Vayanos and Vila, A Preferred-Habitat Model of the Term Structure of Interest Rates

Presented by Jaewon Choi

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Overview

- Term-structure model in which investors with preferences for specific maturities and arbitrageurs
- Solutions are in affine forms
- Results
  - Bond risk premia are negatively related to short rates and positively to the term-structure slope
  - Forward rates under-react to expected short rates
  - Despite the presence of two factors, the first principal component explains about 90% of movement
Continuum of zero-coupon bonds in zero net supply

\[ P_{t,\tau} \] the time-\( t \) price of the bond with maturity \( \tau \)

\[ R_{t,\tau} = -\frac{\log(P_{t,\tau})}{\tau} \] yield of the bond

Short rate is exogenous

\[ dr_t = \kappa_r(\bar{r} - r_t)dt + \sigma_r dB_{r,t} \]

Investors

Each investor demands only a specific maturity

Demand function of the clientele \( \tau \)

\[ y_{t,\tau} = \alpha(\tau)\tau(R_{t,\tau} - \beta_{t,\tau}) \]

Intercept \( \beta_{t,\tau} = \beta_t \)
Arbitrageurs

- Trade a bond portfolio across maturities
- Denoting time $t$ wealth by $W_t$
  
  $$dW_t = \left(W_t - \int_0^T x_{t,\tau} d\tau\right) r_t dt + \int_0^T x_{t,\tau} \frac{dP_{t,\tau}}{P_{t,\tau}} d\tau$$

- Arbitragues solve a standard mean-variance problem
  
  $$\max \left[ E_t(dW_t) - \frac{a}{2} \text{Var}_t(dW_t) \right]$$
  
  Maximization over instantaneous mean and variance.

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\( \beta_t = \beta \) constant.

Short rate \( r_t \) is the only factor

Equilibrium

\[
P_{t,\tau} = e^{-[A_r(\tau)r_t + C(\tau)]}
\]

\[
\frac{dP_{t,\tau}}{P_{t,\tau}} = \mu_{t,\tau} dt - A_r(\tau)\sigma_r dB_{r,t}
\]

\[
dW_t = \left[ W_t r_t + \int_0^T x_{t,\tau}(\mu_{t,\tau} - r_t) d\tau \right] dt - \left[ \int_0^T x_{t,\tau} A_r(\tau) d\tau \right] \sigma_r dB_{r,t}
\]

From the FOC, \( \mu_{t,\tau} - r_t = A_r(\tau)\lambda_r \)

\[
\lambda_r \equiv a\sigma_r^2 \int_0^T x_{t,\tau} A_r(\tau) d\tau
\]

And market clearing, \( x_{t,\tau} = -y_{t,\tau} \)

Risk neutral parameters

\[
\kappa_r^* = \kappa_r + a\sigma_r^2 \int_0^T \alpha(\tau) A_r(\tau)^2 d\tau > \kappa_r
\]

Reverse carry case

\[
\bar{r}^* \equiv \bar{r} + \frac{(\beta - \bar{r})z_{\beta} + z_c}{\kappa_r^*}
\]

\( \bar{r}^* \) increasing in \( \beta \)
Forward rates

\[ f_{t,\tau - \Delta \tau, \tau} = - \frac{\log \left( \frac{P_{t,\tau}}{P_{t,\tau - \Delta \tau}} \right)}{\Delta \tau} \]

\[ f_{t,\tau} = - \frac{\partial \log (P_{t,\tau})}{\partial \tau} \]

Proposition 2: Effect of Short-Rate Expectations

0 < \frac{\partial f_{t,\tau}}{\partial r_t} < \frac{\partial E_t(r_{t+\tau})}{\partial r_t} : Under-reaction of forward rates

\[ \frac{\partial f_{t,\tau}}{\partial r_t} \text{ is decreasing in } \tau \]

\[ f_{t,\tau} = E_t(r_{t+\tau}) + \left[ E^*_t(r_{t+\tau}) - E_t(r_{t+\tau}) \right] - \frac{\sigma_r^2}{2A_r(\tau)^2} \]

Proposition 3: Effect of Investor Demand

0 < \frac{\partial f_{t,\tau}}{\partial \beta} < 1

\[ \frac{\partial f_{t,\tau}}{\partial \beta} \text{ increasing in } \tau \]

Longer maturities are harder to arbitrage
Fama-Bliss regression

\[ \frac{1}{\Delta \tau} \log \left( \frac{P_{t+\Delta \tau, \tau-\Delta}}{P_{t, \tau}} \right) - R_{t, \Delta \tau} = \alpha + \gamma_p (f_{t, \tau-\Delta \tau, \tau} - R_{t, \Delta \tau}) + \epsilon \]

- $\Delta \tau = 1$ year and $\tau = 2, 3, 4, 5$ years
- EH predicts $\gamma_p$ to be zero
- Found that $\gamma_p$ is positive and the standard deviation of predicted risk premia is large (about 1-1.5%)

The model predicts the positive relationship between risk premia and the term-structure slope

- When $r_t$ is low the term structure is upward sloping
- Arbitrageurs are long bonds and borrow short rates because they see $r_t$ will rise and expected future rates will be low
- The trade should have positive risk premium
Risk-Premia and Predictability

- Negative relationship between risk premia and the short rate
  $$\frac{1}{\Delta \tau} \log \left( \frac{P_{t+\Delta \tau,\tau-\Delta \tau}}{P_{t,\tau}} \right) - R_{t,\Delta \tau} = \alpha_s + \gamma_s R_{t,\Delta \tau} + \epsilon$$
  - Model predicts that $\gamma_s = -(\kappa^*_r - \kappa_r)A_r(\tau) < 0$ when $\Delta \tau \to 0$
  - $\lambda_r = \kappa^*_r r^* - \kappa_r \bar{r} - r_t^*(\kappa^*_r - \kappa_r)$
  - Price of risk $\lambda_r$ is affine in $r_t$ and changes the sign

- Campbell and Shiller
  $$R_{t+\Delta \tau,\tau-\Delta \tau} - R_{t,\tau} = \alpha + \gamma_r \frac{\Delta \tau}{\tau-\Delta \tau} (R_{t,\tau} - R_{t,\Delta \tau}) + \epsilon$$
  - EH predicts that $\gamma = 1$. But CS find that $\gamma_r$ is smaller than one
Demand parameter $\beta$ follows an OU process

$$d\beta_t = \kappa_\beta (\beta - \beta_t) + \sigma_\beta dB_{\beta,t}$$

The equilibrium is similar, but solved numerically
Figure 1: Effect of a unit increase in the short rate $r_t$ on the term structure of expected short rates and instantaneous forward rates. The dashed-dotted line represents the effect on expected short rates. The dashed line represents the effect on instantaneous forward rates when $\sigma_\beta = 0$, and the solid line represents the effect when $\sigma_\beta = 0.02$. 

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Figure 2: Effect of a unit increase in $\beta_t$ (decrease in demand) on the term structure of instantaneous forward rates. The dashed line corresponds to the case $\sigma_\beta = 0$, and the solid line to the case $\sigma_\beta = 0.02$.

**Figure**: Russian yield spreads.
Figure 3: The solid line represents the first principal component of bond returns, plotted in yield space and normalized. The dashed line represents the effect of an $r$-shock on yields and the dashed-dotted line represents the effect of a $\beta$-shock.
Figure 4: The coefficient $\gamma_p$ of the Fama-Bliss regression (24). The dependent variable is the excess return of a $\tau$-year zero-coupon bond over an one-year interval ($\Delta \tau = 1$). The independent variable is the difference between the one-year forward rate starting in year $\tau - 1$ and the one-year spot rate. The coefficient $\gamma_p$ is plotted for all maturities $\tau > 1$. The dashed line corresponds to the case $\sigma_{\beta} = 0$, and the solid line to the case $\sigma_{\beta} = 0.02$. 
Discussion

- Nice equilibrium model
  - Affine term structure
- Explains empirical puzzles
- Larger effect of demand on longer maturities - how much of it is mechanical?
- Extensions
  - Demand function $y_{t,\tau} = \alpha(\tau)\tau(R_{t,\tau} - \beta_{t,\tau})$
  - Different demand shocks across maturities