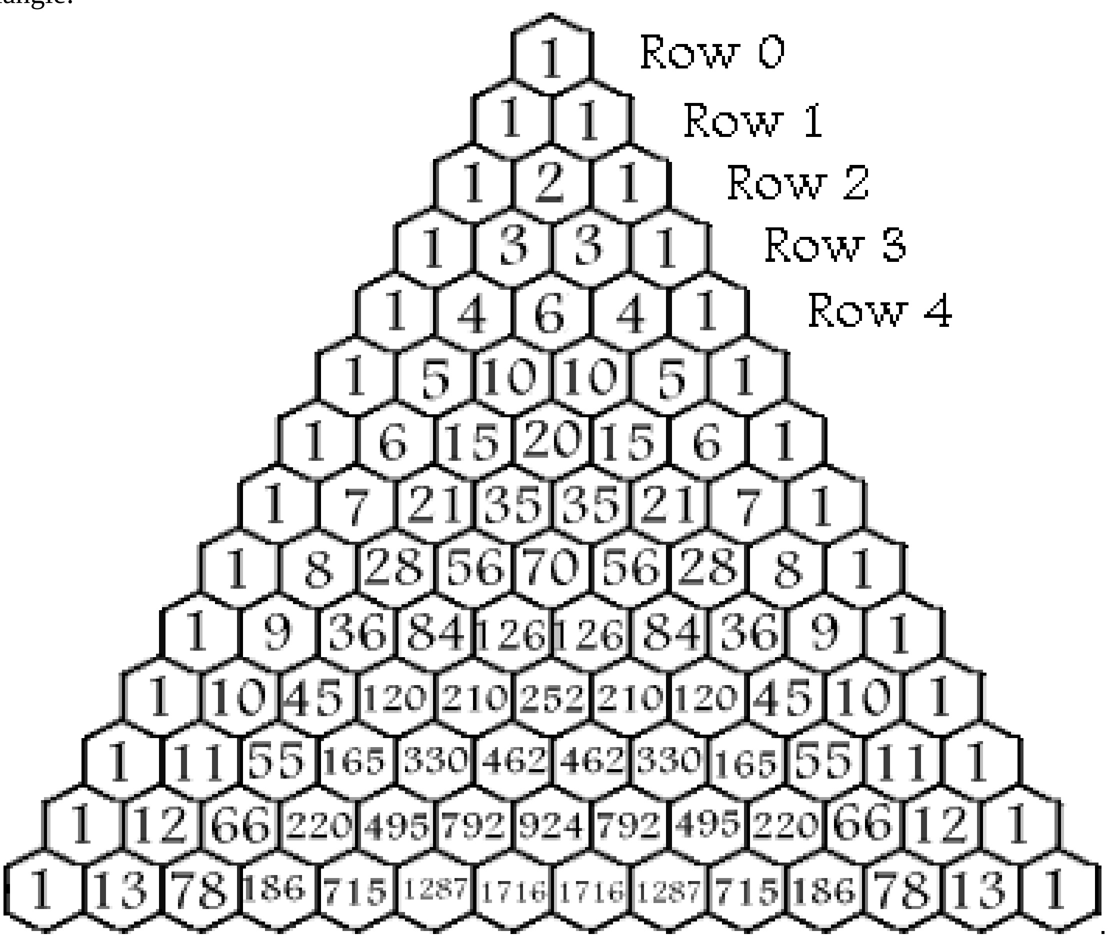
VI. Binomial Coefficients

The result $\binom{N}{K}$ is often called the binomial coefficient because of the useful result

$$(x+y)^N = \binom{N}{0}x^0y^N + \binom{N}{1}x^1y^{N-1} + \binom{N}{2}x^2y^{N-2} + \ldots + \binom{N}{N}x^Ny^0$$

$$= \sum_{K=0}^{N}\binom{N}{K}x^Ky^{N-K}$$

You may (probably should) remember a few of these strings, such as (1,2,1) for N=2 and (1,3,3,1) for N = 3 and (1,4,6,4,1) for N=4. For the mathematically compulsive among you, there is a visual device (method) for obtaining the coefficients in a trivial fashion, known as Pascal's triangle:



The numbers in row N of the triangle are the binomial coefficients. After row 1, each element is found just by taking the sum of the ones above and left and above and to the right. Thus, the 15 in the third and fifth positions of the "Row 6" are found as 5+10 then 10+5, respectively. Pascal's triangle is mystical. See, for example, http://britton.disted.camosun.bc.ca/pascal/pascal.html.