Are Stocks Really Less Volatile in the Long Run?

Pástor and Stambaugh, JF 2009 (forth)
Presented by: Esben Hedegaard\textsuperscript{NYUStern}

October 5, 2009
Outline

1. Introduction
   - Variance Ratios
   - Measures of Variance
   - Some Numbers

2. Model
   - Numerical Illustration
   - Estimation

3. Results
   - Predictive Variance
   - Perfect vs. Imperfect Predictors
   - Predictive vs. True Variance
   - Conditional vs. Unconditional Variance

4. Conclusion

Pástor and Stambaugh
Are Stocks Really Less Volatile in the Long Run?
Common view: Stocks are less volatile in the long run

Wall Street Advice:

- “Stock investors should have an investment horizon of 3 years or more”
- “Long-run investor should have a higher equity allocation than short-run investors”
- “Stocks are safer for long-run investors who can wait out the ups and downs of the market”

Academics have always been sceptical!
Variance Ratios

How volatile are long-horizon returns compared to one-period returns?

Directly from returns:

$$VR(k) = \frac{1}{k} \frac{V(r_{t, t+k})}{V(r_{t, t+1})}$$ (1)
Variance Ratios

How volatile are long-horizon returns compared to one-period returns?

Directly from returns:

\[ VR(k) = \frac{1}{k} \frac{V(r_{t,t+k})}{V(r_{t,t+1})} \]  

1. Variance ratios below 1 are found for long horizons.
2. Hence, Sharpe-ratios are **higher** for long horizons.
3. Stocks are safer in the long run!
Pastor and Stambaugh: Stocks are MORE volatile in the long run!

Example

Suppose returns are iid. with $E(r_t) = \mu$, $V(r_t) = \sigma^2$. Then

$$VR(k) = \frac{1}{k} \frac{V(r_{t,t+k})}{V(r_t)} = \frac{1}{k} \frac{k\sigma^2}{k \sigma^2} = 1 \quad \forall k.$$  \hspace{1cm} (2)
Example

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$$VR(k) = \frac{1}{k} \frac{V(r_{t,t+k})}{V(r_t)} = \frac{1}{k} \frac{k\sigma^2}{k\sigma^2} = 1 \quad \forall k.$$ (2)

However, suppose $\mu$ is unknown! Then

$$V_t(r_{t,t+k}) = E_t(V_t(r_{t,t+k}|\mu)) + V_t(E_t(r_{t,t+k}|\mu))$$

$$= E_t(k\sigma^2) + V_t(k\mu) = k\sigma^2 + k^2V_t(\mu),$$ (3)

so $VR(k) = \frac{1}{k} \frac{k\sigma^2+k^2V_t(\mu)}{\sigma^2+V_t(\mu)} = \frac{\sigma^2+kV_t(\mu)}{\sigma^2+V_t(\mu)}$ increases in $k$!
### Measures of Variance

#### True Unconditional Variance

Conditions on true parameters.

Ex: Sample variance is an estimate of true unconditional variance.

#### True Conditional Variance

Conditions on true parameters, past returns, conditional expected return when returns are predictable.

#### Predictive Variance

1. Incorporates parameter uncertainty
2. Relevant for an investor
5 Components of Predictive Variance

Let $D_t$ be the information available to investors: Full history of returns and predictors, but not $\mu_t$ or the true parameters $\phi$ of the processes.

Main object of interest:

$$V(r_{T,T+k}|D_T) = E(V(r_{T,T+k}|\mu_T, \phi, D_T)|D_T) + V(E(r_{T,T+k}|\mu_T, \phi, D_T)|D_T)$$

(5)
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$$V(r_T, T+k | D_T) = E(V(r_T, T+k | \mu_T, \phi, D_T | D_T) + V(E(r_T, T+k | \mu_T, \phi, D_T | D_T) | D_T)$$

(5)

This is decomposed into five components:

1. i.i.d. uncertainty (+)
2. mean reversion (-)
3. uncertainty about future expected returns (+)
4. uncertainty about current expected return (+)
5. estimation risk (+)
Main Assumption: Time-Varying Expected Returns

Assume expected returns are

1. Time-varying (critical, but not controversial)
2. Predictable
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Even if stock returns are predictable, $\mu_t$ is not known exactly:

Definition

Let $\mu_t = E_t(r_{t+1})$. The predictor $x_t$ is called **perfect** if

$$\mu_t = \alpha + \beta' x_t.$$  \hspace{1cm} (6)

Otherwise the predictor is called **imperfect**.
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$$

(6)

Otherwise the predictor is called **imperfect**.

Imperfect predictors increase uncertainty about current and future $\mu_t$. 
Predictive variance can be calculated with perfect or imperfect predictors. So, 4 measures of variance:

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1 Unconditional true variance</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>2 Conditional true variance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Predictive variance with perfect predictors (known $\mu_t$)</td>
<td>$\sim 1.08$</td>
<td>$\sim 0.45$</td>
</tr>
<tr>
<td>4 Predictive variance with imperfect predictors (unknown $\mu_t$)</td>
<td>$1.45 - 1.70$</td>
<td>$\sim 3.45$</td>
</tr>
</tbody>
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Assume

$$r_{t+1} = \mu_t + u_{t+1}$$  \hspace{1cm} (7)

$$\mu_{t+1} = (1 - \beta)E_r\beta\mu_t + w_{t+1}$$  \hspace{1cm} (8)

1. $\rho_{uw}$: Mean reversion when $\rho_{uw} < 0$: Unexpected low return
   $\iff u_{u+1} < 0 \iff w_{t+1} > 0 \iff \mu_{t+1} \uparrow$.

2. $R^2$: degree of predictability.

3. Let $b_T = E(\mu_T|\phi, D_T)$. With perfect predictors, $\rho_{\mu b} = 1$, otherwise $\rho_{\mu b} < 1$.

4. Note: $\rho_{\mu b} = 0$ gives unconditional variances (no info about $\mu_t$), and $\rho_{\mu b} = 1$ gives variances conditional on $\mu_t$. 

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Distribution of Uncertain Parameters

- $\beta$: Persistence of $\mu_t$.
- $R^2$: degree of predictability.
  - Solid line: $r_{t+1}$ on $\mu_T$
  - Dashed line: $r_{t+1}$ on $b_T$
- $\rho_{uw} < 0$: Controls mean reversion.
- With perfect predictors, $\rho_{\mu b} = 1$, otherwise $\rho_{\mu b} < 1$.

Figure 4. Distributions for uncertain parameters. The plots display the probability densities used to...
Effect of Parameter Uncertainty on $VR(20)$

Table 1 (Based on distributions in Figure 4)

<table>
<thead>
<tr>
<th>fixed (F) or uncertain (U)</th>
<th>$\rho_{\mu b}$ fixed at</th>
<th>$\rho_{\mu b}$ uncertain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$R^2$</td>
<td>$\rho_{uw}$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>U</td>
<td>F</td>
<td>F</td>
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<td>F</td>
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- $\beta$: Persistence of $\mu_t$.
- $R^2$: degree of predictability.
- $\rho_{uw} < 0$: Controls mean reversion.
- With perfect predictors, $\rho_{\mu b} = 1$, otherwise $\rho_{\mu b} < 1$.
- 1. With known params, $VR(20) < 1$
- 2. With unknown params, $VR(20) > 1$. 

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Estimation

Predictive system with three predictors:

1. Dividend yield
2. Bond yield (first diff in long-term high-grade bond yields)
3. Term spread (long-term bond yield minus short-term interest rate)

Choose priors for $\rho_{uw}$, $\beta$ and $R^2$ (see Fig 5).

Use stock market data from 1802-2007 and compute posteriors using MCMC (see Fig 6).

These characterize the parameter uncertainty faced by an investor after updating the priors with 206 years of equity returns.
Estimation: Posterior

Posterior shows evidence of

1. Predictability (posterior for true $R^2$ lies to the right of prior)
2. Persistence of $\mu_t$ (posterior for $\beta$ lies to the right of prior)
3. Mean reversion: $\rho_{uw}$ has mode at $-0.9$ (consistent with observed autocorrelations of real returns)
4. Predictor imperfection ($R^2$ in a regression of $\mu_t$ on $x_t$ is low). Important, as predictor imperfection drives results.
Predictive Variance

Figures show predictive variance and its components with imperfect predictors.

Figure 8. Predictive variance of multiperiod return and its components. Panel A plots the variance...
Figures show predictive variance and its components with imperfect predictors.

1. Predictive variance increases with horizon.
2. Uncertainty about future expected returns has highest effect.
Figures show predictive variances with perfect and imperfect predictors.
Perfect vs. Imperfect Predictors

Figures show predictive variances with perfect and imperfect predictors.

1. Based on non-informative priors.
2. Predictive variance with perfect predictors is almost flat across horizons.
3. Predictive variance with imperfect predictors increases with horizon.
Predictive vs. True Variance

Figures show

1. Sample variance as a measure of true unconditional variance
2. Percentiles for Monte Carlo under iid. returns

The sample variance gets a $p$-val of 1%, supporting the hypothesis that true 30-year variance is $< 1$. 

Figure 10. Sample variance ratios of annual real equity returns, 1802–2007. The plot displays the sample variance ratio and percentiles for Monte Carlo under iid. returns.
Sample variance is a measure of true \textit{unconditional} variance. Appendix A4 show that

\[
VR_u(k) = (1 - R^2) VR_c(k) + \frac{1}{k} \left( \frac{1 - \beta^k}{1 - \beta} \right)^2 R^2
\]  

(9)

So true unconditional variance could be decreasing in $k$, while true conditional variance could \textit{increase}. 

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Pástor and Stambaugh: Are Stocks Really Less Volatile in the Long Run?
Use a predictive system and 206 years of data to form posteriors for model parameters.

Compute long-horizon predictive variances.

Mean-reversion reduces long-horizon variances.

But uncertainty about current and future expected returns, and parameters, offsets this reduction.

Uncertainty about future expected returns has the largest effect.

Imperfect predictors increase uncertainty about current and future returns, and drive results.
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Discussion

Esben Hedegaard

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October 5, 2009
Outline

1. Variance Ratios
2. Predictability
3. Target-Date Funds and Learning

Esben Hedegaard
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1. Variance Ratios
2. Predictability
3. Target-Date Funds and Learning
I missed a summary of different types of variance ratios:

<table>
<thead>
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1. Variance Ratios

2. Predictability

3. Target-Date Funds and Learning
Given that the main focus of the paper is on variance ratios, and not stock return predictability, why not incorporate predictability of second moments?


Second moments seem to be a lot more predictable than first moments!

Regress variance on predictors, or use GARCH models.
Outline

1. Variance Ratios
2. Predictability
3. Target-Date Funds and Learning
Target-Date Funds

Application: Target-date funds.

Target-date funds gradually reduces stock allocation as the target date approaches.

Follows standard advice: “Long-term investor should have higher equity allocation.”

Consider an investor with power utility.
Panel A: Initial and final equity allocation **without** parameter uncertainty. Very similar to real-world target-date funds.
Panel B: Incorporate parameter uncertainty. Implies lower equity allocation. Result is driven by uncertainty about future expected returns.
Panel C: Optimal initial equity allocation, given fixed final allocation.
Discussing investments in target-date funds, the investor bases his investments on the posterior distributions today.

He thus acts as if he will have the same knowledge over the next 30 years!

What if the investor learns and can re-balance every period?
Dynamic programming:

$$\max_{\alpha_t} \sum_{t=1}^{T} U_t(c_t) \text{ s.t.}$$

$$c_T = w_T \quad (1)$$

$$w_{t+1} = \alpha_t(w_t - c_t)(1 + r_{t+1}) + (1 - \alpha_t)(w_t - c_t)(1 + r_{t+1}^f) \quad (2)$$

$$r_{t+1} = \mu_t + u_{t+1} \quad (3)$$

$$x_{t+1} = \phi + Ax_t + \nu_{t+1} \quad (4)$$

$$\mu_{t+1} = (1 - \beta)E_r + \beta \mu_t + w_{t+1} \quad \text{Unobserved} \quad (5)$$

$$\mu_0 \sim N(\hat{\mu}_0, \sigma_{\mu}^2) \quad (6)$$
Idea: Learning and Rebalancing

In an LQG model you could include all five components of long-run variance:

1. iid uncertainty
2. mean reversion
3. uncertainty about future expected return
4. uncertainty about current expected return (predictive system)
5. estimation risk (robust control)

The investor learns about conditional expected return using the Kalman filter. He makes his investment/consumption decision today, knowing that he can rebalance in the next period, and anticipating that he has learned more about the conditional expected return.