OBJECTIVES: (1) To learn basic concepts of discrete time financial models. (2) To develop an understanding of Arbitrage Pricing theory that is based upon probability theory. (3) To build a theoretical foundation for pricing financial derivatives such as options and futures.

PREREQUISITE: B09.2405 or equivalent. This course has significantly lower mathematical requirement than B90.3323. Recommended is familiarity with basic concepts of calculus and matrix algebra (inverse, determinant, rank, etc.).


GRADING: The course grade is determined by a weighted average of homework assignments (30%), mid-term examination (30%), and final examination (40%). Class contribution and regular attendance is strongly urged.

SYLLABUS: The following topics will be covered in the framework of discrete time, single and multiple period securities market. (Chapters 1, 3, 4, and 6.)

1. Model specification
   Arbitrage and risk neutral probability measures
   Valuation of contingent claims

2. Stochastic Processes, conditional expectation and martingales
   Binomial models

3. European and exotic options
   American options
   Forward and future prices
   Bonds and interest rate derivatives
Review of Probability Theory and Matrix Algebra

Sample Space and the $\Omega$, $\omega$ notation:

For mathematical reasons, the set of all possible outcomes on the sample space $\Omega$ is denoted as

$$\Omega = \{ \omega_1, \omega_2, \ldots, \omega_k \}.$$  

Each $\omega_i$ is a distinct event. The sample space is discrete and finite.

Probability Measure

With each event $\omega_i$ is associated a probability, $P(\omega_i)$. What do we require of the probabilities?

Important Different probability measures can be assigned to the events.

Random Variable

A random variable is simply a function of the $\omega_i$. Thus, if $X$ is a random variable defined on the sample space $\Omega$ then it takes the value

$$X(\omega) \text{ with probability } P(\omega).$$

Example 1. (a) Coin tossing. (b) A game of dice. Define the sample space for each, what is the random variable, assign probabilities. Repeat using a common sample space.
Expectation, Variance, and Covariance

We shall define these quantities in both ways, the traditional way and in the $\Omega, \omega$ framework. In the “traditional” framework, suppose the random variable $X$ has $n$ possible outcomes, $x_1, x_2, \ldots, x_k$, and let $P(X = x_i)$ denote the probability that $X$ takes the value $x_i$.

1. **Expectation (mean, average)** of the random variable $X$ is given by

$$E(X) = \sum_{i=1}^{k} x_i P(X = x_i)$$

equivalently

$$E(X) = \sum_{i=1}^{k} X(\omega_i) P(\omega_i) .$$

- For any constants, $a$ and $b$, $E(aX + b) = a E(X) + b$.
- For any two random variables, $X$ and $Y$, $E(X+Y) = E(X) + E(Y)$.

2. **Variance** of the random variable $X$ is defined as

$$V(X) = \sum_{i=1}^{k} (x_i - E(X))^2 P(X = x_i) .$$

Equivalently, ______________________________________________________________________

- $V(X) = E(X^2) - E(X)^2$.
- $V(aX + b) = a^2 V(X)$.
- $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$.
- $V(X+Y) = V(X) + V(Y) + 2 \text{Cov}(X,Y)$.
- Standard deviation, $\sigma(X) = \sqrt{V(X)}$.

3. **Standard deviation**

For constants $a$ and $b$, $\sigma(aX + b) = a \sigma(X)$.

4. **Covariance** between two random variables is the linear association between them. $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$. Also equal to, $E((X-E(X)) (Y – E(Y))$.

- If $> 0$, then $X$ and $Y$ tend to move together.
- If $< 0$, the $X$ and $Y$ tend to move in opposite direction.
- Independent random variables have $\text{Cov} = 0$.
- Correlation between two random variables $X$ and $Y$ (assuming that their standard deviations are strictly positive) is defined as, $\rho(X,Y) = \text{Cov}(X,Y)/(\sigma(X)\sigma(Y))$. 
Example 2. What is the expected payoff and variance of the return for a dollar spent on the New York lottery?

Example 3. The following table gives the predicted gains of two stocks under different conditions. Both stocks are priced at $10 per share today. What is your best investment strategy? Is diversification always beneficial?

<table>
<thead>
<tr>
<th>Internet Stocks</th>
<th>Probability</th>
<th>Stock 1</th>
<th>Stock 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hot</td>
<td>0.5</td>
<td>$10</td>
<td>-$6</td>
</tr>
<tr>
<td>Cool</td>
<td>0.5</td>
<td>-$6</td>
<td>$10</td>
</tr>
</tbody>
</table>

Matrices

Matrix equations and inequalities: $Ax = b, Ax \leq b$.

Rank of a matrix = number of independent rows
($=$number of independent columns)

$Ax = b$ has a “solution x” for every “b” if all rows are independent.
Single Period Markets

Model Specification

1. At time $t = 0$

   (1) State of the world $\omega$ at time $= 1$ is unknown, but can take one of $k$ possible states,
   \[ \Omega = \{ \omega_1, \omega_2, ..., \omega_k \}. \]
   Subjective Probability (agree these are the only outcomes)
   Probabilities $P_1, P_2, ..., P_k$ (or, equivalently, $P(\omega_i)$
   \[ P_i > 0, i = 1,2,...,k. \]
   \[ \sum_i P_i = 1. \]

   (2) Bank account process: $B_0 = $1 (value per unit).

   (3) Investment is possible in $N$ securities or risky assets.

   (4) Price process: $S_0 = (S_1(0), S_2(0), ..., S_N(0))$ are the prices per share of the $N$ securities.

   (5) Trading strategy (T.S.): $H = (H_0, H_1, ..., H_N)$, where
   $H_0$ is the number of units of bank account;
   for $1 \leq n \leq N$, $H_n$ is the number of shares of security $n$
   $H_n < 0$ means “borrowing” or “short selling.”

   (6) Value process: $V_0 = H_0B_0 + \sum_{n=1}^{N} H_nS_n(0)$.

2. At time $t = 1$

   (1) One state $\omega$ from $\Omega = \{ \omega_1, \omega_2, ..., \omega_k \}$ is realized.

   (2) Bank account process: $B_1 \geq $1 (per unit)
   risk-free interest rate: $r = B_1 - 1$ (can be deterministic or stochastic).
   $r$ is greater than or equal to zero.

   (3) Values of the $N$ securities change according to the state of the world.

   (4) Price process: $S_1 = (S_1(1), S_2(1), ..., S_N(1))$.

   (6) Value process: $V_1 = H_0B_1 + \sum_{n=1}^{N} H_nS_n(1)$. 
(7) Gain process: Let $\Delta S_n = S_n(1) - S_n(0)$.

$$G \equiv H_0 r + \sum_{n=1}^{N} H_n \Delta S_n.$$ (what is the gain?)

Verify: $V_1 = V_0 + G$.

3. Discounted Processes: Objective is to make comparisons at time $t = 0$.

Discounted stock prices: $S_n^\ast (1) = S_n(1) / B_1 = \frac{S_n(1)}{1 + r}, \quad n = 1, 2, ..., N.$

(define $\Delta S_n^\ast = S_n^\ast (1) - S_n(0)$.)

Discounted value process: $V_1^\ast = H_0 + \sum_{n=1}^{N} H_n S_n^\ast (1) = \text{verify} \frac{V_1}{B_1} = \frac{V_1}{1 + r}.$

Discounted gain process: $G^\ast = \sum_{n=1}^{N} H_n \Delta S_n^\ast = \text{verify} V_1^\ast - V_0.$

Notice that $G^\ast \neq \frac{G}{B_1}$.

Example 1 There are two states of the world. One risky security.
$k = \quad N = \quad r = 1/9$ in both states.

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>$S_1$</td>
</tr>
<tr>
<td>$S_0 = $5$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$40/9.$</td>
</tr>
</tbody>
</table>

$B_1 =$

You are given $H_1 = 100$ and $V_0 = 400$.

Compute the value at time 1. Gain at time 1.

Repeat for the discounted value and gain.
Self-assignment  Memorize the notation. Go over examples 1.2 and 1.3 in the book carefully.

**Game of Trading and Pricing**

1. Trading side of the story

What is an arbitrage opportunity? Intuitive and mathematical definitions.

Intuitive

Mathematical: define using the notation developed so far ($H_0$, $H_1$), $V_0$, $V_1$, and $G$.

Subtle but important distinction between arbitrage opportunity, dominant trading strategy, and the law of one price.

**Example 2**  Discounted prices. Single security.

<table>
<thead>
<tr>
<th>Case a</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case b</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Example 3**  $r = 1/9$.  $N = 2$.  $k = 3$.  

<table>
<thead>
<tr>
<th></th>
<th>$t=0$</th>
<th>$t=1$ (undiscounted price)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State</td>
<td>Price Pair</td>
</tr>
<tr>
<td>(5,10)</td>
<td>$\omega_1$</td>
<td>(60/9, 40/3)</td>
</tr>
<tr>
<td></td>
<td>$\omega_2$</td>
<td>(60/9, 80/9)</td>
</tr>
<tr>
<td></td>
<td>$\omega_3$</td>
<td>(40/9, 80/9)</td>
</tr>
</tbody>
</table>
**Example 4**  \( k = 3, \ N = 1, \ r = 1/9 \)

<table>
<thead>
<tr>
<th>t=0</th>
<th>t=1 (undiscounted price)</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>Price Pair</td>
</tr>
<tr>
<td>24</td>
<td>( \omega_1 )</td>
</tr>
<tr>
<td></td>
<td>( \omega_2 )</td>
</tr>
<tr>
<td></td>
<td>( \omega_3 )</td>
</tr>
</tbody>
</table>

**Example 5**  \( k = 3, \ N = 2, \ r = 1/9 \)

<table>
<thead>
<tr>
<th>t=0</th>
<th>t=1 (undiscounted price)</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>Price Pair</td>
</tr>
<tr>
<td>(24,30)</td>
<td>( \omega_1 )</td>
</tr>
<tr>
<td></td>
<td>( \omega_2 )</td>
</tr>
<tr>
<td></td>
<td>( \omega_3 )</td>
</tr>
</tbody>
</table>