1. a. 1 is obviously better – higher mean and lower standard deviation
   b. \(0.5 \mu_1 + 0.5 \mu_2 = 0.9\). The variance will be \(0.5^2(0.12^2) + 2(0.5)(0.5)(-0.8)(0.1)(0.12) = 0.0013\). The standard deviation is 0.036
   c. \(0.8(1) + 0.2(0.8) = 0.96\). The variance will \(0.8^2(0.12^2) + 0.2^2(0.12^2) + 2(0.8)(0.2)(-0.8)(0.1)(0.12) = 0.003904\). The standard deviation will be 0.062482
   d. return = \(w*1 + (1-w)*0.8\)
   risk = \(sqr(w^2*0.01 + (1-w)^2*0.0144 + 2*w*(1-w)*0.1*0.12*(-0.8))\)

2. Suppose \(x\) is distributed uniformly from 0 to 1. What is the density of \(z = \exp(x^2)\)?

   \(f(x) = 1, 0 \leq x \leq 1\).
   \(z = \exp(x^2), 1 \leq z \leq e\). so \(x^2 = \log z\) and \(x = (\log z)^{1/2}\). \(dx = \frac{1}{2} * (\log z)^{1/2} * 1/z\) \(dz\)
   \(f(z) = 1 * \text{Jacobian} = \frac{1}{2} * (\log z)^{1/2} * 1/z\)
3. If the random variable $x$ has gamma density with $\lambda$, $P = \frac{1}{2}$, $\frac{1}{2}$, what is the density of $x^2$? What is the density of $\log(x)$?

$$z = x^2 \text{ so } x = z^{1/2}. \text{ dx = } \frac{1}{2} z^{-1/2} \text{ dz is the Jacobian.}$$

$$f(z) = (1/2)^{(1/2)/\Gamma(1/2)} \exp(-1/2 z^{1/2})(z^{1/2})^{1/2-1} (\frac{1}{2} z^{1/2}) \text{ Now collect terms}$$

$$f(z) = (1/2)^{3/2}/\Gamma(1/2) \exp(-1/2 z^{1/2}) z^{-3/4}. \text{ Collect terms.}$$

$$z = \log(x) \text{ so } x = \exp(z) \text{ and } dx = \exp(z) \text{ dz is the Jacobian.}$$

$$f(z) = (1/2)^{(1/2)/\Gamma(1/2)} \exp(-1/2 \exp(z)) \exp(z)^{1/2-1} \exp(z)$$

$$= (1/2)^{(1/2)/\Gamma(1/2)} \exp(-1/2 \exp(z)) \exp(z)^{1/2} \text{ Collect terms.$$

4. Suppose $x$ has standard normal distribution, $\mu = 0$, $\sigma = 1$. Derive the covariance between $x$ and $x^2$.

Covariance is $E[x \times x^2] - E[x]E[x^2] = E[x^3] - 0(1)$. But, the normal distribution is symmetric, so the third moment about the mean of zero is zero. They are uncorrelated. The covariance is zero.

5. The exact probability for any $N$ is $100C_N (1/6)^N (5/6)^{100-N}$.

For 100 tosses, the expected number of 6s is $N\pi = 100(1/6) = 16.67$. The variance is $100(1/6)(5/6) = 13.89$, so the standard deviation is 3.73.

.a. Prob($15 < x < 20$) = Prob($16 \leq x \leq 19$). Using the continuity correction, we use 15.5 and 19.5.

Using the central limit theorem, we then use Prob($15.5 \leq x^* \leq 19.5$) from the normal distribution.

Standardizing, this is Prob($15.5 - 16.67)/3.73 \leq z \leq (19.5 - 16.67)/3.73) = .7760 - .3769 = .3991$.

.b. The sum of the face values has mean $100*3.5 = 350$ and variance $100*2.917$ or standard deviation $17.078$, Prob[$\sum < 300$] = Prob[$(\sum - 350)/17.078 < (300 - 350)/17.078$] = Prob[$z < -2.928$] = .00171

6. The central limit theorem can be used to analyze round-off error. Suppose that the round-off error is represented as a uniform random variable on $[-1/2,+1/2]$. If 100 numbers are added, approximate the probability that the round-off error exceeds (a) 1, (b) 2, (c) 5.

The mean of the sum of 100 variables is 100*0 since all have mean 0. The variance is the sum of the variances, which is 100 * 1/12 = 8.333. The standard deviation is the square root, 2.887.

.a. The probability that the sum is greater than 1 is Prob[$x > 1$] = Prob[$(x - 0)/2.887 > (1 - 0)/2.887$] = .3645

If you interpret this to mean that the absolute value of the sum is > 1, then the probability is doubled.

.b. Prob[$x > 2$] = Prob[$z > 2/2.887$] = .245 (or .490).

.c. Prob[$x > 5$] = Prob[$z > 5/2.887$] = .0416 (or .0892).

7. Expected loss on any game is zero (fair game) = -5*(.5) + (+5)(.5)

Variance of loss on any .5*(-5-0)^2 + .5*(5-0)^2 = 25.

Expected winning on 50 games is zero. Variance of winning is 50*25 = 1250. Standard deviation is 35.35.

Prob[$\sum < -75$] = Prob[$z < (-75 - 0)/35.35$] = Prob[$z < -2.12$] = .01695.
8. Consider a variable with exponential distribution, \( f(x) = \frac{1}{\alpha}e^{-x/\alpha}, x \geq 0, \alpha > 0 \). We have found that \( E[x] = \alpha \) and \( \text{Var}[x] = \alpha^2 \). Consider the random variable \( Y = X \) if \( X \leq T \), and \( Y = T \) if \( X > T \). \( Y \) is a censored version of \( X \). A different random variable is \( Z = X \) if \( X \leq T \). \( Z \) is a truncated version of \( X \).)

Find the density, mean and variance of \( Z \).

Solution of this problem will require repeated use of the following two integrals (search online for ‘table of integrals’ if you are interested in the sources):

\[
\int xe^{cx}dx = \frac{e^{cx}}{c^2}(cx - 1) \quad \text{and} \quad \int x^2e^{cx}dx = e^{cx}\left(\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}\right)
\]

For the censored version, \( E[y] = \text{Prob}[y < T]*E[y|y < T] + \text{Prob}[y \geq T]*T \)

\( \text{Prob}[y < T] = 1 - \exp(-T/\alpha) \)

\( \text{Prob}[y \geq T] = \exp(-T/\alpha) \)

The density of \( y \) has the two parts. When \( x < T \), \( y = x \). So that part of the density is \( f(y|y < T) = f(x| x < T) = f(x)/\text{Prob}(x < T) \). To get the first term in \( E[y] \), the two probabilities will cancel, leaving

\[
E[y] = \left[ \int_0^T x \frac{1}{\alpha}e^{-x/\alpha} \, dx \right] + \left\{ T \times e^{-T/\alpha} \right\}
\]

The integral is one of the two listed above. Plugging in the terms, we have \( c = -\frac{1}{\alpha} \)

\[
E[y] = \frac{1}{\alpha} \left\{ \frac{e^{-T/\alpha}}{(-1/\alpha)^2}\left(-\frac{T}{\alpha} - 1\right) \right\} + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \frac{1}{\alpha} \left\{ \frac{e^{-T/\alpha}}{(-1/\alpha)^2}\left(-\frac{T}{\alpha} - 1\right) \right\} + \frac{1}{\alpha} \left\{ \frac{e^{-0/\alpha}}{(-1/\alpha)^2}\left(-\frac{0}{\alpha} - 1\right) \right\} + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \frac{1}{\alpha} \left\{ \frac{e^{-T/\alpha}}{(-1/\alpha)^2}\left(-\frac{T}{\alpha} - 1\right) \right\} + \frac{1}{\alpha} \left\{ \frac{1}{1/\alpha^2} \right\} + \left\{ T \times e^{-T/\alpha} \right\}
\]

\[
= \alpha \left\{ 1 - e^{-T/\alpha} (1 + T/\alpha) \right\} + Te^{-T/\alpha}
\]

The variance of \( y \) would be obtained by using the preceding to compute

\( E[y^2] \) in the same fashion.

\[
E[y^2] = \left[ \int_0^T x^2 \frac{1}{\alpha}e^{-x/\alpha} \, dx \right] + \left\{ T^2 \times e^{-T/\alpha} \right\}
\]

Use the integral forms given above again,

\[
E[y^2] = \frac{1}{\alpha} \left\{ x^2(-\alpha) - 2x(-\alpha)^2 + 2 - (-\alpha)^3 \right\} + \left\{ T^2 \times e^{-T/\alpha} \right\}
\]

Collecting terms, \( E[y^2] = 2\alpha^2 - \frac{1}{\alpha}e^{-T/\alpha}\left[\alpha T^2 + 2T\alpha^2 + 2\alpha^3\right] + e^{-T/\alpha}T^2 \)

\[
= 2\alpha^2 + e^{-T/\alpha}\left[T^2 - T^2 - 2T\alpha - 2\alpha^2\right] = 2\alpha^2 - 2\alpha e^{-T/\alpha}[T + \alpha] \]

\[
= 2\alpha^2 \left\{ 1 - \frac{1}{\alpha}e^{-T/\alpha}[T + \alpha] \right\} = 2\alpha^2 \left\{ 1 - e^{-T/\alpha}[1 + T/\alpha] \right\}
\]

Finally, the variance is \( \text{Var}[y] = E[y^2] - \{E[y]\}^2 \), which has a lot of terms, and no intuition.
The truncated distribution is manipulated in the same way. The integrals are simpler, as the density has only one part:

\[ f(x \mid x < T) = \frac{f(x)}{\text{Prob}(x < T)} = \frac{(1/\alpha)e^{-x/\alpha}}{1 - e^{-T/\alpha}} \]

The mean is

\[ \frac{1}{1 - e^{-T/\alpha}} \int_0^T (1/\alpha)xe^{-x/\alpha}dx = \frac{(1/\alpha)}{1 - e^{-T/\alpha}} \left[ \frac{e^{-(1/\alpha)x}}{(-1/\alpha)x - 1} \right]_0^T \]

\[ = \frac{(1/\alpha)}{1 - e^{-T/\alpha}} \left[ e^{-(1/\alpha)T} \{(1/\alpha)T - 1\} - 1(0-1) \right] \]

\[ = \frac{-1}{1 - e^{-T/\alpha}} \left[ e^{-(1/\alpha)T} \{(1/\alpha)T - 1\} + 1 \right] = \frac{1}{1 - e^{-T/\alpha}} \left[ e^{-(1/\alpha)T} \{(1/\alpha)T - 1\} - 1 \right] \]

The variance is found similarly, by obtaining \( E[y^2] \) as before.