Sessions 1&2: Course Overview and Introduction to Derivatives

**Futures and Options**

**Course number:** FINC-UB.0043 Futures and Options

**Course description:** This course is designed to introduce Finance students to the theoretical and real world aspects of financial futures, options, and other derivatives. Over the last 40 years, the markets for these versatile instruments have grown enormously and have generated a profusion of innovative products and ideas, not to mention periodic crises. Derivatives have become one of the most important tools of modern finance, from both the academic and the practical standpoint. The subject is inherently more quantitative than other business courses, but the emphasis in this course is not on the math and theory, but always on developing your intuition. The goal is for you to understand the principles of how these instruments and markets work, not to derive models and prove theorems.

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**Office:** KMEC 9-64  
**Office hours:** Tuesday/Thursday after class (5:00-6:30); and by appointment

**Course website:** Course materials and announcements will be posted on the course website.
This session begins the second portion of the course, which focuses on option valuation theory and practice and more specialized "advanced topics."

The development of option theory is one of the major triumphs of modern finance. It has contributed to enormous growth and proliferation of trading in derivative securities, as well as the use of new theoretical valuation tools for derivatives valuation and risk management.

In 1997 the Nobel Prize was awarded to Myron Scholes and Robert Merton in recognition of the importance of the option pricing model. Fischer Black would surely have shared the award, but he unfortunately died in 1995, before it was given.
Put-Call Parity and Option Properties from Portfolio Dominance

A Very Important Arbitrage Trade

Consider a call and a put on XYZ stock, with the same maturity, 1 month, and the same strike price, 100. (Both options are European.)

If XYZ is above 100 in a month, the 100 strike call will be exercised.
If XYZ is below 100 in a month, the 100 strike put will be exercised.

What if you buy the call and also write the put?

• Suppose XYZ ends up at 105. You choose to exercise the call: you pay 100 and you get the stock. (The put is out of the money and expires without being exercised.)
• What if XYZ is at 95? The put will be exercised by your counterparty and you will have to buy the stock for 100. (The call is worthless.)

In other words, no matter what happens with the stock price, at maturity you will pay 100 and you will own the stock.
Put-Call Parity

This position returns exactly what you would have if you just bought the stock today for $S = 100$, and borrowed the present value of 100 to finance a large part of the position. (On option maturity day, you would own the stock and pay out 100.)

With two different ways to achieve exactly the same payoff, there is a possible arbitrage trade. If these two positions do not cost exactly the same to set up, arbitrageurs will buy the position in the way that is cheaper and sell it (i.e., take the opposite position) in the way that is more expensive. The difference in cost between them is an arbitrage profit that is locked in. Arbitrage trading will continue until prices come into line (meaning: into line closely enough that there is no more arbitrage profit after transactions costs).

In our example, arbitrage will force

$$Call \ price - Put \ price = S - PV(100)$$

The general Put-Call parity relationship that must hold in equilibrium is

$$Call \ price - Put \ price = Asset \ price - PV(Strike \ price)$$

or

$$C - P = S - PV(X)$$
Put-Call Parity and Option Properties from Portfolio Dominance

**How to Exploit Violations of Put-Call Parity**

**Example:** $S = 100$, $X = 100$, 3 month Call = $5.00$, 3 month Put = $3.50$, $r = 8.00\%$

What should we do?

1. Does put-call parity hold? $C - P = 1.50$
   
   $S - PV(X) = 100 - 100 / 1.02 = 1.96$
   
   No. $C - P < S - PV(X)$. The put is too expensive relative to the call.

2. Arbitrage trade: Buy what is cheap and sell what is expensive
   
   $\Rightarrow$ (Buy the call, write the put) and (sell short the stock - lend at the riskless rate)
   
   $\Rightarrow$ initial cost: $(5 - 3.5) + (-100 + 98.04) = -0.46$, negative "cost" = net cash inflow

3. At expiration in $T = 3$ months:

<table>
<thead>
<tr>
<th></th>
<th>$S_T &lt; 100$</th>
<th>$100 \leq S_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>0</td>
<td>exercise (buy stock, pay 100)</td>
</tr>
<tr>
<td>Put</td>
<td>exercised (buy stock, pay 100)</td>
<td>0</td>
</tr>
<tr>
<td>Stock</td>
<td>use stock to cover short sale</td>
<td>use stock to cover short sale</td>
</tr>
<tr>
<td>Riskless asset</td>
<td>riskless bonds pay off 100</td>
<td>riskless bonds pay off 100</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Put-Call Parity and Option Properties from Portfolio Dominance

How to Exploit Violations of Put-Call Parity

Example: $S = 100, \ X = 100, \ 3\text{ month Call} = 4.00, \ 3\text{ month Put} = 1.50, \ r = 8.00\%$

What should we do?

1. Does put-call parity hold?
   \[ C - P = 2.50 \]
   \[ S - PV(X) = 100 - \frac{100}{1.02} = 1.96 \]
   No. $C - P > S - PV(X)$. The put is too cheap relative to the call.

2. Arbitrage trade: Buy what is cheap and sell what is expensive
   ⇒ (Write the call, buy the put) and (buy the stock - borrow PV(X) at the riskless rate)
   ⇒ initial cost: $(-4 + 1.5) + (100 - 98.04) = -0.54$ (net cash inflow)

3. At expiration in $T = 3$ months:

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<tr>
<td><strong>Riskless asset</strong></td>
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</tr>
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<td><strong>Total</strong></td>
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Put-Call Parity and Option Properties from Portfolio Dominance

Beyond Arbitrage: Portfolio Dominance

Put-call parity is a fundamental relationship.

- It must hold for (European) options.
- It shows up all over in option strategies and price relationships.

The style of proof is very powerful.

- Two positions that are equivalent to one another no matter what happens, must always cost the same.

The principle of "Portfolio Dominance" (no portfolio dominance, actually) is a generalization of "no-arbitrage."

- If two positions are such that their payoffs are not always equal, but the first is never less than the second and can be greater under some circumstances, then the first position must cost more than the second.

Portfolio dominance leads to more general properties of option prices than we can get from option pricing models like Black-Scholes.

- We use it to prove a variety of price relationships, which seem intuitively obvious, must in fact hold for all options under all circumstances. The proof is similar to that for put-call parity.
- We can also prove some option properties that aren't so obvious, such as the fact that an American call on a non-dividend paying stock should be worth no more than a European call with the same terms.
**Example: Portfolio Dominance Proof That**

**A Lower Strike Price Makes A Call Option More Valuable**

Consider the payoffs at expiration on two calls on the same stock, that are identical except that they have different strike prices, with \( X_1 < X_2 \) (e.g., \( X_1 = 95 \), \( X_2 = 100 \))

<table>
<thead>
<tr>
<th>Payoff at Expiration Date ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_T \leq X_1 )</td>
</tr>
<tr>
<td>Call 1</td>
</tr>
<tr>
<td>Call 2</td>
</tr>
<tr>
<td>( (Call 1 \cdot Call 2) )</td>
</tr>
<tr>
<td>( (Positive) )</td>
</tr>
</tbody>
</table>

The payoffs can both be zero, or else Call 1 pays more than Call 2.

**Therefore, Call 1 must always sell for more than Call 2:** \( C(X_1) \geq C(X_2) \)

(They can be equal, but only if the probability is zero that the stock price at expiration is above \( X_1 \), in which case they are both worthless).
Option Properties from Portfolio Dominance

Call value increases with
- Higher stock price
- Lower strike price
- Longer time to expiration
- Higher interest rate
- Higher volatility
- Lower dividend payout

Put value increases with
- Lower stock price
- Higher strike price
- ??? time to expiration
- Lower interest rate
- Higher volatility
- Higher dividend payout
Put-Call Parity and Option Properties from Portfolio Dominance

More Option Properties that Can Be Derived from Portfolio Dominance

- No early exercise of American calls unless the asset makes cash payouts (dividends, coupon interest, etc.)

- Dividend payout can make it rational to exercise an American call early (just before the stock goes ex-dividend).

- Options are “convex” functions of both the asset price and the strike price.

- "An option on a portfolio is worth less than a portfolio of options."
Put-Call Parity and Option Properties from Portfolio Dominance

Using Put-Call Parity to Understand a Call Option's Value

The market price is $6.43 for a 95 strike 1 month call option on a stock whose market price is 100. What do you get from buying the call today instead of the stock?

We can use put-call parity to decompose this into three intuitive elements.

Rearranging the put-call parity equation gives

\[ C = S - PV(X) + P \]

\[ C = (S - X) + (X - PV(X)) + P \]

\[
\begin{align*}
\text{Call value} &= \left( \text{intrinsic value} \right) + \left( \text{interest saved by not paying 95 until expiration} \right) + \left( \text{the right not to own the underlying if its price falls below 95} \right) \\
6.43 &= 5 + .61 + .83
\end{align*}
\]

95 strike call intrinsic value 1 month interest on 95 at 8% 95 strike put
### Rearranging Put-Call Parity

- **S** = Price of Underlying Asset
- **C** = Price of Call on S with Strike Price X
- **P** = Price of Put on S with Strike Price X

#### Put-Call Parity

<table>
<thead>
<tr>
<th>&quot;Synthetic Long&quot;</th>
<th>Buy on Margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C - P )</td>
<td>( S - PV(X) )</td>
</tr>
<tr>
<td>Buy and Write Call</td>
<td>Buy and Borrow Asset</td>
</tr>
</tbody>
</table>

Rearranging the Put-Call Parity equation produces a variety of new positions

#### Covered Call

\[
\text{Buy and Write Stock and Call} \quad \Rightarrow \quad \text{Write Call} \\
\begin{align*}
\text{S} & \quad \text{C} \\
\text{Buy and Write Stock} & \quad \text{C}
\end{align*}
\]

#### Cash-Secured Put

\[
\text{Hold and Write Cash and Put} \quad \Rightarrow \quad \text{Write Put} \\
\begin{align*}
\text{PV(X)} & \quad \text{P} \\
\text{Hold and Write Cash} & \quad \text{Put}
\end{align*}
\]
Put-Call Parity: Bonus Tracks

Rearranging Put-Call Parity, continued

**Protective Put**

\[
S + P = \begin{cases} 
\text{Buy and Buy} \\
\text{Stock and Put} 
\end{cases}
\]

**90/10 Strategy**

\[
C + PV(X) = \begin{cases} 
\text{Buy and Hold} \\
\text{Call and Cash} 
\end{cases}
\]

**"Conversion"**

\[
S - C + P = \begin{cases} 
\text{Buy and Write and Buy} \\
\text{Stock and Call and Put} 
\end{cases}
\]

**Riskless Investment**

\[
PV(X) = \begin{cases} 
\text{Hold} \\
\text{Cash} 
\end{cases}
\]

**"Reverse Conversion"**

\[
-S + C - P = \begin{cases} 
\text{Short and Buy and Write} \\
\text{Stock and Call and Put} 
\end{cases}
\]

**Riskless Borrowing**

\[
- PV(X) = \begin{cases} 
\text{Borrow} 
\end{cases}
\]
Option Pricing Models

Option Pricing Models

Option properties derived from portfolio dominance are very general, but not very precise. Many people tried to find a pricing formula that could say exactly how many dollars a given option is worth. But for years, no one succeeded.

Fischer Black and Myron Scholes, along with Robert Merton, did it in the early 1970s, using a kind of mathematics borrowed from physics to model the behavior of stock prices.

The new theory was very hard to explain to students (or to finance professors). Just for pedagogical purposes, in his 1977 Investments textbook, William Sharpe introduced the Binomial model, a very simplified framework in which an option could be easily understood and priced.

However, it turned out that the simple classroom model could be extended and developed into an extremely powerful and practical valuation tool, with flexibility to solve some kinds of problems, like pricing an American put, that the Black-Scholes model can not handle. It has been successfully extended to Trinomial and more complicated lattices for even more power and flexibility.

You have already seen the one-period Binomial. The next few slides review that material. We then extend the Binomial, and look at an important property known as Risk Neutral Valuation.
Option Pricing Models: The Binomial Model

Theoretical Underpinning of Derivatives Pricing Models

There are two types of valuation models for securities in modern finance theory, **Equilibrium models** and **Arbitrage-based models**. We have seen this already for futures.

- The Expectations Model is an equilibrium model. So is the Capital Asset Pricing Model.
- The Cost of Carry Model is an arbitrage-based model.

**Equilibrium Models**

- The security price is determined by aggregate supply and demand in the market.
- The model value for a security should include a risk premium.
- Investors value each security relative to the entire set of available instruments.
- Prices are brought into line with theoretical values because investors search out undervalued securities and bid up their prices. They sell off overvalued securities, driving their prices down until they become cheap enough to be worth holding.
- All securities must be in equilibrium for the model to apply.
Option Pricing Models: The Binomial Model

Theoretical Underpinning of Derivatives Pricing Models, p.2

Arbitrage-Based Models

• The price is determined by the possibility of arbitrage between the derivative security and its underlying asset.
• The derivative is only valued relative to its underlying (and the riskless interest rate)
• The model price does not include a risk premium. All investors should value the security the same way regardless of how risk averse or risk tolerant they are.
• Prices are brought into line by arbitrage. In theory, because mispricing creates the opportunity for riskless arbitrage, it only takes a single energetic arbitrageur to force the market price to the model value.
• Because the model only says how the derivative should be priced relative to the underlying, it doesn't matter whether the underlying or any other security is in equilibrium with respect to other traded assets.

Examples: The Binomial Model, the Black-Scholes Model, and the Cost of Carry Model for futures
Option Pricing Models: The Binomial Model

The Binomial Model in Symbols and in Numbers

\[ S = 100 \]

\[ uS = 150 \]

\[ dS = 50 \]

\[ C_d = 0 \]

\[ C_u = 50 \]

\[ C = ?? \]

\[ u = 1.5, \quad d = .5. \]

\[ S = \text{asset price at } t=0. \]

We want the value of the call \( C \) at \( t=0 \)

Over the next period, the stock can go to only two possible prices:

- up to \( uS \) \((u > 1)\), or
- down to \( dS \) \((d < 1)\).

Suppose \( S = 100 \), and it can go to either 150 or 50 next period. \( u = 1.5, \ d = .5 \).

A 100-strike call \( C \) that matures next period will either pay off 50, if the stock goes to 150, or 0, if the stock is at 50.

There is also a third asset we can trade: a riskless bond that costs 1 and pays a total (principal plus interest) of \( R \) at \( t=1 \). You can borrow money by selling the bond short.

Let \( R=1.1 \) in the example, i.e., 10% interest.
Option Pricing Models: The Binomial Model

The trick is to construct a position using the stock and the bond that has the same payoff as the call option in both the "up state" and the "down state." Such a position is called a "replicating portfolio."

Buy \( h \) units of the asset and invest \( B \) dollars in bonds (negative \( B \) means borrowing). Choose \( h \) and \( B \) to make the portfolio values in the up and down states be \( C_u \) and \( C_d \).

\[
\begin{align*}
\text{Up state:} & \quad h \cdot uS + RB = C_u \quad (h \cdot 150) + 1.1B = 50 \\
\text{Down state:} & \quad h \cdot dS + RB = C_d \quad (h \cdot 50) + 1.1B = 0
\end{align*}
\]

Solving the two equations in two unknowns gives

\[
\begin{align*}
   h &= \frac{C_u - C_d}{uS - dS} = \frac{50}{150 - 50} = 0.5 \\
   B &= \frac{1}{R} \left[ \frac{uS}{uS - dS} C_d - \frac{dS}{uS - dS} C_u \right] = -22.73
\end{align*}
\]

Since they have the same payoffs, to avoid a riskless arbitrage, the call option and the "replicating portfolio" must have the same price.

\[
C = hS + B = 0.5 \cdot 100 - 22.73 = 27.27
\]
Substituting for $h$ and $B$ and simplifying leads to

$$C = \frac{1}{R} \left[ \frac{R - d}{u - d} C_u + \frac{u - R}{u - d} C_d \right]$$

The valuation formula can be simplified further if we define a new variable

$$p = \frac{R - d}{u - d}$$

The final valuation equation is

$$C = \frac{1}{R} \left[ p \, C_u + (1 - p) \, C_d \right]$$

This is called the "backward recursion" equation because it shows how to "roll back through the tree" from the end to the start. The same equation is used in every step to price the option at an earlier node when the values at the two nodes it branches to are known.

"Backward" means you begin the valuation process at expiration and move backward through the tree to the starting date; "recursion" means you use the same formula over and over again at every step.
Plugging in \( u = 1.5, \ d = 0.5, \ R = 1.10, \ C_u = 50 \) and \( C_d = 0 \), from our example gives

\[
p = \frac{R - d}{u - d} = \frac{1.10 - 0.5}{1.5 - 0.5} = 0.6
\]

which leads to an option value of

\[
C = \frac{1}{R} \left[ p \ C_u + (1 - p) \ C_d \right] = \frac{1}{1.10} \left[ 0.6 \times 50 + (1 - 0.6) \times 0 \right]
\]

\[
C = 27.27
\]
Delta in the Binomial Model

The delta of an option is the number of units of the underlying one would use to create a riskless hedge of the option.

Delta in the binomial is simply $h$, the number of shares in the replicating portfolio.

In our example, this is

$$h = \frac{C_u - C_d}{uS - dS}$$

$$h = \frac{50 - 0}{150 - 50} = 0.5$$
Let's check that a delta hedge of buying the option and shorting 0.5 shares does produce a riskless position in our example.

The initial cost of the position is $C - hS = 27.27 - 0.5 \times 100 = -22.73$

There is a cash inflow from the market at the beginning.

\[
\text{Payoff} = \begin{cases} 
\text{Up state:} & C_u - h uS = 50 - 0.5 \times 150 = -25 \\
\text{Down state:} & C_d - h dS = 0 - 0.5 \times 50 = -25 
\end{cases}
\]

The position has the same value in both states, so it is riskless. Whether the stock goes up or down, we have to pay out 25 at maturity.

What this trade amounts to is riskless borrowing of 22.73 from the market at $t = 0$, with repayment (including 10% interest), of 25 at $t=1$. 
Option Pricing Models: The Binomial Model

Two Important Properties of the Binomial Model

1. The backward recursion valuation method works essentially the same way for any contingent claim within the binomial framework. There is nothing in the way the pricing equation was derived that limits it to calls.

2. The valuation equation does not involve the actual probabilities attached to the up and down branches. The option value is the same for all probabilities. (The replicating portfolio matches the option value in both possible states, so it doesn't matter how likely each one is. The "probabilities" in the formula are artificial numbers that come out of the solution to the "two equations in two unknowns" problem.)
Risk Neutral Valuation

Recall the Cost of Carry model for pricing gold futures. If $S$ is the spot price for gold, say $1600$ per ounce, and $r$ is the riskless interest rate, say 10%, what should the futures price be for a 1 year futures contract?

$$F_{\text{1 year}} = S \left( 1 + r \frac{T}{365} \right) = 1600 \times 1.10 = 1760$$

Suppose investors are so risk averse that they only hold gold if they expect the price to go up 25% a year. For them to pay $1600$ an ounce for gold today, investors must be expecting it to go up to $1600 \times 1.25 = $2000$ an ounce next year. What is the equilibrium gold futures price in that case? It is the same 1760! The gold futures market obeys the cost of carry model because you can buy gold and hedge away all the risk with futures, leaving a riskless position that must earn the riskless interest rate, no matter how risk averse investors are. (Where does the risk go?)

Suppose investors are completely indifferent to risk: They are "risk neutral." Then any asset, risky or not, should be priced to earn the same expected return. If gold is at 1600 in the spot market when investors are risk-neutral, it is because the market expects the price to go up to $1600 \times 1.10 = $1760$ next year. What is the futures price? Still 1760, of course. But now, the cost of carry model and the expectations model give the same answer.

"Risk neutral valuation" means the derivatives price in an arbitrage-based model is the same as what it would be in the expectations model in a world of risk neutral investors.
Risk Neutral Valuation

Risk Neutral Valuation for Options

Because of arbitrage, the option value in the Binomial Model is the same under all probabilities. But it's often easier to solve a valuation problem if you know the probabilities. Risk neutral valuation means that we can pick any convenient probabilities to solve the valuation problem, and the answer we get must be the value under all possible probabilities.

An especially useful assumption is that investors are indifferent to risk and care only about an asset's expected return. In this "risk neutral" world, all assets, including stocks and options, will be priced to have the same expected return as a risk free asset.

In our Binomial world if $p$ is the probability of an up step, then

$$E[S_1] = [p u S + (1 - p) d S]$$

And in a risk neutral world, the current price $S$ is just the expected value of the next period stock price $S_1$ discounted by $(1 + \text{the riskless rate of interest})$.

$$S = \frac{E[S_1]}{R}$$

Combining the two, we have that the stock price in the Binomial follow the same equation that the option does.

$$S = \frac{1}{R} [p u S + (1 - p) d S]$$
Let's solve for the value of the "risk neutral probability" $p$ using the stock prices in the market.

In the example: $S = 100$, $R = 1.1$, $u = 1.5$, $d = 0.5$

$$ S \cdot R = \left[ p \cdot uS + (1 - p) \cdot dS \right] $$

$$ 100 \cdot (1.10) = p \cdot (150) + (1 - p) \cdot (50) $$

$$ 110 = 50 + p \cdot (150 - 50) $$

$$ p = \frac{110 - 50}{150 - 50} = \frac{60}{100} = 0.6 $$

In general: Solving the general equation for the risk neutral probability $p$ gives

$$ p = \frac{R - d}{u - d} $$
Risk Neutral Valuation

Risk Neutral Valuation for the Option

The current option value is the expected value of next period's price discounted at the riskless rate. p is the risk neutral probability of an up step, so

\[
C = \frac{1}{R} \left[ p C_u + (1 - p) C_d \right]
\]

This is exactly the form of the general valuation function we derived in solving the option replication problem using arbitrage arguments!

\[
C = \frac{1}{1.10} \left[ 0.6 \times 50 + (1 - 0.6) \times 0 \right]
\]

\[
= \frac{30}{1.10} = 27.27
\]

Risk neutral valuation is a general property of all derivatives pricing models that are based on arbitrage, like the Cost of Carry model for futures. It is an extremely useful and powerful tool for obtaining valuation equations.
Extending the Binomial Model

To turn the Binomial into a practical valuation tool, it must allow more realistic asset price behavior.

This is done by building a tree out of multiple binomial steps. The time period of each one can be made as short as one likes, so that the asset price can be made to go to as many possible values as one likes in any given length of time, simply by subdividing the interval.

Price movements are multiplicative in the Binomial model, so an up move followed by a down yields $duS$, which is the same price as a down followed by an up, $udS$. This means that rather than having the number of nodes going up by $2, 2^2, 2^3, 2^4, ...$ as each new time period is added, the lattice recombines, and the number of nodes goes up as $2, 3, 4, ...$, which keeps the procedure computationally manageable as the number of time steps grows.
Extending the Binomial Model

Two Period Binomial Model

\[ \begin{align*}
S & \quad \text{uuS} \\
uS & \quad \text{udS} \\
dS & \quad \text{ddS} \\
C & \quad \text{C}_{uu} \\
C_u & \quad \text{C}_{ud} \\
C_d & \quad \text{C}_{dd}
\end{align*} \]
Extending the Binomial Model

Pricing a Two-Period Call: \( S=100, X=100, u=1.5, d=.5, R=1.1 \)

\[
\begin{align*}
\text{Cu} &= (.6 \times 125 + .4 \times 0)/1.1 \\
&= 68.18 \\
C &= (.6 \times 68.18 + .4 \times 0)/1.1 \\
&= 37.19
\end{align*}
\]
**Pricing a Put:** Use the same parameters to price a 100-strike European put option.

\[
C_{u} = (0.6 \times 0 + 0.4 \times 25)/1.1 = 9.09
\]

\[
C_{\text{put}} = (0.6 \times 9.09 + 0.4 \times 40.91)/1.1 = 19.83
\]

\[
C_{d} = (0.6 \times 25 + 0.4 \times 75)/1.1 = 40.91
\]
To illustrate how useful the principle of risk neutral valuation can be, let's see how the two period European call and put would be priced.

There are 3 possible states after 2 periods:

- uu (2 up moves): probability = prob(uu) = p × p = 0.6 × 0.6 = 0.36
- ud (1 up and 1 down): probability = prob(up then down) + prob(down then up)
  = p (1 - p) + (1 - p) p = 2 p (1 - p) = 2 × 0.24 = 0.48
- dd (2 down moves): prob(dd) = (1 - p) × (1 - p) = 0.4 × 0.4 = 0.16

2-period discount rate: \( R^2 = 1.1 \times 1.1 = 1.21 \)

**Risk Neutral Prices:**

**Call:**
\[
\frac{1}{1.21} \times (0.36 \times 125 + 0.48 \times 0 + 0.16 \times 0 )
\]
\[
= 0.36 \times 125 / 1.21
\]
\[
= 45 / 1.21 = 37.19
\]

**Put:**
\[
\frac{1}{1.21} \times (0.36 \times 0 + 0.48 \times 25 + 0.16 \times 75 ) = (12 + 12) / 1.21
\]
\[
= 24 / 1.21 = 19.83
\]
Early Exercise of American Puts

We have seen that a European put can have a fair value below its intrinsic value. In that case, you would like to exercise it early, but you can't. This situation can't happen with an American put. There is an "early exercise boundary" on the asset price, such that if the price falls below the boundary, an American put should be immediately exercised.
Using the Binomial Model to Value American Puts

Early Exercise of American Puts

It is rational to exercise an American put early. Time value has two components, one due to "optionality" and the other due to the interest that can be earned on the exercise price. These are both positive for a call option, so you won't exercise early. Optionality is also positive for a put, but the interest effect is negative: If you are going to exercise the option and receive the exercise price in cash, the longer you have to wait, the smaller is the present value of that cash. Time value goes to zero for a deep in the money put, at which point it is better to exercise it than to hold it any longer.

An American put is therefore worth more than a European put, because early exercise has economic value. But valuing the early exercise feature of an American put is mathematically tricky, so that there is no "simple" closed-form formula for the American put price.

Put exercise is more likely with

- high intrinsic value, X - S
- short time to maturity
- low volatility
- high interest rate
- high dividend yield (payment of a dividend tends to delay exercise until after ex-dividend date)
Using the Binomial Model to Value American Puts

One of the great advantages of the Binomial is that it can be modified easily to deal with early exercise and other features that are hard to handle in the Black-Scholes framework.

An American put should be exercised at any time prior to expiration if the intrinsic value that would be received immediately is greater than the option value based on holding it over the next period.

This is easy to build into the Binomial model. Since the rational option holder will make the exercise choice that gives the larger value, simply compare the two values at each node and put the larger one into the tree.

The backward recursion formula becomes

\[
C_{\text{American put}} = \max \left[ \frac{1}{R} \left( p C_u + (1-p) C_d \right), X - S \right]
\]
Pricing an American Put: Use the same parameters to price a 100-strike American put option.

\[
C_d = \text{Max}[(.6 \times 25 + .4 \times 75)/1.1 , 100 - 50] = \text{Max}[40.91, 50] = 50 \text{ (exercise early)}
\]

\[
C_u = \text{Max}[(.6 \times 0 + .4 \times 25)/1.1 , 100 - 150] = \text{Max}[9.09, -50] = 9.09
\]

\[
C_{\text{put}} = (.6 \times 9.09 + .4 \times 50)/1.1 = 23.14
\]
Incorporating a Dividend Payout in the Binomial Model

To price options on actual securities, the model must be adjusted to allow the underlying to pay dividends or some other type of cash payout over time.

For a constant proportional dividend yield \( q \), this is easily done by adjusting the risk neutral probability \( p \).

\( R \) is 1 plus the riskless interest rate per time step. Incorporate a proportional dividend yield by dividing \( R \) by 1 plus the dividend rate per time step. Set

\[
p = \frac{R/(1 + q) - d}{u - d}
\]

**Examples:**

- **Stock index portfolio:** \( q = \text{percent dividend yield} \)
- **Foreign currency:** \( q = \text{foreign riskless interest rate} \)
- **Futures** \( q = \text{riskless interest rate (set } R/(1+q) \text{ to 1)} \)**
Using the Binomial Model

Setting Binomial Model Parameters Based on Market Data

To use the model for pricing real-world options, we must set its parameters to match market values.

- Initial stock price, $S$, strike price $X$, and time to maturity $T$ are already known.
- Let $\Delta t$ be the length of one time step. In an $N$-step tree, $\Delta t = T / N$.
- If $r$ is the (continuously compounded!) annual interest rate and $y$ is the annual continuously compounded proportional dividend yield, set $R = e^{r \Delta t}$ and $(1+q) = e^{y \Delta t}$.
- Volatility per step $v = \sigma \sqrt{\Delta t}$ where $\sigma$ is the annualized volatility. This must be turned into values for $u$ and $d$. There are several common ways to do this.
  - Easiest: Set $u = e^v$ and $d = e^{-v}$
  - Fastest convergence:
    
    \[
    u = e^{r \Delta t - \frac{v^2}{2} + v} \quad \text{and} \quad d = e^{r \Delta t - \frac{v^2}{2} - v}
    \]

    This builds the riskless interest rate into the tree as the mean return on the stock.
The Binomial model allows us to approximate option values to as close a degree of accuracy as we want. It is also much more flexible than Black-Scholes, so it can be used to solve many valuation problems, such as pricing an American put, that the Black-Scholes model can't handle.

However, the Binomial is just an approximation, not a "closed-form" solution. A closed form solution is an equation that gives the exact option value as a function of a set of input parameters.

Also, the Binomial is less efficient than a closed-form equation. It may take a long time to converge to the exact option value as the number of time steps in the tree is increased, and convergence is not monotonic (for example, a 200 step tree may give a less accurate answer than a 100 step tree).

This session introduces the Black-Scholes model. This classic model is a closed-form solution to the option pricing problem. It is universally used in practice (although not in the way the theory says it should be used) and it points the way towards similar valuation equations for other types of derivative instruments with option features.
Convergence to the True Option Value in the Binomial Model

Note the strong even-odd pattern and also that convergence isn't uniform: you can be closer to the right answer with 50 steps than with 80 here.
Dynamic Hedging to Replicate Option Payoffs

The Binomial Model set up a framework in which the underlying asset and the riskless bond could be combined to create a position that exactly replicates the payoff on the option.

The Black-Scholes model is derived in a similar way: The option and the stock are combined to create a hedged position that is like a riskless bond.

Like the bond, the riskless option-stock hedged position must return the riskless rate of interest. This leads to a fair price for the option.

Black and Scholes' assumptions permit a much more realistic price process than the Binomial for the underlying asset, while still allowing a riskless hedge to be constructed over the option's entire lifetime. However, this requires a dynamic hedging strategy, because the position is only riskless over the next instant in time, and then it must be rebalanced.
The Standard Model for Asset Returns

Price changes for securities like stocks have several stylized features that need to be incorporated into whatever model is used for the "returns process:"

- the price is observable (more or less) continuously
- random fluctuations occur all the time
- prices follow a "random walk," meaning that the random fluctuations are independent from one period to the next, even at the shortest intervals
- the distribution of percentage returns is (approximately) normal

These properties are expressed formally in the form of a "lognormal diffusion process." This is the standard assumption for derivatives modeling, particularly options.

We will look at the lognormal diffusion model more closely later in the course, when we get to options. For now, we will just assume some of the properties of the model hold, without getting into details.

**Key assumptions:** Over a period of time of length T,

- the (continuously compounded) return follows a normal distribution
- security prices follow a lognormal distribution (the logarithm of price is normally distributed)
- **The "Square Root of T Rule" for volatility:**
  - the variance of the return is proportional to the length of the time interval
  - \( \text{VAR}[R] = \text{VAR}[\ln(S_T/S_0)] = \sigma^2 T; \text{ standard deviation of return } = \sigma \sqrt{T} \)
- returns measured over non-overlapping time periods are statistically independent
The Standard Model for Asset Returns

Standard Normal Probability

About 2/3 of the probability falls within plus or minus 1 standard deviation of the mean.
The Standard Model for Asset Returns

Lognormal Probability

Probability of stock price in 1 year
Initial price = 100
Mean return = 6%
Volatility = 25%

Expected value of price in t = one year
= $V_0 \, e^{(r + \sigma^2/2)t}$
= $100 \, e^{(.06 + .25^2/2)}$
= 109.55
The Standard Model for Asset Returns

The Asset Price Process

The Black-Scholes model assumes the price of the underlying asset, $S$, follows a "lognormal diffusion" process:

$$dS = \mu S \, dt + \sigma S \, dz$$

where
- $dS$ = the change in stock price over the next instant
- $\mu$ = the "drift," that is, the average rate of capital gains as a continuously compounded annualized figure
- $dt$ = an "instant"
- $\sigma$ = the volatility, expressed as an annual rate
- $dz$ = "Brownian motion," a very small random shock to the price over the next instant.

dz has mean zero and variance $1 \, dt$. The standard deviation of $dz$ is $1 \sqrt{dt}$ (which is a lot bigger than $dt$ when $dt$ is very small. This means the volatility term dominates the drift term over short time periods.)

Current $dz$ is independent of all $dz$ in previous (and future) periods.
The lognormal diffusion model of stock price changes has several important implications:

• Price paths are continuous. It is impossible for the price to jump from one level to another without passing through every price in between.

• Prices fluctuate randomly at all points in time, but the randomness is independent from period to period. The price behavior is often called a "random walk." (This is intuitive but imprecise mathematically. The exact description is a "semi-martingale": price changes are random and independent, but there can be a nonrandom drift and variance can change from period to period.)

• Given a starting value $S_0$, the log price change over the period from 0 to $T$ is given by $R = \ln \left( \frac{S_T}{S_0} \right)$.

• $R$ has a normal distribution, with: mean $= \mu T$ and standard deviation $= \sigma \sqrt{T}$

• $S_T$ has a lognormal distribution with expected value $= S_0 e^{(\mu + \sigma^2 / 2) T}$
Option Pricing Models: Black-Scholes

Underlying Assumptions of the Black-Scholes Option Pricing Model

- Options are European

- "Perfect" markets -- no transactions costs, no taxes, no constraints on short selling with full use of the proceeds, no indivisibilities, etc. (doesn't say anything about "efficient" markets)

- No limits on borrowing or lending at a known risk free rate of interest

- The price of the underlying asset follows a "lognormal diffusion" process

- The return volatility of the underlying asset is known

- No dividends or cash payouts from the underlying asset prior to option maturity
The call valuation model derived by Black and Scholes gives the option price as a function of 5 variables (6, if one allows dividends, which the original BS model did not).

\[ \text{Call} = C(S, X, T, r, \sigma) \]

where,
- \( S \) = stock price
- \( X \) = strike price
- \( T \) = time to option expiration
- \( r \) = riskless interest
- \( \sigma \) = volatility

The sensitivity of the option value with respect to a change in one of the parameters is given by the relevant partial derivative.

We normally think of the stock price and time as changing, while the other parameters are fixed. But it can be useful to know how much the option price would be affected by a slightly different volatility parameter or riskless interest rate, so those partial derivatives are of interest as well.
Option Pricing Models: Black-Scholes

The Partial Derivatives of the Call Value Function

A partial derivative is like an ordinary derivative in calculus. If one is interested in a function \( f(x) \), defined for a single variable \( x \), the derivative of \( f(x) \) with respect to \( x \) tells how much the function value will change per unit change in \( x \).

We write the derivative as

\[
\frac{d \ f}{d \ x}
\]

For a function of more than one variable, the same concept applies. The partial derivative with respect to one of the variables in the function tells how much the function will change per unit change in the variable in question, holding all of the other variables in the function constant. If \( f(x,y) \) is a function of two variables \( x \) and \( y \), we write the partial derivatives as

\[
\frac{\partial \ f}{\partial \ x} \quad \text{and} \quad \frac{\partial \ f}{\partial \ y}
\]

Example: Consider the function \( f = 2 \ x^2 \ y \) :

\[
\frac{\partial f}{\partial x} = 4 \ xy \ ; \quad \frac{\partial f}{\partial y} = 2 \ x^2 \ ; \quad \frac{\partial^2 f}{\partial x^2} = 4 \ y \ ; \quad \frac{\partial^2 f}{\partial xy} = 4 \ x \ ; \quad \frac{\partial^2 f}{\partial y^2} = 0
\]
The Black-Scholes model equation is the solution to a partial differential equation (PDE). Such an equation exists for every derivative, in many cases differing only in the boundary conditions it must satisfy. The PDE can be used for derivative pricing with numerical approximation techniques even when there is no closed form solution.

Here is a quick look at the fundamental PDE for the Black-Scholes model and how it leads to the Black-Scholes option pricing equation. (The next few slides are provided exclusively for your viewing pleasure. This material will not appear on any homework or exam.)

The price $S$ of the underlying asset is assumed to follow the lognormal diffusion process

$$dS = \mu S \, dt + \sigma S \, dz$$

where

- $\mu$ is the instantaneous mean return
- $\sigma$ is the volatility of the return.
- $dz$ is standard Brownian motion
Ito's Lemma

The mathematical tool for dealing with diffusion processes is Ito's Lemma.

We are interested in a variable C that is a function of S and time: $C = C(S,t)$.

Ito's Lemma gives the equation for the diffusion process followed by C.

\[
\text{Ito's Lemma} \quad \frac{dC}{dt} = \frac{\partial C}{\partial S} \frac{dS}{dt} + \frac{\partial C}{\partial t} \frac{dt}{dt} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS)^2
\]

The term $(dS)^2$ in this expression is evaluated using the 3 multiplication rules:

\[
dt \, dz = 0; \quad (dt)^2 = 0; \quad (dz)^2 = dt
\]

which come from the fact that as $dt$ goes to zero, any higher power of $dt$, like $(dt)^2$ and $(dt)^{3/2}$ becomes infinitesimally small relative to $dt$ and drops out of the equation.

(For example, suppose $dt$ is 1/1000th of a year (about 1/3 of a day). $(dt)^2$ would be 1/1,000,000th of a year: a thousand times smaller.)
Applying Ito's Lemma to the option value equation $C(S,t)$ gives

$$dC = \frac{\partial C}{\partial S} \, dS + \frac{\partial C}{\partial t} \, dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \, (dS)^2$$

$$= \frac{\partial C}{\partial S} \, \mu \, S \, dt + \frac{\partial C}{\partial S} \, \sigma \, S \, dz + \frac{\partial C}{\partial t} \, dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \, \sigma^2 \, S^2 \, dt$$

Notice that the first term is the option's "delta" times the change in the stock price. The last term comes from using the multiplication rules from the previous page in calculating $(dS)^2$. 
The Fundamental Partial Differential Equation of Derivatives Pricing

The value of any derivative instrument $C(S,t)$, such as a call option, only varies with the price of the underlying asset and time. It can be hedged over the next instant by a short position in the underlying asset.

Hedge: Sell delta = $\frac{\partial C}{\partial S}$ units of the asset for each call.

The hedged position is worth $V = C - \left(\frac{\partial C}{\partial S}\right)S$. The dynamics of the hedge portfolio are given by

$$dV = dC - \frac{\partial C}{\partial S} dS = \frac{\partial C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dt$$

Notice that both the $dz$ term and the term involving the stock's mean return $\mu$ have canceled out. Because the position is perfectly hedged, there is no risk and no risk premium in this expression. (Sounds a lot like risk neutral valuation: Risk aversion doesn't enter the valuation equation.)

The position is perfectly hedged over the next instant $dt$, so it must earn the riskless interest rate. Otherwise there would be an arbitrage. Therefore, we also have

$$dV = rV dt = r \left( C - \frac{\partial C}{\partial S} S \right) dt$$
The Fundamental Partial Differential Equation of Derivatives Pricing

Combining the two relationships that $dV$ must satisfy leads to the following partial differential equation (PDE), that must hold for every derivative security:

$$rC - rS \frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

The solution to a partial differential equation is a function. In this case, that function is the Black-Scholes valuation model for the derivative security $C$.

This same equation holds for a call and a put both. What makes the formula price one rather than the other are 3 boundary conditions.

To solve this PDE for a particular derivative security, one must add boundary conditions, that specify what happens to the value at maturity date $T$, what happens at some date $t < T$ before maturity if $S$ goes to zero, and what happens when $S$ grows infinitely large.

Boundary conditions for a European Call Option:

*at option maturity:* \[ C(S, T) = \text{Max} \ (S_T - X, 0) \]

*if the stock goes to 0:* \( \lim_{S \to 0} C(S, t) = 0 \)

*if the stock goes very high:* \( \lim_{S \to \infty} \frac{\partial C(S, t)}{\partial S} = 1 \)

Boundary conditions for a European Put Option:

*at option maturity:* \[ P(S, T) = \text{Max} \ (X - S_T, 0) \]

*if the stock goes to 0:* \( \lim_{S \to 0} P(S, t) = \text{PV}(X) \)

*if the stock goes very high:* \( \lim_{S \to \infty} P(S, t) = 0 \)
Option Pricing Models: Black-Scholes

The Black-Scholes Model

Let

- \( S \) = asset price
- \( X \) = strike price
- \( r \) = riskless rate
- \( T \) = maturity
- \( \sigma \) = volatility

Assume the option is European and the underlying asset pays no dividends.

**Call option value:**

\[
C = S \, N[d] - X \, e^{-rT} \, N[d - \sigma \sqrt{T}]
\]

where

\[
d = \frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}}
\]

**Call delta:**

\[
\delta_{\text{CALL}} = N[d]
\]
The Black-Scholes Model

Under the same assumptions as above, the Black-Scholes model value for the European put can be derived in the same way, but it is obtained more easily directly from put-call parity.

**Put option value:**

\[
P = X e^{-rT} N\left[-d + \sigma \sqrt{T}\right] - S N\left[-d\right]
\]

**Put delta:**

\[
\delta_{\text{PUT}} = -N\left[-d\right]
\]
Modern option pricing models following Black-Scholes all have a major role for the volatility of the underlying asset's return. This is the one parameter in Black-Scholes that is not observable, so naturally there is a lot of dispersion around the average volatility expectation in the market.

In Black-Scholes, volatility is assumed to be a constant known parameter. But in the real world it is neither constant nor known. This session will review what we know about volatility, how it goes into option prices, and how the wide variety of volatility-based products can be used to speculate or hedge volatility.
"Stylized Facts" about Volatility

Here are some common findings from research on the behavior of asset volatility

- volatility is not constant; it changes substantially over time
- periods of high volatility and periods of low volatility cluster together
- there appears to be "mean reversion" in volatility; periods of unusually high or low volatility tend to be followed by a reversion to more normal behavior
- in equity markets, volatility increases when stock prices fall, and (may) decrease when prices rise (this is often called the "leverage effect")
- implied volatility has a regular structure across options with different strike prices, known as the "smile" or the "skew"
- implied volatility also shows systematic "term structure" effects for options with different maturities
Estimating Realized Volatility

Three Techniques for Estimating / Forecasting Volatility from Historical Data

1. Historical volatility:
   - Compute K log returns from past prices: \( R_{t-k} = \ln( \frac{S_{t-k}}{S_{t-k-1}} ) \), for \( k = 1,\ldots,K \)
   - Volatility estimate = annualized standard deviation of \{R\}

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<th>stock</th>
<th>log return</th>
<th>return squared</th>
</tr>
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<td>102</td>
<td>0.985%</td>
<td>9.70677E-05</td>
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<tr>
<td>t-1</td>
<td>101</td>
<td>4.041%</td>
<td>0.001632931</td>
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</table>
Estimating Realized Volatility

Three Techniques for Estimating / Forecasting Volatility from Historical Data

2. Exponentially weighted moving average
   • Compute log returns from past data as above
   • Downweight data as it ages, by multiplying each squared deviation by \( w^k \), for some weight \( w < 1.0 \). (Riskmetrics uses \( w = 0.94 \).)
   • Volatility estimate = annualized value of
     \[
     \sqrt{\max_{k=0}^{k=max} w^k \sum_{k=0}^{k=max} w^k R_{t-k}^2 / \sum_{k=0}^{k=max} w^k}
     \]

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3. **GARCH (Generalized Autoregressive Conditional Heteroskedasticity)**

Model variance at date t as a combination of

- last period's variance, $\sigma^2_{t-1}$
- last period's squared random price shock $\epsilon^2_{t-1}$

The simplest GARCH model has two equations:

Return equation:  
$$r_t = \mu + \epsilon_t, \quad \epsilon_t \text{ is distributed as Normal (0,} \sigma^2_t\text{)}$$

Variance equation:  
$$\sigma^2_t = C + a \sigma^2_{t-1} + b \epsilon^2_{t-1}$$

(Typical values are 0.90 to 0.95 for $a$ and 0.05 to 0.08 for $b$. The sum should be < 1.0.)

For stocks an asymmetry term is added to make volatility go up when the stock price falls.

The GARCH model for stock returns:  
$$\sigma^2_t = C + a \sigma^2_{t-1} + b \epsilon^2_{t-1} + d \epsilon^2_{t-1} \text{ (if } \epsilon_{t-1} < 0)$$

Note that in the GARCH framework, time is not continuous.
Estimating Realized Volatility

Practical Issues in Estimating Volatility from Past Prices

If the price of the underlying asset did follow a lognormal diffusion with constant mean and volatility, as assumed by Black and Scholes, you would get the best volatility estimate by using as much past data, sampled at as fine an interval, as you could get. But with real world prices, several practical issues arise:

• what observation interval to use (daily? monthly? intraday?)
  • suggestion: higher frequency is better, so use daily data; intraday returns require special handling

• whether to estimate the mean
  • NO!

• how much past data to include
  • as much as possible, but not from much different economic environments

• how to deal with "outliers," i.e., events like October 19, 1987
  • there is no perfect answer; use judgment and consider how sensitive the results are to the outliers
Conclusions on Forecasting Volatility: Historical Data

Here are several general conclusions I have reached based on my research on volatility prediction:

- Different methods should be compared in terms of **out of sample forecasting** performance.
- Accuracy of all methods is low.
- Using data sampled at very short intervals (daily or less) requires careful adjustment for "noise" arising from the trading process (e.g., "bid-ask bounce").
- It is better to assume the mean is 0 than to take deviations around the sample mean.
- Simpler models, such as straightforward use of measured historical volatility over a long sample period seem to be about as accurate as more complicated models, and are more robust.
- Volatility forecasts for long horizons seem to be more accurate than for short horizons.
- GARCH models appear to work well over very short horizons, but only if there is a lot of data available to estimate model parameters.
- GARCH seems to work best for equities.
Implied volatility is the value of the volatility input to an option pricing model that makes the model value equal the option price observed in the market.

$$IV \text{ is the solution to } C(S, X, T, r, IV) = C_{\text{market}}$$

(The actual value for IV must be found by a search process.)

IV should impound "the market's" forecast of volatility. IV is felt by many to be the best volatility estimate possible, because the market has access to much more information than any model can incorporate.

There is a one-to-one correspondence between implied volatility and option price. In some markets, options are quoted in terms of implied volatility, rather than price. OTC foreign currency options are an important example.

Note that implied volatility depends on the option pricing model used to calculate it. IV as commonly reported is always computed from the Black-Scholes model (with dividend correction, and sometimes with an adjustment for American exercise).
SPX Volatility smiles from AUG calls, 7/31/2013 and 8/1/2013

Implied volatility July 31

July 31 Closing price

Implied volatility August 1

August 1 Closing price

July 31 volatility smile from calls

Aug. 1 volatility smile from calls
The VIX Volatility Index

The CBOE computes, publishes, and now trades futures and options based on a composite index known as the VIX. The VIX index is designed to measure the standard deviation of the S&P500 stock index over the next month, as implied by the current market prices of SPX index options. The index is an interpolated value for a one month horizon, extracted from the nearest to expiration and the next nearest contracts.

The original formula for computing the VIX was changed in 2003. The "Old VIX," (still computed and published as the VXO), was based on 8 at-the-money calls and puts on the OEX index and used the Black-Scholes model to extract the implied volatilities. These IVs were combined into a weighted average 30-day at-the-money implied volatility. (There was also a technical problem in annualizing the index, which made it significantly biased upward.)

The new VIX uses all out of the money SPX index calls and puts expiring just before and just after 30 days and extracts, not individual IVs, but the whole risk neutral probability distribution (without using Black-Scholes or any other pricing model). The implied volatility is the standard deviation from this implied distribution.

Futures contracts on the VIX began trading in 2004 and VIX options in 2006.

Although it is supposed to be an estimate of volatility, the VIX is widely thought of and referred to, as "the market fear gauge".
Historical and Implied Volatility for S&P 500 Index

- SPX
- VIX
- Historical Volatility (last 6 months)
The Volatility Surface

Volatility Surface E-mini S&P 500 Futures Options: 2015-02-23

Red dot indicates front month underlier price
Volatility: Flowchart of How Volatility Gets into Option Prices and How it is Extracted

Historical price data

Forecasts and other information

Hedging (risk aversion)

Statistical Estimate of Future Returns

Investors' Estimate of Future Returns

Demand for Options

Invert the Black-Scholes Model

Market Option Prices

Model-free Volatility calculation

"Risk-neutral" Probabilities

Implied BS Volatilities

Volatility smile

Volatility surface

The original VIX Index VXO

Risk Neutral Density

Model-free Implied Volatility; the VIX Index

What a financial econometrician calculates

What the market actually predicts: the "P" density also known as "empirical" or "real world" probabilities

What the options market embeds in option prices, including risk premia: the "Q" density,
Implied Volatility

The Real Mystery: Where Does the Volatility Smile Come From?
A Variety of Possible Explanations:

1. The underlying returns distribution is not Normal. ("Fat tailed" distributions)
   • too many "big" returns (both positive and negative); a Student-t distribution with about 7 degrees of freedom fits the data better

2. Volatility of a stock depends on the ratio of equity to debt in the firm's capital structure, and that changes when the stock price moves. (The "leverage effect")

3. Volatility is stochastic
   • GARCH (volatility is a function of the asset price change)
   • Two-factor models (in which volatility is subject to random changes that are at least partly independent of the price change)

4. Stock prices can make large jumps (non lognormal "jump-diffusion processes")

5. Investors are "Crash-o-phobic" and are willing to pay extra for the protection of out of the money puts. In the money calls (that have strike prices below the current stock price, like out of the money puts) must also have high implied volatilities or else they would violate put-call parity.
There are now multiple ways to trade volatility as an investible asset. The easiest is a **variance swap**. This is a forward contract on the difference between the realized variance of returns over some period ending at a future date and a strike level set at the beginning.

Example: A trader who expects high volatility in the stock market until the end of the year 2016 could "buy" a variance swap with a strike variance rate of $(0.20)^2$ and maturity Dec. 31. The quote is often done in terms of the volatility implied by the given variance, which is 20% in this case.

Realized variance is calculated as the annualized mean of $R_t^2$, where $R_t$ is the log return, for dates $t$ from now through Dec. 31. The difference between realized variance and the strike is multiplied by a notional principal to get the payoff in dollars.

A **volatility swap** is essentially the same thing, except the payoff is in terms of the difference between realized volatility and the strike.

A volatility swap seems to make more sense, since volatility is what investors care about, but valuation is trickier mathematically, meaning it is a lot harder to hedge than a variance swap, so vol swaps are less popular.
In 2004, the CBOE introduced trading in **futures contracts based on the VIX**. Contract size is $1000 times the VIX index, with minimum tick size of 0.01 (1 volatility basis point).

There is no cost of carry pricing model for VIX products, since you can't store the VIX.

In 2006, the CBOE added **VIX options**, i.e., options whose payoffs are equal to $100 * Max(VIX$_T$ – X, 0), where T is option maturity and X is the strike level.

Volatility products have become immensely popular. There are now dozens of related volatility contracts for different commodities and time horizons.

There are also numerous **exchange traded funds** that offer exposure to volatility (of stocks, or oil, or gold, or...) just like a mutual fund. These trade like stocks, not futures or options. But they all use VIX futures to create their payoffs.
Trading the VIX and financial products based on the VIX is very different from trading stocks or commodities, because the VIX itself cannot be bought and held. It is not investible. Exchange-traded VIX-based products like ETFs are all based on VIX futures.

Gold futures are tied to the current price of gold, because you can buy gold today, hedge by selling futures, and lock in the return on the trade. Similarly, you can buy a portfolio of stocks, sell short index futures, and carry that hedged position to futures maturity.

By contrast, the VIX is more like a temperature. The current level can be observed all the time, but you can't buy the VIX, carry it over time and deliver it against a future or option contract.

This means:
1. The VIX future is based on expectations about what the VIX will be at futures maturity. There is no arbitrage-based pricing model for the VIX.

2. There is therefore no direct connection between what the spot VIX does and how any VIX products behave, just as today's temperature tells us little about what the temperature will be 30 days from now.
Implied Volatility

The Information Content of Implied Volatility

Question: Is implied volatility from an option's market price an efficient forecast of future volatility of the underlying asset?

- Is IV an unbiased forecast?

- Does IV impound all of the information contained in historical volatility?

This issue has been examined many times in the literature for different markets. Most researchers find IV biased, but that it contains information, and generally more than historical volatility.
The Information Content of Implied Volatility

The idea that implied volatility is an efficient forecast of future realized volatility involves a joint hypothesis.

1. The implied volatility IV is equal to the market's volatility forecast:

   \[ IV = E_{MKT}[\sigma] \]

2. The market's forecast is rational

   \[ \sigma = E_{MKT}[\sigma] + \epsilon \]

where \( \epsilon \) has mean 0 and minimum variance given the currently available information.
Implied Volatility

The Information Content of Implied Volatility: Standard tests

1. "Rationality test regression:" If F is a rational forecast of volatility $\sigma$, then

$$\sigma = F + \varepsilon$$

To test this, run the regression

$$\sigma_t = \alpha + \beta F_t + u_t$$

Test $\alpha = 0$ and $\beta = 1.0$

2. "Encompassing regression:" With multiple forecasts $F_1$ and $F_2$, run this regression

$$\sigma_t = \alpha + \beta_1 F_{1t} + \beta_2 F_{2t} + u_t;$$

If the $F_{1t}$ forecast impounds all of the information contained in $F_{2t}$, then $\alpha = 0.0$, $\beta_1 = 1.0$, and $\beta_2 = 0$
The Information Content of Implied Volatility: Typical Test Results

1. "Rationality test regression:"

   \[ \sigma_t = \alpha + \beta F_t + u_t \]

   Test results: \( \alpha \approx 0.06 \) and \( \beta \approx 0.65 \)

2. "Encompassing regression:"

   With multiple forecasts, run this regression

   \[ \sigma_t = \alpha + \beta_1 \text{ImpliedF}_{1t} + \beta_2 \text{HistoricalF}_{2t} + u_t; \]

   Test results: \( \alpha \approx 0.06 \) and \( \beta_1 \approx 0.65 \), and \( \beta_2 \approx 0 \)
Implied Volatility

Conclusions on Volatility Prediction: Implied Volatility

• The volatility smile shows that the basic Black-Scholes option pricing model does not fully explain how options are priced in the market.

• Implied volatility nearly always contains information about the volatility that will occur in the future, but it is biased as a forecast.

• Even when IV is shown to contain a significant amount of information about future realized volatility, if it is biased, it will not necessarily be an accurate forecast unless the bias is corrected.

• More sophisticated option pricing models can be constructed that are consistent with the existence of a volatility smile, but are they the true explanation for it?

• Despite all of these issues, market makers prefer to use implied volatility in their models, because they want the model values to match the prices they are seeing in the market. This leads to the use of "practitioner Black-Scholes," which is the Black-Scholes equation, but with a different volatility input for each option.
The Partial Derivatives of the Call Value Function

The partial derivative of the call value with respect to a small change in the stock price is the option's delta. Delta is often written using the Greek letter delta, either \( \delta \) (lower case) or \( \Delta \) (upper case). We will use \( \delta \).

\[
\delta = \frac{\partial C}{\partial S}
\]

Delta serves the same function as the hedge ratio in a futures hedge. It tells how many units of the underlying asset one should trade in order to hedge the market risk exposure of the option.

For example, if \( \delta = 0.50 \) for a given call option, the position that is long one call and short 0.50 shares of stock will be hedged against a (small) change in the stock price up or down.

This is called a "delta hedge" and the hedged position is said to be "delta neutral."
The Partial Derivatives of the Call Value Function

A very important problem with delta hedging is that delta changes as the stock price moves. The delta of a call option ranges from 0, for an option that is very far out of the money, to 1.0 for a call that is very deep in the money. Delta hedging requires rebalancing the proportions of stock and the option continuously, as the stock price moves and as time elapses.

How much delta changes as S moves is given by the partial derivative of delta with respect to S. This is the second partial derivative of the option value with respect to S. This concept is called by another Greek letter, \textbf{gamma} (\(\gamma\)). Gamma is related to the curvature of the option value function. Positive gamma means the function is convex.

\[
\gamma = \frac{\partial \delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}
\]

The other partial derivatives of the option function are also of importance in hedging, so the Greek alphabet is well represented. These measures of exposure to different types of risk affecting option value are commonly known as "the Greeks."
Delta Hedging and Beyond

"Greek Letter" Risk Exposures for Options

**Delta** (δ) - The change in the option value produced by a 1 point change in the price of the underlying asset. Delta measures exposure to Market Risk.

**Gamma** (γ) - The change in the delta produced by a 1 point change in the price of the underlying asset. Gamma measures Convexity, which turns into risk for a delta neutral hedge.

**Vega** - The change in the option value produced by a 1 percentage point change in the volatility of the underlying asset. Vega measures exposure to Volatility Risk (despite not being a true letter in the Greek alphabet). Vega is sometimes written as Λ (which is actually upper case lambda) or as ν, (which is a Greek nu).

**Theta** (θ) - The change in the option value produced by a 1 day drop in the time to maturity. Theta measures Time Decay.

**Rho** (ρ) - The change in the option value produced by a 1 percentage point change in the interest rate. Rho measures exposure to Interest Rate Risk.
Greek Letter Risk Exposures for Option Positions

Computing the total Greek letter risk exposures for a position containing several options is straightforward.

In calculus, the derivative (or partial derivative) of the sum of two functions is the sum of the (partial) derivatives of the individual functions. That is,

\[
\frac{\partial}{\partial x} \left( f(x, y) + g(x, y) \right) = \frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial x} g(x, y)
\]

For example, if our position consists of \( N_1 \) options of type \( C_1 \) (calls, puts, or any other kind of derivative) and \( N_2 \) options of type \( C_2 \), the delta of the combined position is just:

\[
\text{Combined delta} = N_1 \delta_1 + N_2 \delta_2
\]

and the same for all of the other Greek (and near-Greek) letters.
In-Class Problems

Here are the current market prices for XYZ stock and two XYZ options. The Greek letter risk exposures come from the Black-Scholes model. The interest rate is 8% and the implied volatility is 0.25.

<table>
<thead>
<tr>
<th></th>
<th>Market price</th>
<th>delta</th>
<th>gamma</th>
<th>vega</th>
<th>theta</th>
</tr>
</thead>
<tbody>
<tr>
<td>XYZ Stock</td>
<td>100</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>XYZ Call 105 strike, 1 month</td>
<td>1.25</td>
<td>0.29</td>
<td>0.047</td>
<td>0.099</td>
<td>-0.044</td>
</tr>
<tr>
<td>XYZ Put 95 strike, 1 month</td>
<td>0.83</td>
<td>-0.21</td>
<td>0.039</td>
<td>0.084</td>
<td>-0.030</td>
</tr>
</tbody>
</table>

You are long the 105 call on 100,000 shares.

1. How would you set up a delta hedge for this position?

2. What would the overall hedged position be worth? (What is the net cost to set it up?)

3. What are the Greek letter exposures for the overall position?
Delta Hedging and Beyond

In-Class Problems

<table>
<thead>
<tr>
<th>XYZ Stock</th>
<th>Market price</th>
<th>delta</th>
<th>gamma</th>
<th>vega</th>
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<td>-0.044</td>
</tr>
<tr>
<td>XYZ Put 95 strike, 1 month</td>
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<td>-0.21</td>
<td>0.039</td>
<td>0.084</td>
<td>-0.030</td>
</tr>
</tbody>
</table>

1. Position delta is $100,000 \times 0.29 = 29,000$. Hedge by **shorting** 29,000 shares.

2. Cost to set up is negative:
   - Calls $100,000 \times 1.25 = 125,000$
   - Stock $-29,000 \times 100 = -2,900,000$
   - Total $= -2,775,000$

3. delta: $100,000 \times 0.29 + (-29,000) \times 1 = 0$
   - gamma: $100,000 \times 0.047 + (-29,000) \times 0 = 4,700$
   - vega: $100,000 \times 0.099 + (-29,000) \times 0 = 9,900$
   - theta: $100,000 \times -0.044 + (-29,000) \times 0 = -4,400$
Delta Hedging and Beyond

In-Class Problems

Tomorrow, XYZ stock opens at 95. Here is the new set of option prices and Greek letters.

<table>
<thead>
<tr>
<th></th>
<th>Market Price</th>
<th>delta</th>
<th>gamma</th>
<th>vega</th>
<th>theta</th>
</tr>
</thead>
<tbody>
<tr>
<td>XYZ Stock</td>
<td>95</td>
<td>1.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>XYZ Call 105 strike, 1 month</td>
<td>0.30</td>
<td>0.10</td>
<td>0.025</td>
<td>.047</td>
<td>-.021</td>
</tr>
<tr>
<td>XYZ Put 95 strike, 1 month</td>
<td>3.35</td>
<td>-0.46</td>
<td>0.044</td>
<td>.108</td>
<td>-.052</td>
</tr>
</tbody>
</table>

4. If you liquidated right now, what would the profit or loss on the hedged position be?

5. If you don't liquidate, what stock trade will you need to do to become delta neutral again?
Delta Hedging and Beyond

In-Class Problems

Answers

4. If you unwind at the new prices your profit is:

\[
\begin{align*}
P\&L \text{ on Calls} & \quad 100,000 \times (0.30 - 1.25) & = -95,000 \\
\text{Stock} & \quad -29,000 \times (95 - 100) & = +145,000 \\
\text{Total} & & = +50,000
\end{align*}
\]

5. If you wanted to rehedge, with the new delta, you should only be short

\[100,000 \times 0.10 = 10,000 \text{ shares}.\]

You have to buy back 19,000 of the shares you shorted.
Delta Hedging and Beyond

Hedging Greek Letter Risks

Derivatives risk management begins with the basic delta-neutral hedge, but it does not end there. Serious derivatives users try to minimize the (unintended) exposure of their positions to all sources of price variability, and at the lowest possible cost. This includes:

- changes in the price of the underlying asset (delta)
- changes in the delta for large asset price changes (gamma)
- changes in volatility (vega)
- time decay (theta)
- changes in interest rates (rho)

and quite probably other things as well.

Each of these Greek letters is a partial derivative with respect to some parameter of the option pricing equation (a calculus derivative, that is).
Delta Hedging and Beyond

Principles of Generalized Hedging

1. The Greek letter risk exposures "add up." The total risk exposure of a position is the sum of the exposures of the component securities.

   - To hedge a given type of risk fully, the aggregate exposure of all hedging instruments must be equal in magnitude and opposite in sign to the aggregate exposure to that risk for the position being hedged.

2. In general you need at least one hedge instrument per type of risk.

   - For example, to hedge both delta and gamma, you need a minimum of two hedge instruments.

3. Not every instrument can be used for every type of risk.

   - For example, the bond can’t hedge delta, and neither the bond nor the stock can hedge gamma.
   - Two options may have the same value for two different Greek letters (for example, a European and call and put with the same maturity and strike have the same gamma and the same vega). In that case, you can’t use the two options to hedge those two risks separately.
4. If there are more hedge instruments than risks to be hedged, the solution is not unique. This allows optimizing on other aspects of the hedge. Things to optimize on include

- minimizing the overall cost
- maximizing the (theoretical) expected profit from selling overvalued options and buying undervalued ones
- minimizing the amount of future rebalancing that will be required
- etc.

Such goals can be pursued using linear programming techniques.
Designing and evaluating option positions requires use of a software implementation of an option pricing model. Rather than use a canned package like the one that comes with the Hull textbook, we have an Excel spreadsheet with the Black-Scholes option pricing model laid out in a few cells. This allows you to develop your own customized analysis tool.

[Tricks you can do with the Option Calculator Spreadsheet]
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation for Dividend Payout

The original Black-Scholes model applies to options on a non-dividend paying stock. But many underlying assets make cash payouts. It is not difficult to adjust the B-S equation. In addition to dividend-paying stocks, this also leads to option pricing equations for many other underlying assets, including

- stock indexes
- foreign currencies
- futures
- interest rates
- commodities
- and lots of other things
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation for Dividend Payout

Discrete dividends: Suppose the stock will pay a dividend of \$D and it goes ex-dividend on date \( t_{DIV} \) (which is sometime prior to option expiration day).

Replace the stock price \( S \) in the formula by \( S^* \)

\[
S^* = S - D e^{-r t_{DIV}}
\]

Call option value:

\[
C = S^* N \left[ d \right] - X e^{-rT} N \left[ d - \sigma \sqrt{T} \right]
\]

\[
\ln \frac{S^*}{X} + \left( r + \frac{\sigma^2}{2} \right) T
\]

where \( d = \frac{\ln \frac{S^*}{X} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \)

Call delta:

\[
\delta_{CALL} = N \left[ d \right]
\]

Multiple discrete dividends: The adjustment is the same. \( S^* \) is set equal to \( S \) minus the present value of all dividends to be paid over the option's lifetime.
Extending Black-Scholes and Option Replication

Early Exercise of American Calls

An American call should not be exercised early, in theory, except possibly just before the underlying goes ex-dividend (early exercise gives away the option's time value).

- Even if you think the stock is about to go down, sell the option, don't exercise it. (Unfortunately, this works in theory but not in practice! In the real world, the best available bid in the market may be below intrinsic value. Exercise in that case.)

Theory shows that it may be rational to exercise an American call just before ex-dividend day. Exercise if the intrinsic value is more than the value of a European call with the same maturity and strike price, but at a stock price that is below the current price by the amount the stock price will fall when it goes ex-dividend.

- Exercise if \[ S_t - X > C_{EUR}(S_t - \text{div}, X, T-t) \]
  (assuming the stock falls by the full amount of the dividend)

Early exercise is more likely with

- high intrinsic value
- large dividend \[ \text{big ex-dividend price drop} \]
- short time remaining to maturity
- low volatility
- low interest rate \[ \text{low remaining time-value} \]
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation for Dividend Payout

Continuous proportional payout (the Merton Model): Suppose the underlying is like a stock index portfolio that is most easily modeled as paying a dividend flow at a continuous proportional rate $q$. For example, if the annual dividend yield on the S&P 500 Index is currently 2.5%, set $q = 0.025$.

Replace the stock price $S$ in the formula by $S^*$:

$$S^* = S e^{-qT}$$

Call option value:

$$C = S e^{-qT} N\left( d \right) - X e^{-rT} N\left( d - \sigma \sqrt{T} \right)$$

where

$$d = \frac{\ln \frac{S}{X} + \left( r - q + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

Call delta:

$$\delta_{\text{CALL}} = e^{-qT} N\left( d \right)$$

Note the change in delta!
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation to Price Options on Other Assets

European options on other underlying assets can be priced using the suitable variant of the Black-Scholes equation. The formulas are simply the Merton continuous dividend equation with appropriate values for q.

Options on Foreign Currencies (Garman-Kohlhagen Model): The price of the underlying is the exchange rate (in $ per unit of FX). The underlying pays interest at the foreign riskless rate, so set $q = r_{FOR}$. The riskless rate $r$ is the domestic rate.

Replace the stock price $S$ in the formula by $S^*$:

$$S^* = S \, e^{-r_{FOR} \, T}$$

Call option value:

$$C = S \, e^{-r_{FOR} \, T} \, N[ \, d \, ] - X \, e^{-r \, T \, N[ \, d - \sigma \sqrt{T} \, ]}$$

where

$$d = \frac{\ln S/X + \left( r - r_{FOR} + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

Call delta:

$$\delta_{CALL} = e^{-r_{FOR} \, T} \, N[ \, d \, ]$$
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation to Price Options on Other Assets

Options on Futures (Black 1976 Model): The underlying is the futures contract, so $S$ in the equation is the futures price, call it $F$. Since a position is taken in the underlying without any cash having to be invested, the value for $q$ is the riskless interest rate: Set $q = r$.

Replace the stock price $S$ in the formula by the discounted value of the futures price $F$:

$$F e^{-rT}$$

Call option value:

$$C = F e^{-rT} N \left[ d \right] - X e^{-rT} N \left[ d - \sigma \sqrt{T} \right]$$

where

$$d = \frac{\ln F/X + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$$

Call delta:

$$\delta_{\text{CALL}} = e^{-rT} N \left[ d \right]$$
Extending Black-Scholes and Option Replication

Adjusting the Black-Scholes Equation to Price Options on Other Assets

Options on Short Term Interest Rates, or other Noninvestible Assets ("Black Model"): Like futures, cash is not invested in the underlying. The Black futures option model is typically used. The riskless rate \( r \) here is the current short term rate.

Replace the stock price \( S \) in the formula by the discounted value of the interest rate \( R \) that is the underlying for the option (e.g., 3 month LIBOR at option maturity one year from today):

\[
R \ e^{-rT}
\]

Call option value:

\[
C = R \ e^{-rT} \ N[d] - X \ e^{-rT} \ N[d - \sigma \sqrt{T}]
\]

where

\[
d = \frac{\ln R/X + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}
\]

Call delta:

\[
\delta_{\text{CALL}} = e^{-rT} \ N[d]
\]
Adjusting the Black-Scholes Equation to Price Options on Other Assets

Commodity Options:

- Options on commodity futures are priced like other futures options.

- Options on physical commodities like wheat must take account of the carrying costs for the underlying.
  - Fixed costs (e.g., an inspection fee, the cost of transporting the deliverable commodity to the delivery point, etc.) should be handled like negative discrete dividends, that is, $S^*$ equals $S$ plus the present value of the fixed costs.
  - Continuous charges (e.g., daily storage costs) should be treated like negative continuous dividends, that is,

  $$S^* = Se^{qT}$$
Extending Black-Scholes and Option Replication

Adjusting the Put-Call Parity Equation for Payouts

The Put-Call parity equation is altered when the underlying asset makes cash payouts. The price of the underlying, \( S \), is replaced in the put-call parity formula by \( S^* \) as defined above for the particular underlying.

- **discrete dividends:** \[ C - P = S - PV(\text{divs}) - PV(X) \]
- **continuous payout:** \[ C - P = S e^{-qT} - PV(X) \]
- **foreign currencies \( S = \) exchange rate:**
  \[ C - P = S e^{-R_{\text{for}}T} - X e^{-R_{\text{dom}}T} \]
- **futures:**
  \[ C - P = F e^{-rT} - X e^{-rT} \]
  \[ = e^{-rT} ( F - X ) \]
Put Valuation with Payouts

The fair values for put options for each of the above cases come from put-call parity. The put delta is also affected in many cases. (The variable d in these formulas is defined as it is in the appropriate call pricing equation shown earlier.)

Discrete dividends: \[ P = C - S + PV(\text{divs}) + PV(X) \]
\[ \delta_{\text{put}} = N \left[ d \right] - 1 \]

Continuous payout: \[ P = C - S e^{-qT} + PV(X) \]
\[ \delta_{\text{put}} = e^{-qT} N \left[ d \right] - e^{-qT} \]

Foreign currencies (S = exchange rate): \[ P = C - S e^{-R_{\text{for}}T} + X e^{-R_{\text{dom}}T} \]
\[ \delta_{\text{put}} = e^{-R_{\text{for}}T} N \left[ d \right] - e^{-R_{\text{for}}T} \]

Futures: \[ P = C - F e^{-rT} + X e^{-rT} \]
\[ \delta_{\text{put}} = e^{-rT} N \left[ d \right] - e^{-rT} \]
\[ = -e^{-rT} N \left[ -d \right] \]
Options on Futures

Many futures contracts have associated options trading on the same exchange.

Active markets

- Agricultural: Corn, soybeans, wheat, cotton, sugar
- Oil: Crude oil, natural gas
- Gold
- Interest rates: T-Bonds, T-Notes, Euro$, Bund
- S&P 500 and other stock indexes
- VIX volatility index
Unique Features of Futures Options

The underlying is a futures contract
- exercise of a futures call leaves you long a futures contract, that is immediately marked to market
- exercise of a futures put leaves you with a short position in the future

Pricing model
- the "Black 1976" variant of Black-Scholes, with continuous dividend set equal to the riskless interest rate

Simpler delivery process
- no issue of what is cheapest to deliver
- futures options are more liquid and easier to hedge than options on many spot commodities and assets

American exercise
- American futures options (which most are) will be exercised early, both puts and calls. However, the difference in value between American and European options with the same terms is much smaller for futures options than for stock options.
Extending Black-Scholes and Option Replication

Unique Features of OTC Foreign Currency Options

Price quotes are in terms of "vols," i.e., the volatility parameter to be input into the Garman-Kohlhagen pricing equation.

"Moneyness," if expressed in dollar terms, is quoted relative to the forward rate.

- For example, if the spot rate on the Euro is 1.35 and the three month forward is 1.36, a three month "at the money forward" (ATMF) call will be struck at 1.36.

Usually, though, an option's moneyness is defined in terms of its delta.

- A "25-delta" call, will be one whose delta according to the formula is 0.25. A 25-delta put has delta of -0.25.

Quotes (in vol terms) are often given on combination positions:

- Straddle: Long a call and a put with the same maturity, both struck at the money forward.
- Strangle: Long a 25-delta call and a 25-delta put (i.e., both are out of the money).
- Risk Reversal: Long a 25-delta put, short a 25-delta call (also called a "Range Forward" or a "Collar").
Structuring Portfolio Payoffs: Creating a Protective Put Position

Investors particularly like a payoff pattern that resembles a protective put position: unlimited upside potential but a floor on the downside. They will pay a premium for structured products that offer this pattern.
Buying Put Protection with Options on a Specific Portfolio

Securities firms will create and sell options that their customers want to buy, including a put option on a specific portfolio.

Consider buying a protective put on a portfolio that is currently worth $V_0 = 100$ million. The objective is to place a floor of $96$ million on the total portfolio value in 1 year.

The parameters of the problem are:

\[ V_0 = 100 \]
\[ \text{Floor} = 96 \]
\[ \text{Horizon} = 1 \text{ year} \]
\[ \text{Riskless interest} = 8.00\% \]
\[ \text{Portfolio volatility} = 0.20 \]
\[ \text{No dividends (dividends to be received are simply included in the value of the portfolio to be guaranteed)} \]

There are several different ways to set the problem up. It is important to distinguish clearly between changes in the amount of stock in the portfolio from purchases and sales, versus changes in the prices of the stocks. We only want to be protected against price changes.
Buying Put Protection with Options on a Specific Portfolio, p.2

The underlying asset for the put option is the whole portfolio, whose value $V$ is assumed to follow the standard Black-Scholes lognormal diffusion. That is, one put is based on one "share" of portfolio, whose price is the portfolio's current market value. Setting the asset value $V_0 = 100$, and the put strike price equal to the desired floor, $X = \text{Floor} = 96$, the Black-Scholes value for the put is

$$P(V_0, X) = P(100, 96) = \$3.230 \text{ million}$$

Buying this put would guarantee that if the value of our $100$ million portfolio is below $96$ million a year from now, the protective put will make up the difference.

A problem with this solution is that it costs more than we have to invest:

$\text{Cost} = V_0 + P(V_0, X) = 100 + 3.230 = 103.230$. If we sell off some stock to buy the put, we don't have $100$ million invested anymore. A 96 strike put is not so far out of the money relative to our new smaller portfolio, so it will cost more than what we just calculated.

It is not hard to find the solution to this problem by iteration. We will find that if $100$ million is all we can invest, the protective put that gives a floor of 96 will cost $4.506$ million and we will only have $95.494$ left to invest in stocks.
Extending Black-Scholes and Option Replication

Creating the Protective Put Payoff Synthetically with a "Portfolio Insurance" Strategy

A big problem with buying a put on a specific portfolio is that such options are not widely traded. We have to find a securities firm willing to write the puts, and then negotiate an acceptable price. (This used to be a bigger problem than it is today.)

In the late 1970s, two Berkeley finance professors, Hayne Leland and Mark Rubinstein, proposed, and began marketing, an idea they called "Portfolio Insurance." It amounted to a dynamic trading strategy that would replicate the desired protective put payoff, simply by trading between the original portfolio and riskless bonds.

The concept is simple. The basic replicating strategy for any option position is simply to hold an amount of the underlying that will have the same delta as the desired position and to borrow or lend enough of the riskless asset so that the total replicating portfolio has the same cost. THIS IS A KEY POINT TO REMEMBER!

Important: The following two steps give the general procedure to replicate an option:

1. Use the underlying (or some other derivative tied to the underlying, such as a futures contract) to produce the same delta as the option one is trying to replicate.
2. Use riskless borrowing or lending to make the cost of the replicating position equal to the theoretical price of the option.
Creating the Protective Put Payoff Synthetically using "Portfolio Insurance"

We have calculated that for our example, the amount of stock to keep, $V_p$, is

\[ V_p = V_0 - P(V_p, X) = 100 - 4.506 = 95.494 \text{ million} \]

For the purpose of calculating a delta, we treat our stock position $V_p$ as being one "share" of portfolio with a current price of $95.494$ million.

- Using the Black-Scholes model, with $S = 95.494$, the put's delta is $-0.3233$.

- Just as if it were 1 share of stock, the delta of $V_p$ is $1.0$. 
Creating the Protective Put Payoff Synthetically using "Portfolio Insurance," p.2

The replicating portfolio for the put alone is

- short 0.3233 units of Vp: \(-0.3233 \times 95.494 = -30.873 \text{ million of V}_{p}\)
- lend at the riskless rate to make the total out of pocket cost = the price of the put we are trying to replicate, which is 4.506:
- \(\Rightarrow\) lend \(4.506 + 30.873 = 35.379 \text{ million.}\)

The overall portfolio insurance protected portfolio is therefore:

- \(95.494 - 30.873 = 64.621 \text{ million of V}_{p}\) (sell off $35.379 million of the original stocks)
- $35.379 million in riskless bonds

This position will have to be rebalanced regularly to maintain the right delta as stock prices change. This may be hard to do. It requires selling stocks to reduce effective market exposure when stock prices are falling, and buying stocks when prices are rising. (Ask about what happened on October 19, 1987.)
Extending Black-Scholes and Option Replication

A Hybrid Security for Retail Investors: The Equity-Linked CD

An equity-linked CD is a type of certificate of deposit, whose payoff is tied to the stock market, but with downside protection of principal.

Typical structure: The deposit has a maturity of five years, during which time it pays no interest. At maturity, the initial deposit is returned, plus interest equal to the percentage increase in the level of the S&P 500 stock index over the five years. If the S&P goes down, the initial principal is returned in full but there is no additional interest payment.
Designing an Equity-Linked CD

Suppose you have the following data:

- S&P index: 1000
- S&P dividend yield: 2.9%
- S&P volatility: 0.137
- Issuing bank's normal interest rate on a 5 year zero coupon CD: 5.30%

1) Per $100 invested, what does it cost the bank in present value terms to provide this payoff structure?
Extending Black-Scholes and Option Replication

Designing an Equity-Linked CD

To get downside protection of principal, buy a five year zero coupon bond with 100 face value. At 5.30 percent interest,

\[
5\text{-year zero, } \$100 \text{ face value} = \frac{100}{(1.0530)^5} = 77.24
\]

To get the upside price appreciation on the S&P, buy a 5-year call option.

- The underlying is $100 worth of the S&P portfolio. As before, think of this as "one unit of an underlying asset whose current price is 100."
- The call should be at-the-money, so set the strike to 100.

(Note: Because we are doing everything in terms of rates of return, the current level of the S&P is not used in solving this problem.)

\[
5\text{-year at the money call on } \$100 \text{ of the S&P } 500 = 15.39
\]

This leaves an immediate profit to the bank:

\[
\text{Profit} = \text{Deposit} - \text{bond price} - \text{call price} = 100 - 77.24 - 15.39 = 7.36
\]
Designing an Equity-Linked CD

$7.36 per $100 deposited is a high profit rate. Competition will surely drive it down.

Suppose the bank limits its fee to 2 percent of principal, taken out at the beginning. The remainder of the deposit is then used to lock in the minimum payment at maturity and to provide equity market exposure.

One alternative would be for the account to pay a small interest rate on the deposit, in addition to 100 percent of the appreciation of the stock index.

Alternative #1: Full matching of the S&P on the upside plus payment of a fixed interest rate if the S&P falls.

<table>
<thead>
<tr>
<th>Deposit</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>less cost of call option</td>
<td>-15.39</td>
</tr>
<tr>
<td>less bank profit</td>
<td>-2</td>
</tr>
<tr>
<td>equals amount available</td>
<td></td>
</tr>
<tr>
<td>to buy zeros</td>
<td>82.61</td>
</tr>
</tbody>
</table>

Buying $82.61 of 5-year zeros produces $82.61 \times 1.0530^5 = $106.94 at maturity. This corresponds to:

**annual interest rate** = 1.35%
Extending Black-Scholes and Option Replication

Designing an Equity-Linked CD

Alternatively, the bank can limit its fee to 2 percent of principal and use the extra funds to give greater than 100 percent of the appreciation on the index, if it goes up.

**Alternative #2:** Guaranteed return of principal without interest if the market goes down, and more than 100% of the S&P capital gain if it goes up.

<table>
<thead>
<tr>
<th>Deposit</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>less cost of 5-year zeros</td>
<td>-77.24</td>
</tr>
<tr>
<td>less bank profit</td>
<td>- 2</td>
</tr>
<tr>
<td>equals amount available to buy calls</td>
<td>= 20.76</td>
</tr>
</tbody>
</table>

By buying calls on more than $100 worth of the S&P 500, the bank can guarantee return of $100 principal plus

\[
\frac{20.76}{15.39} = 134\% \text{ of the S&P capital gain.}
\]
Some of the most important and most actively used derivatives are those based on interest rates. These allow management of interest rate risk on loans and bonds.

We first review how interest rate futures and forwards work and how to set up hedges with these contracts. We then go on to consider interest rate swaps and to extend the ideas from option valuation to interest rate derivative products with option features such as caps and floors.

The most basic model for interest rate options is the Black model, a variant of Black-Scholes. Unfortunately, the empirical evidence shows that real world interest rate processes are considerably more complicated than simple logarithmic diffusions.

We discuss the problems with the Black model framework and look briefly at alternative models of the interest rate process. As with other instruments, interest rate models come in both equilibrium and arbitrage-free formulations, with pros and cons attached to each.
Interest Rate Derivatives and Interest Rate Models

Important Concepts in this Section

• Types of interest rate derivatives
• LIBOR and the Eurodollar futures contract
• Hedging with FRAs and Eurodollar futures
• Swaps
• The Black model for interest rate options
• How interest rate caps, floors, and collars work
• Problems with the Black model for pricing interest rate derivatives in the real world
• Alternative pricing models for interest-dependent securities
  ° equilibrium models (e.g., Vasicek model)
  ° arbitrage-free models (e.g., the LIBOR Market Model)
Main Types of Interest Rate Derivatives

The underlying is always an interest rate applied to a "notional" principal amount for a specified time period ("tenor").

The simplest interest rate derivatives are basic forward and option contracts, with a single maturity date.

**Forward Rate Agreement (FRA):** A forward rate agreement is a kind of forward contract. A FRA fixes the interest rate to be paid on the notional principal at a specified strike value. The payment period (the tenor) begins on the contract's maturity date. If the market rate on that date is above the strike rate, the long FRA counterparty receives a payment from the short equal to the difference in interest cost between the two rates. If the market rate is lower than the strike rate, the long pays the short the difference in interest.

**Interest rate call (or put) option, "caplet" (or "floorlet"):** Like a FRA, because it has a single maturity date, but the payoff is like an option: If the market rate is above the strike at maturity, the call buyer receives the difference from the writer, but if the market rate is below the strike, the option expires worthless. A "caplet" is a single call option in a cap contract, and a "floorlet" is a single put in a floor.

Examples: If the strike interest rate is 5% on a 3 month FRA or call option with $1 million notional principal and 6 month tenor, and the actual 6-month rate 3 months from today is:

- 6%: The FRA and the call both receive \((0.06 - 0.05)(1/2)(1,000,000) = 5000\)
- 4%: The FRA pays \((0.04 - 0.05)(1/2)(1,000,000) = -5000\); there is no payoff on the call.
**Main Types of Interest Rate Derivatives**

The most important interest rate derivatives involve repeated payments at regular intervals over time. They are like a set of FRAs or options with the same terms and sequential maturities.

**Swap:** A swap consists of a series of FRAs with the same strike rate and periodic maturities (e.g., every 3 months). A swap is useful for turning a loan with a fixed interest rate into one with a floating rate tied to the underlying rate for the swap, or vice versa.

**Cap, Floor:** Like a swap, a cap (floor) contract is a series of interest rate calls (puts) with the same strike and sequential maturities. A cap can be used to place a maximum on the interest rate one has to pay on a floating rate loan, without locking in that rate if the actual market rate turns out to be lower. A floor can be used by a floating rate lender to lock in a minimum rate that will be received.

**Swaption:** An option to enter into a swap at a swap rate equal to the strike of the swaption. A "2 by 5" swaption, is a two-year option to enter into a 5-year swap.
The next several slides review hedging a single future cash flow with interest rate futures or FRAs.

The most basic interest rate derivative is the forward rate agreement (FRA). A FRA fixes the level of some interest rate, such as 90-day LIBOR, to be paid on the notional principal at a specified strike value.

The Eurodollar futures contract is effectively the same thing, except that it is marked to market daily. We will see that setting up a hedge correctly with FRAs can be easy but hedging with Eurodollar futures becomes a little trickier than one might first imagine.
Recall that:

- Like other short term "money market" rates, LIBOR is quoted on a 360 day year. If the quoted rate is 2.00%, interest accrues at the rate of 2.00%/360 per calendar day. Interest is not compounded when the holding period is a year or less.

- At 2.00%, a loan of 100 for 90 days would earn interest of \((90/360) \times (.02) \times 100 = $0.50\). A one year loan would pay \((365/360) \times (.02) \times 100 = $2.028\).

- The Eurodollar futures price is defined by: \(F = (100 – \text{Annualized Forward LIBOR Rate})\), the underlying is 90-day LIBOR, and the notional is $1 million.

- This makes the "dollar value of a basis point" (called DV01) equal $25 per contract. If the Eurodollar futures price goes from 98.20 to 98.25, this corresponds to the annualized forward interest rate falling 5 basis points, from 100 - 98.20 = 1.80% to 1.75%. The long position would get a 5 x $25 = $125 mark-to-market cash inflow. The short would lose $125.

- Eurodollars are NOT Euros. They are deposits in non-U.S. banks that are denominated in dollars. Originally, Eurodollar deposits were at banks in London; now they can be anywhere.
Interest Rate Derivatives and Interest Rate Models

Eurodollar Futures September 2, 2016
Chicago Mercantile Exchange

Underlying instrument
• Special index of 90 day Euro$ deposit rates (LIBOR)

Futures Prices
• Quoted as 100 minus interest rate
• Tick = 0.01 = $25.00 (half ticks are used now because rates are so low, and quarter ticks for near maturities.)

Quantity
• $1 million ("notional principal")

Expiration dates
• Monthly for next 4 months, then every March, June, September, December
• 2 London business days before 3rd Wednesday of the expiration month.
• Contracts currently traded for maturities up to 10 years.

Delivery
• Cash settlement only
• No delivery options
Example: Hedging the Repricing of a Swap Payment with Eurodollar Futures

Suppose your firm is paying a fixed rate of 2.70 percent and receiving 6 month US dollar LIBOR on a $50 million swap. Repricing is every 6 months.

[What is an interest rate swap? A swap is a contract in which periodically (e.g., every 6 months), the two counterparties exchange ("swap") two cash amounts calculated as the interest for that period on a given "notional" principal (e.g., $50 million) using two different interest rates. Generally one rate is fixed and the other is floating (e.g, 2.70% fixed annual rate versus 6 month LIBOR).]

At next March's repricing, the floating rate will be reset to the level of 6 month LIBOR in the market on that date. You want to use Eurodollar futures to hedge the interest rate risk on the swap payment that will be based on that rate.
An important first question

Considering the risk and the instruments involved and how the Eurodollar futures contract works, do you want to buy Eurodollar futures or sell Eurodollar futures?

Is figuring out the answer to this obvious question harder than you might have thought? One way to unravel the complexity is to apply "Figlewski's Rule."
"Figlewski's Rule"

A Rule of Thumb for Avoiding Really Stupid Mistakes in Hedge Design

To avoid selling futures when you really ought to buy them, or buying futures when you really should sell, break the thought process into two parts:

1. **Figure out what you are afraid might happen that will hurt the position you want to hedge.**

Then,

2. **Take a futures position that will make money if what you are afraid of in step 1 actually does happen.**

How does Figlewski's Rule work in this case?
Example: Hedging with Eurodollar Futures, p.3

How many contracts should you trade?

A Eurodollar futures contract = $1 million principal value and the swap notional principal is $50 million. Do you trade 50 contracts?

No. We need "dollar equivalence". Since the repricing interval is 6 months, a 1 basis point change in 6 month LIBOR translates to a dollar change in the floating payment equal to

$$DV01 = 0.0001 \times \frac{180}{360} \times 50,000,000 = 2500$$

A 1 basis point change in the Eurodollar futures price is

$$0.0001 \times \frac{90}{360} \times 1,000,000 = 25 \text{ per contract}$$

To achieve dollar equivalence, so that the futures hedge offsets the change in dollar value of the swap payment, you need to trade

$$\frac{2500}{25} = 100 \text{ contracts}$$
In setting up a simple interest rate hedge, there are three relevant dates:

- today,
- the date on which the cash flow you are trying to hedge will occur,
- and the date on which the uncertainty over that cash flow is resolved.

Dollar equivalence requires that the cash flow on the hedge position should be equal in size and opposite sign, as of the same date. Getting this right when the cash flow and the resolution of uncertainty are on different dates involves present-valuing or future-valuing the cash flow from the hedge to get it to match up at the same time with the cash flow being hedged.

Futures and forwards are basically the same kind of contract, but because futures are marked to market every day, their cash flows begin immediately as soon as the interest rate changes, while a forward contract does not pay until it reaches maturity. (There might be adjustments in the collateral requirements for the FRA, but this doesn't involve cash payments to the counterparty.)

This key difference leads to different hedge design for the two.
Here are futures quotes for the next 8 quarters and the forward interest rates extracted from those futures quotes. The discount function computed from these rates, PV($1), is used for discounting future cash flows.

### Spot interest rate:

\[ r_0 = 5.00\% \]

### Notional:

\[ V = $100,000,000 \]

To compute DV01s for a 1 basis point change in the interest rate, we consider two possibilities: either the rate changes for just one future period and all the others stay the same, or else the whole yield curve moves and all future rates go up a basis point.
Hedging with a FRA

Hedging the quarterly interest payment on a floating rate loan that will occur on date \( t_4 \).

At \( t_4 \) the cash flow will be: (notional) \( \times \) (rate at \( t_3 \)) \( \times \) (interval from \( t_3 \) to \( t_4 \))

At the current forward rate this is: \[ 100,000,000 \times 5.75\% \times 0.25 = \$1,437,500 \]

WHEN IS THE UNCERTAINTY RESOLVED? At \( t_3 \) when the interest rate that determines the size of the interest payment is set. So we need our hedge to mature at \( t_3 \).

Suppose the rate at \( t_3 \) goes up 1 b.p.: \[ 100,000,000 \times 5.76\% \times 0.25 = \$1,440,000 \]

The DV01 as of \( t_4 \) is therefore: \[ \$1,440,000 - \$1,437,500 = \$2500. \]

To offset the risk, hedge with a $100 million FRA that fixes a rate for the period from \( t_3 \) to \( t_4 \). But if the FRA's cash flow occurs at \( t_3 \), the timing of the cash flows doesn't match up.

Real world FRAs are often designed so that a perfect hedge of the interest payment is possible. When date \( t_3 \) arrives, the payoff on the FRA is set equal to the present value of \( (r_{t_3} - s) \), where \( s \) is the strike rate on the FRA. The discounting is done at the \( t_3 \) market rate \( r_{t_3} \). That way the cash flow on the FRA exactly offsets the extra interest above the strike rate that is caused by the realized rate \( r_{t_3} \).
**Hedging with Eurodollar Futures**

Hedging the quarterly interest payment on a floating rate loan that will occur on date $t_4$.

At $t_4$ the cash flow will be: $100,000,000 \times 5.75\% \times 0.25 = 1,437,500$

The uncertainty is resolved at $t_3$ so we use the futures contract that matures at $t_3$ (or immediately after).

The DV01 on the loan payment as of $t_4$ is: $2500$.
The DV01 on a Eurodollar futures contract (as of $t_0$) is: $25$.

If the futures price changes, the futures cash flow begins immediately. To bring the $2500 loan DV01 back to the present, multiply by the $t_4$ discount factor $0.94801$ to get

The DV01 on the loan payment as of $t_0$ is: $2500 \times 0.98401 = 2370$.
The DV01 on a Euro$ future is (as of $t_0$): $25$

Hedge the interest on the $100$ million loan with:

$(2370 / 25) = 94.8 \implies 95$ $t_3$ Eurodollar futures contracts.

The extra discounting needed when hedging with futures is called "tailing the hedge".
What is a Swap?

A Swap is an agreement between two counterparties to exchange periodic cash payments in the future, based on some prespecified formula.

Key features:

- agreement between counterparties: a swap is a kind of over-the-counter derivative;
- cash flows are exchanged: both counterparties have a liability to pay (although only the net difference actually changes hands);
- periodic: a swap normally entails a sequence of future payments.
- under Dodd-Frank regulations, swaps with standard features may be set up OTC, but they now must be cleared through a Central Clearing CounterParty (CCP)

The most common type of swap is a fixed-for-floating interest rate swap.

(Note that in recent years, the word "swap" has come to be used more broadly than this. In some contexts, a "swap" is just another term for a forward contract. Interest rate swaps are as described in the next few slides.)
Example of a Swap

The counterparties A and B agree that every 6 months for the next 3 years, A will pay to B the interest on a "notional" principal amount of $100 million at the fixed rate of 10%. B will simultaneously pay to A the interest on the same notional $100 million at a floating interest rate equal to 6-month LIBOR (as of the beginning of each 6 month period) plus 50 basis points. In practice, the two cash flows are netted and the counterparty with the larger liability simply pays the net difference to the other counterparty.

Important point
The $100 million notional principal never changes hands and is never at risk. Its purpose is only to turn an interest rate into a dollar payment amount.
### Sample Swap Payment Schedule

<table>
<thead>
<tr>
<th>Date</th>
<th>A owes B</th>
<th>LIBOR (%)</th>
<th>B's interest rate</th>
<th>B owes A</th>
<th>Net Payments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>—</td>
<td>8.00</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>t = 6 months</td>
<td>$5 million</td>
<td>8.00</td>
<td>8.50</td>
<td>$4.25 million</td>
<td>$0.75 million</td>
</tr>
<tr>
<td>12 months</td>
<td>$5 million</td>
<td>8.50</td>
<td>8.50</td>
<td>$4.25 million</td>
<td>$0.75 million</td>
</tr>
<tr>
<td>18 months</td>
<td>$5 million</td>
<td>9.00</td>
<td>9.00</td>
<td>$4.50 million</td>
<td>$0.50 million</td>
</tr>
<tr>
<td>24 months</td>
<td>$5 million</td>
<td>9.50</td>
<td>9.50</td>
<td>$4.75 million</td>
<td>$0.25 million</td>
</tr>
<tr>
<td>30 months</td>
<td>$5 million</td>
<td>9.75</td>
<td>10.00</td>
<td>$5.00 million</td>
<td>—</td>
</tr>
<tr>
<td>36 months</td>
<td>$5 million</td>
<td>—</td>
<td>10.25</td>
<td>$5.125 million</td>
<td>—</td>
</tr>
</tbody>
</table>
Why Swap?

A swap often seems to offer both counterparties lower borrowing costs than are available to them otherwise.

Example: Suppose the following are the normal borrowing costs for A and B.

<table>
<thead>
<tr>
<th>Floating rate</th>
<th>3 year fixed rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>A - LIBOR + 120 b.p.</td>
<td>A - 11.0%</td>
</tr>
<tr>
<td>B - LIBOR + 100 b.p.</td>
<td>B - 10.0%</td>
</tr>
</tbody>
</table>

Firm A would like to borrow for three years at a fixed interest rate. The market would charge a firm with A's credit quality 11% to do this.

Firm B would like to borrow for three years at a floating rate. The market rate for B would be LIBOR + 100 basis points.

But if A borrows in the market at the floating rate and B borrows at the fixed rate, and they then enter into a swap with each other in which A pays 10.0% fixed rate to B while B pays LIBOR + 50 b.p. to A, they both can reduce their overall borrowing costs.
EFFECTIVE BORROWING COSTS BEFORE THE SWAP

A borrows fixed and B borrows floating

Floating Rate Market

B pays LIBOR + 100 b.p.

Fixed Rate Market

A pays 11% fixed rate
How does this work?

A takes out a 3 year $100 million floating rate loan at LIBOR + 120 b.p. and enters into a pay-fixed-receive-floating interest rate swap with B.

- The swap payments received from B will cover LIBOR plus 50 b.p.. A adds an extra 70 b.p. and pays LIBOR plus 120 b.p. to its lender, and 10.0 percent to B.
- This effectively turns the floating rate loan into a fixed rate loan, with a total interest cost equal to 10.70 percent.

B takes out a fixed rate 3 year $100 million loan at 10.0 percent and enters into a pay-floating-receive-fixed interest rate swap with A.

- The 10.0% fixed swap payments from A cover B's interest payments to the market.
- The swap turns the fixed rate loan into a floating rate loan at LIBOR + 50 b.p. (paid to A).

A saves 30 b.p. and B saves 50 b.p. in borrowing costs on 3 year financing.
EFFECTIVE BORROWING COSTS AFTER THE SWAP

A borrows floating, B borrows fixed, and they swap

Floating Rate Market

A pays LIBOR + 120 b.p.

(In LIBOR + 50 b.p. comes from B; A adds 70 b.p.)

In the swap, A pays 10% fixed rate to B

Fixed Rate Market

A

B

In the swap, B pays LIBOR + 50 b.p. to A

B pays 10% fixed rate received from A

The result of swapping is that, effectively, A pays 10.70 percent fixed rate and B pays LIBOR + 50 b.p.
Are Swaps a Free Lunch?

How can a swap offer a profit to both counterparties? The most logical explanation focuses on the fact that, unlike a normal loan, the notional principal is never at risk.

An ordinary loan carries a default premium because the borrower may default and not repay the principal. The premium for a long term loan is greater than for a short term loan, and the difference is bigger the less creditworthy is the borrower.

The more creditworthy counterparty therefore has a comparative advantage borrowing at a long maturity and the less creditworthy counterparty has a comparative advantage (that is, a smaller disadvantage) borrowing at a short maturity.

In a swap, neither counterparty pays a premium for the risk that they will default on the principal amount, so swapping allows them to exploit each one's comparative advantage and divide the net improvement in borrowing terms between them.
Pricing a Swap

The Swap Rate: Like a forward price, the "swap rate" in the market is the fixed interest rate for a given maturity at which a swap against LIBOR can be set up such that neither counterparty has to make a payment to the other at the beginning.

How is the swap rate determined? The future cash flows on the swap can be thought of in two equivalent ways:

1. A swap is a series of forward contracts (forward rate agreements, actually).
   - The "strike price" is the fixed interest rate and the "underlying" is the floating rate. In our example, the swap between A and B is like 6 forwards with sequential maturities; the strike on each one is the $5 million fixed rate payment and the underlying is the floating market value of 1/2 year's interest on $100 million computed as market LIBOR 6 months before the payment date, plus 50 b.p.

2. A swap is like being long one bond and short another with the same face value.
   - Paying fixed and receiving floating gives the same cash flows as being long a floating rate bond and being short (i.e., issuing) a fixed rate bond. In our example, A's position is the same as issuing a $100 million face value 10 percent coupon 3 year bond and using the proceeds to buy $100 million of bonds paying a floating interest rate of LIBOR plus 50 b.p. B's position is the reverse, long the fixed rate bond and short the floating rate bond.
Pricing a Swap

Compare a swap to pay a fixed rate $C$ and receive LIBOR on future dates $t_1$, $t_2$, $t_3$ with the payments on a series of FRAs all struck at rate $C$, and with payments on the bond portfolio that is long a date $t_3$ floating rate bond paying LIBOR and short a fixed rate bond with coupon rate $C$.

<table>
<thead>
<tr>
<th>Fixed for Floating Swap</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C$</td>
<td>$C$</td>
</tr>
<tr>
<td></td>
<td>LIBOR($t_0$)</td>
<td>LIBOR($t_1$)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_1$ FRA</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C$</td>
<td></td>
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<tr>
<td></td>
<td>LIBOR($t_0$)</td>
<td>$C$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_2$ FRA</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LIBOR($t_1$)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t_3$ FRA</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
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</table>

<table>
<thead>
<tr>
<th>Long floating rate bond</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>LIBOR($t_0$)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Short fixed rate bond</th>
<th>Pay</th>
<th>Receive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$C$</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
The Value of a Swap
Like both of these positions, the value of a swap struck at the market swap rate is zero initially. It will become positive or negative as market interest rates move. A rise in the floating rate in the market increases the value of the swap for the fixed rate payer (counterparty A), and lowers it for the receiver of the fixed rate (counterparty B). The change in value is easy to compute from the effect on the two bond values.

Risk Exposures
The exposure to interest rate risk, as measured by duration, convexity, DV01, and other sensitivities can be obtained easily by considering the equivalent bond or position in forwards.
Variations on the Theme

The concept of swapping is very powerful and many new types of swaps and related contracts have become commonplace.

- **Currency swaps** (e.g., dollars vs. Euro)
- **Asset swap** (exchange payments on some asset against a floating riskless rate)
- **Amortizing swaps** (notional principal varies over time, like a mortgage)
- **Yield spread swaps**
  - basis swaps (e.g., T-bill rate vs. LIBOR)
  - yield curve swaps (e.g., 10 year rate vs. 3 month)
  - diff swaps (US $ interest vs. Euro-zone interest)
- **Equity swaps** (e.g., S&P return vs. fixed rate)
- **Commodity swaps** (e.g., oil prices vs fixed rate)
- **Total return swap** (can transform any instrument into a different one)
Adjusting the Black-Scholes Equation to Price Interest Rate Options

The underlying is an interest rate so, like a futures contract, one does not invest cash to "buy and hold" a position in it. The basic valuation model is a modified version of the "Black '76" futures option model, so it is known as the Black model.

One important issue is that there is no longer a riskless interest rate. Still, discounting a cash flow from option maturity at the (stochastic) interest rate is easily accomplished: simply multiply it by the price of a zero coupon bond maturing on that date. This makes use of today's price of a traded security to capture all of the uncertainty about the future course of the discount rate. Because the zero coupon bond price can simply be observed in the market, the stochastic behavior of the discount rate does not have to be modeled at all.

This is an example of an extremely useful and powerful technique for derivatives valuation, known as a "change of numeraire." Here, the interest rate option's payoff is in dollars at option expiration, but we change what we are thinking of as the unit of account--the numeraire--from "dollars on date T in the future " (which we don't know how to present value) to "date T maturity zero coupon bonds" (which the market is present valuing for us).

When it is possible, a suitable change of numeraire reduces the number of random factors that have to be dealt with and can significantly simplify option pricing problems.
The Black Model for Interest Rate Options

The underlying rate, call it $R$, is the value of a specified interest rate as of option maturity date $T$, for example, 3-month LIBOR. The assumption is that today's forward rate for date $T$, call it $F$, is the expected value as of today (time 0) of $R$ on date $T$, that is, $F = E_0[R_T]$.

The probability distribution for $R$ on date $T$ is assumed to be lognormal with expected value $F$ and standard deviation $\sigma \sqrt{T}$.

Discounting in the formula at a stochastic "riskless" rate is handled by replacing $e^{-rT}$ by $B(0,T)$, the time 0 market price for a zero coupon bond maturing at date $T$.

Call option value: 

$$C = B(0,T) \left( F N \left[ d \right] - X N \left[ d - \sigma \sqrt{T} \right] \right)$$

where 

$$d = \frac{\ln \frac{F}{X} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}$$

Call delta: 

$$\delta_{\text{CALL}} = B(0,T) N \left[ d \right]$$
The Black Model for Interest Rate Puts

The formulas for interest rate put options come directly from put-call parity (which is also modified when the underlying is an interest rate).

Put-Call Parity: \[ C - P = B(0, T)(F - X) \]

Put option value: \[ P = B(0, T) \left( -F N[-d] + X N[-d + \sigma\sqrt{T}] \right) \]

where \[ d = \frac{\ln(F/X) + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}} \]

Put delta: \[ \delta_{PUT} = -B(0, T) N[-d] \]

(These equations make use of the property that the normal distribution is symmetric, so \[ 1 - N[d] = N[-d]. \])
Interest Rate Options and Interest Rate Models

Problems with the Black Model

• The "riskless" interest rate changes randomly and it is correlated with changes in the price of the underlying.

• Mean reversion in interest rates; they are not a random walk

• The theoretical term structure of interest produced by the model is not consistent with the current yield curve observed in the market

• Because there is only one source of risk, some commonly observed interest rate behavior is inconsistent with the model
  ○ twists in the yield curve (e.g., short rates rise and long rates fall at the same time)
  ○ changes in curvature of the term structure

• Forward rate volatility is different for different maturities
Valuation Models for Interest-Dependent Securities

A large number of theoretical pricing models have been developed for bonds and other interest-dependent securities, and derivatives based on them. These are some of the most complex valuation models around.

As is the case for other kinds of derivatives, there are two basic types of models: "equilibrium" models and "arbitrage-free" models. Both try to model the evolution of interest rates, from which pricing equations for particular securities can be derived.

In other words, all interest-dependent securities are treated as derivatives based on the underlying interest rates.
Interest Rate Options and Interest Rate Models

Valuation Models for Interest-Dependent Securities

Equilibrium interest rate models

- the short term interest rate is assumed to follow a stochastic process with plausible features (as in the Black-Scholes and Black '76 models)

- long term rates (the term structure) are derived as functions of the short rate process

- the short rate may depend on several random factors; typically, it is assumed to revert towards a long term mean and have stochastic volatility, both of which might follow their own stochastic processes

- but, model values for actual bonds need not match their current prices in the market

- An example: the Vasicek model: \[ dr = K (\mu - r) \, dt + \sigma \, dz \]

  - change in rate
  - speed of adjustment
  - long term mean
  - volatility
Arbitrage-free Valuation Models for Interest-Dependent Securities

• Traders do not like pricing models that say current market prices are wrong (i.e., that there are arbitrage opportunities available among existing bonds)

• Arbitrage-free models take the current term structure of interest rates in the market as an input and derive interest rate processes that are consistent with it.

• The **Ho-Lee model** was the first arbitrage-free model of the term structure. It was essentially a binomial model which assumed that over the next time step, the entire yield curve could move to just one of two possible new shapes. It was a theoretical breakthrough, but it was obviously limited in how the term structure could behave, and had the unfortunate feature that interest rates could go negative in the model.

• The **Heath-Jarrow-Morton (HJM)** family of models eliminated the problems of the Ho-Lee formulation. These model the evolution of the whole market yield curve of forward interest rates in such a way that there are no arbitrage opportunities either in the current structure of interest rates or in future rates that are possible within the model.

• HJM is mathematically elegant, but quite hard to use in practice.
Interest Rate Options and Interest Rate Models

Arbitrage-free Valuation Models for Interest-Dependent Securities

Interest rate products may be valued and risk-managed in practice using an equilibrium-type model that is adapted to incorporate the current term structure.

One model that does this is Hull and White's "Extended Vasicek" model:

\[
dr = K (\theta(t) - r) \, dt + \sigma \, dz,
\]

where \(\theta(t)\) specifies an expected drift function for the short interest rate that can be tweaked to produce expected values for future short rates that are consistent with the current term structure observed in the market.

\(\theta(t)\) is calibrated to current bond prices.

Unfortunately, volatility is a fixed parameter, so the model can match market prices for bonds, but not for interest rate options, and also the rate can go negative.
Arbitrage-free Valuation Models for Interest-Dependent Securities

A way to extend the Extended Vasicek model, that is common in practice, is the **Black-Karasinski** model.

\[
d \log(r) = ( \theta(t) - a(t) \log(r) ) \, dt + \sigma(t) \, dz,
\]

One key difference from the Extended Vasicek model is that this model is written in terms of the log of the interest rate, so the rate can't go negative. Another is that volatility and the speed of mean reversion are now allowed to vary over time. This makes it possible to calibrate the model to both the current yield curve and also the current volatility surface in the market, even though there is still only one source of risk in the model, dz.

The fact that short term interest rates in a number of countries are now (Spring 2017) negative causes lots of problems for many interest rate models, like this one.
Interest Rate Options and Interest Rate Models

Arbitrage-free Valuation Models for Interest-Dependent Securities

**LIBOR Market Model:** The Brace, Gatarek, and Musiela (BGM) model has nearly become the industry standard at this point. The model focuses on the short interest rate at each one of a set of relevant future dates, in particular the future repricing dates \( \{t_1, t_2, \ldots, t_M\} \) for a given swap.

\[
df_m = \sigma_m(t) f_m dZ_m
\]

where

- \( f_m \) is the forward rate as of date \( t \) for the future period beginning at date \( t_m \)
- \( \sigma_m(t) \) is the volatility of \( f_m \) as of date \( t \)
- \( dZ_m \) is the \( m \)-th element of an \( M \)-dimensional vector of Brownian motions, with mean vector 0 and covariance matrix \( \Omega \)

The key assumption is that each rate follows its own lognormal diffusion process--\( M \) sources of risk for a swap with \( M \) payment dates--but they must satisfy several constraints:

- The current rates are consistent with the observed forward rates in the market
- The expected drift of each forward rate is zero. The forward rate is the expected value of the future spot rate—the Expectations Model holds. This imposes a consistency condition between the volatilities and the drifts for the LIBORs in the model.

Even so, there can be a lot of parameters to calibrate and stochastic variables to simulate in a Monte Carlo analysis. Variants of the BGM model impose further constraints to reduce the computational burden.
Bonds and Mortgages

Bond Options
Most interest rate derivatives like swaps, caps, and floors are based on rates. But there are also option contracts written on bonds and bond futures, as well as a variety of optional features, like callability or convertibility that may be embedded in the bonds themselves.

Bond Option Contracts
- exchange-traded and over the counter contracts
- options on specific Treasury bonds are traded over-the-counter by government bond dealers
- options on T-Bond futures are traded at the Chicago Board of Trade.

Embedded Options
- callable bonds
- mortgage prepayment option (the borrower's right to repay a mortgage early is a kind of call option)
- convertible bonds (some corporate bonds can be exchanged for shares in the issuing firm at the bondholder's option)
Callability in Bonds and Mortgages

Callable Bonds

Yields on callable bonds are evaluated in terms of the "Option-Adjusted Spread".

To compute the Option-Adjusted Spread (OAS), first value all embedded options and subtract the total value of the optionality from the bond's market price. Compute the yield to maturity on this option-free price. The spread relative to the yield to maturity for the comparable maturity Treasury bond is the OAS.

Example: PDQ Corporation has an outstanding bond issue with the following terms:

- Maturity: 11 years
- Coupon: 7.30 percent
- Face value: 100
- Callable at a price of 103, beginning in year 5
- Current market price: 92.00
- Quoted yield (at P = 92.00): 8.43 percent
- Yield on 11 year Treasuries: 6.20 percent

Suppose value of the call feature is estimated to be 2.10 per $100 face value. That is, if it were not callable the same bond would be expected to sell at a price of \(92.00 + 2.10 = 94.10\).

Option adjusted yield (at P = 94.10) = 8.12 percent

Option-adjusted spread (OAS) = 8.12 - 6.20 = 1.92 percent
Mortgages

Mortgages and Mortgage-Backed Securities

In the U.S., a mortgage loan is like a long term bond, except

- it is collateralized by the value of the house
- the principal is gradually repaid over the whole life of the loan, rather than in a single lump sum payment at maturity.

There is an enormous total volume of mortgage debt outstanding: over $13.5 trillion by 2015.

Problems with mortgages

- Mortgage loans are illiquid (Loans are small, costly to service, and closely tied to the value of the property and specific characteristics of the borrower.)
- The homeowner always has the right to pay off the loan early. This makes mortgage loans effectively callable at any time (which is a bother for the lender).

Mortgages are often pooled and pass-through securities, like GNMA's, and other mortgage-backed securities (MBS), are issued against the pool. These represent a different and extremely important new class of derivative instrument. By the end of 2008, more than $7.5 trillion of the outstanding mortgage loans were held in pools underlying mortgage-backed securities. Since the market meltdown in 2008, securitization of mortgages has gone way down. By mid-2012 only $2.9 trillion was outstanding.
Prepayment Risk

The future cash flows from a pool of mortgage loans depend on the prepayment experience. Prepayment is hard to predict because it will depend on whether interest rates go up or down in the future and also on "noneconomic" factors.

Noneconomic factors include

- people move
- borrowers default
- transactions costs affect the refinancing decision
- "nonrational" reasons, such as lack of information, may cause suboptimal prepayment behavior

New types of derivatives were created to manage the impact of prepayment risk.
Prepayment Risk and Valuation

Prepayments depend on the path taken by interest rates, so the value of the option in a mortgage-backed security becomes **path-dependent**.

- Prepayments for economic reasons (refinancing to get a lower interest rate) increase when market interest rates fall below the rate on the mortgage
- This effect is strongest the first time market rates fall to a new level; if they then bounce up, the next time they fall to the same level there will be fewer prepayments because the most interest-sensitive borrowers will have already prepaid. This is called "**burnout**."

Therefore to price the mortgage or mortgage-backed security properly, you need to know not just the current interest rate, but the entire past history of rates since the security was issued.

Path dependence requires **simulation methods** (e.g., "**Monte Carlo simulation**") to compute theoretical values for mortgage-backed securities.
Mortgage-Backed Securities

Mortgage Loans

Banks make individual mortgage loans.

- Expertise in evaluating property in the local market
- Low default risk, because the house is collateral
- High servicing requirements for the loans produce income for the lender (i.e., higher interest rate than on a bond with comparable risk)

**Individual Mortgage Loans**

But mortgage loans are illiquid and prepayment risk is hard to manage well.
Mortgage-Backed Securities

Mortgage Pass-Throughs

Following the "Credit Crunch" of 1966, the Government National Mortgage Association (GNMA, known as "Ginnie Mae") was created in 1970 to provide a new mechanism for financing mortgage loans.

- The new financing idea is known as securitization.
- The new financial instrument was the mortgage pass-through security.

Mortgage lenders "originate" loans (they set them up). Once a number of similar mortgage loans have been made, they are bundled together into a mortgage pool and GNMA pass-through securities are issued and sold in the securities market.

A mortgage pass-through is similar to a bond. Each month, as the homeowners make their mortgage payments,

- the originating bank retains a fee for servicing the loan
- the rest of the funds are passed through to the holders of the pass-through securities
- to be eligible for inclusion in a GNMA pool, a mortgage loan must be insured by the government (e.g., the Veteran's Administration). There is no risk to the lender from a borrower default.
- prepayments of principal from loans paid off early by borrowers or from government payoffs of defaulted loans are also passed through. This makes monthly cash flow irregular and hard to predict.
Mortgage Pass-Throughs

Cash flows from the mortgage pool are allocated proportionally to the pass-through securities.

The GNMA pass-through revolutionized the mortgage market. Funding of home loans no longer depended on the ability of savings banks and S&Ls (savings and loan institutions) to attract deposits: funds could be obtained as needed in the bond market.

But prepayments still create substantial risk for the investor.

- Cash flows are less predictable than with regular bonds
- Prepayments increase when interest rates are low, to the disadvantage of the lender
Collateralized Mortgage Obligations (CMOs)

Before long further innovations were introduced. More complex structures called CMOs were introduced, with different "tranches" (French for "slice") of securities.

Each month when the homeowners make their mortgage payments

- holders of the "A" tranche get their promised payments first--these are extremely predictable
- then "B" tranche holders are paid from the remaining funds, and so on
- uncertainty in the total cash flows from the pool due to prepayment risk is concentrated in the lower priority classes

Cash flows from mortgage pool are allocated to different priority classes (tranches) of mortgage-backed securities

Mortgage Pool

A Tranche
B Tranche
C Tranche
Z Tranche
"toxic waste"
"waterfall"
Mortgage-Backed Securities

Pricing MBS using Monte Carlo Simulation

The payoffs on basic forwards, futures and European options depend on the price of the underlying asset at the expiration of the contract. It doesn't matter how the price gets from today's value to the final value. The payoff is independent of the path.

American options are different because you might want to exercise early, depending on where the asset price is on one or more dates before expiration. The option is path-dependent, but only in a limited way. At each date you can decide whether to exercise or not, but that decision will depend only on how the price might evolve from that date forward. It doesn't matter how it got to its present level.

Mortgages and mortgage-backed securities are path-dependent in a more complicated way. The timing and amounts of their payoffs are a function of the prepayment experience on the underlying mortgage loans. It does matter what path interest rates have followed in getting to today's level. That makes it impossible to price MBS with the standard backward recursion technology we have considered so far.
Consider an MBS that is backed by a pool of mortgage loans with a 6% fixed interest rate. We may know that the rate in the market is 6.5% today, but that's not enough to value the MBS. Its future cash flows will be affected by how many of the underlying mortgages have already prepaid, and that depends on how interest rates have moved before now.

If the rate has remained above 6% since the pool was formed, there may have been comparatively few prepayments. But if rates have dropped to 4% and then gone back up to today's 6.5%, many of the original borrowers will have prepaid their 6% loans and refinanced at lower rates. The number of mortgages left in the pool will be much lower.

Path-dependent securities like MBS have to be priced approximately, by Monte Carlo simulation.
Monte Carlo simulation starts with a model, like one of the interest models we have looked at, for how the underlying risk factor(s) behave.

Interest rate model: \[ dr = \kappa (\mu - r) \, dt + \sigma \, dz \]

Discretized version: \[ r_{t+1} - r_t = \kappa (\mu - r_t) \, \Delta t + \sigma \, z_t \sqrt{\Delta t} \]

A possible path for future interest rates \( \{r_1, r_2, ..., r_T\} \) is simulated by drawing random numbers from a standard normal distribution (or whatever distribution one thinks is most appropriate) and plugging them into the equations for the stochastic \( \{z_t\} \) terms. Once a full path of rates from the present to maturity is generated, a prepayment model is used to compute expected prepayments from the pool along that interest rate path.

\[ V_i = V(\{r_1, r_2, ..., r_T\}_i, \text{prepayments}) \]

The present value of the simulated cash flows along the \( i \)th simulated interest rate path gives one observation, \( V_i \), for the possible value of the MBS consistent with the interest rate model and the assumed prepayment behavior.
This process is then repeated, maybe $N = 100,000$ times. In the end, one has 100,000 possible outcomes that are consistent with the interest rate process and the prepayment model. These are used to compute the mean, standard deviation, and other necessary statistics for the returns distribution, from which the fair value $V$ and risk parameters are obtained.

$$V = \text{mean}( \{ V_i, i = 1, \ldots, N \} )$$

Monte Carlo simulation is heavily used for solving models in:

- weather forecasting
- nuclear weapons design
- financial derivatives
Up to this point, our focus has been on hedging and managing exposure to the risk of adverse fluctuations in the market values of our assets or liabilities. But for many financial institutions like banks, the risk that worries them most is not that the present value of the repayments on a loan will go down, but that a borrower will not repay the loan at all, i.e., credit risk.

An early insight from modern option valuation technology was that securities issued by a corporation, like stocks and bonds, are actually derivatives whose values are derived from the value of the underlying firm. Option pricing theory can help us understand the risk of bankruptcy embedded in corporate securities.

Thinking of the stock as an option on the firm can help us value risky debt and also clarifies how limited liability gives shareholders the incentive to increase risk exposure at the firm level. The current "structural" and "reduced form" approaches to evaluating credit risk have developed from this insight.

Finally we take a look at two important new types of derivative instruments that have been developed specifically for managing credit risk: credit default swaps (CDS) and collateralized debt obligations (CDOs).
Credit Risk: Default as an Option and Credit Derivatives

Models of Default Risk

There are two basic derivatives-related approaches to analyzing and valuing risky debt (i.e. debt with a risk of default): "Structural" models (based on ideas first published in a paper by Merton in 1974), and "Reduced Form" models.

Structural Models

• The true underlying asset is the whole firm, with current value $V_t$.

• $V_t$ follows a Black-Scholes type diffusion process

• Bonds with face value $F$, maturing at date $T$, are paid off if $V_T > F$

• If $V_T < F$, the firm defaults and the bondholders take over the firm (they get $V_T$)

• This makes equity a kind of call option on the underlying firm value $V$
The ability to default is an option

Consider a firm with a very basic capital structure. It has issued stock and a single zero coupon bond.

Let

\[ V = \text{Value of the entire firm.} \]
\[ F = \text{Face value of bonds to be paid off at date T.} \]

\( V \) is assumed to follow the standard lognormal diffusion process

\[
\frac{dV}{V} = \mu \, dt + \sigma \, dz
\]

Consider the payoffs on the different securities on date T as a function of the firm value on that date, \( V_T \).
Default in a Structural Model

Value of Firm

$V_0$

Face value of debt $F$

Firm defaults here

Bond maturity date

$t = T$
### Credit Risk: Default as an Option and Credit Derivatives

#### Security Payoffs at Bond Maturity, Date $T$

<table>
<thead>
<tr>
<th>$V_T &lt; F$</th>
<th>$F &lt; V_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. STOCK</strong></td>
<td><strong>V_T - F</strong></td>
</tr>
<tr>
<td>Firm is bankrupt</td>
<td>Firm is solvent. Bonds are paid in full.</td>
</tr>
<tr>
<td>0</td>
<td>$V_T - F$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F &lt; V_T$</th>
<th>$F &lt; V_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>2. BONDS</strong></td>
<td><strong>F</strong></td>
</tr>
<tr>
<td>Firm is bankrupt. Assets distributed to bondholders</td>
<td>Firm is solvent. Bonds are paid off at face value.</td>
</tr>
<tr>
<td>$V_T$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

**Note:** The bond payoff is the same as (yet another example of put-call parity!)

Default-free bond

<table>
<thead>
<tr>
<th>$F$</th>
<th>$F$</th>
</tr>
</thead>
</table>

Writing a put option on the firm value $V$ with strike price $F$

<table>
<thead>
<tr>
<th>$- (F - V_T)$</th>
<th>$0$</th>
</tr>
</thead>
</table>

| $V_T$ | $F$ |
Implications of Modeling Stock and Bonds as Contingent Claims

The stock is an option on the firm and option value is enhanced by higher volatility.

- The shareholders have an incentive to increase firm risk. (This shifts value from the bondholders to the stockholders.)
- One way to increase risk is to distribute firm assets to the shareholders (paying dividends); this is typically restricted by covenants in the bond indenture agreement.
- But sometimes other opportunities arise to transfer firm value from the bondholders to the stockholders, e.g., RJR Nabisco in the late 1980s (see the book and movie "Barbarians at the Gates")

This approach is one of the major ways of addressing credit risk on bonds within the standard contingent claims valuation paradigm.

- KMV (now Moody's KMV) is major firm currently doing credit analysis based largely on the structural framework.
Problems with Structural Models of Default Risk

Firm value is hard to determine for real-world firms

Actual debt contracts are much more complicated than what Merton models
  • "bonds" are coupon debt, with many small periodic payments before maturity, call provisions and other special features
  • firms may have issued multiple classes of debt with different maturities and other terms (senior debt, junior debt, bank loans, lines of credit, commercial paper, etc., etc.)
  • priority rules for who gets paid off first are often violated in real world bankruptcy proceedings
  • it is difficult to model the firm's optimal default strategy

The structural approach is little help in valuing many popular kinds of credit derivatives whose payoffs are tied to a change in bond rating or yield spread, not to actual default
Reduced Form Models of Default Risk

The other major class of default models does not try to look carefully inside the corporation, but just focuses on overall probabilities. These are known as "reduced form" models.

- Default is a random event, like a lightning strike, that could happen to any firm at any time.

- Default is modeled as a Poisson process (a probability model for infrequently occurring big events; almost the exact opposite of a diffusion process).

- Default intensity (the probability of a default within a given span of time) can be modeled as exogenous, or as a function of firm value and other variables.

- Payoff on bonds if default occurs is not $V_T$, but some exogenously specified recovery rate (empirical recovery rates are quite variable, but average around 40 - 50%).
Credit Risk: Default as an Option and Credit Derivatives

Moody's KMV Historical Credit Ratings Transition Matrix

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>From/To:</td>
<td>Aaa</td>
</tr>
<tr>
<td>Aaa</td>
<td>87.480%</td>
</tr>
<tr>
<td>Aa</td>
<td>0.833%</td>
</tr>
<tr>
<td>A</td>
<td>0.056%</td>
</tr>
<tr>
<td>Baa</td>
<td>0.036%</td>
</tr>
<tr>
<td>Ba</td>
<td>0.006%</td>
</tr>
<tr>
<td>B</td>
<td>0.008%</td>
</tr>
<tr>
<td>Caa</td>
<td>0.000%</td>
</tr>
<tr>
<td>Ca-C</td>
<td>0.000%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>From/To:</td>
<td>Aaa</td>
</tr>
<tr>
<td>Aaa</td>
<td>52.910%</td>
</tr>
<tr>
<td>Aa</td>
<td>2.263%</td>
</tr>
<tr>
<td>A</td>
<td>0.202%</td>
</tr>
<tr>
<td>Baa</td>
<td>0.177%</td>
</tr>
<tr>
<td>Ba</td>
<td>0.035%</td>
</tr>
<tr>
<td>B</td>
<td>0.026%</td>
</tr>
<tr>
<td>Caa</td>
<td>0.000%</td>
</tr>
<tr>
<td>Ca-C</td>
<td>0.000%</td>
</tr>
</tbody>
</table>

Source: Moody's KMV. "Default and Recovery Rates of Corporate Bond Issuers, 1920-2015."
Beginning in the mid-1990s, derivatives based on credit risk began to appear. The simplest is the **credit default swap (CDS)**, a derivative contract that functions very much like an insurance policy against default by a bond issuer.

The **collateralized debt obligation (CDO)** uses securitization and tranching, as in mortgage-backed securities, to reallocate the incidence of the default risk in a portfolio of risky bonds. An MBS concentrates prepayment risk into a small fraction of the new securities, leaving the rest with virtually none. In the same way, a CDO concentrates default risk into a small fraction of the CDO tranche securities, leaving most of the tranches at AAA quality or above, even when the underlying bonds in the pool are much riskier.

Unfortunately, valuation models for credit derivatives are complicated and hard to test empirically because defaults are such rare events (luckily!). Pricing in the real world does not seem to be entirely consistent with the theoretical models.
Credit Default Swaps

A CDS is a derivative contract based on default risk. Like a futures contract, a CDS transfers risk from the owner of the risky security to the derivatives counterparty.

How it works (originally):
Counterparty A (the protection buyer) commits to make regular premium payments to Counterparty B. The rate is $S$ basis points per year, applied to a notional face value $F$.

Counterparty B (the protection seller) commits to make the following payments to A:

- If there is no default by the "reference entity" (the bond issuer) before the CDS expires, B pays nothing.
- If the reference entity defaults, B must compensate A's loss. Either
  - A delivers bonds issued by the defaulting entity, and B pays A their face value $F$, or
  - B pays a cash amount to A, equal to the difference between the post-default price of the bonds and face value $F$. The price is determined in a special auction 1 month after the credit event.

Market conventions changed in 2009: The premium payments are now standardized, so most CDS pay 100 basis points annually (high risk CDS pay 500 b.p.), and an upfront payment adjusts for the precise level of default risk. If equilibrium spread $S$ for a particular issuer would be below 100 b.p., the protection seller will be receiving too much premium, so the seller pays the buyer the fair value of the difference up front. If equilibrium $S$ would be above 100 b.p., the buyer pays the seller up front. Maturity dates are also standardized. Both of these changes are big improvements to liquidity in the secondary market for CDS.
Credit Risk: Default as an Option and Credit Derivatives

An Extremely Important Innovation: The Credit Default Swap

For many institutional investors, market price risk is much less important than risk of default. The CDS is a way to buy (or sell) insurance against default.

The protection buyer pays a regular quarterly premium to the protection seller. If there is a default, the protection seller must pay the protection buyer. The amount of compensation that is determined at a special auction of the defaulted bonds about a month later.
An Extremely Important Innovation: The Credit Default Swap

In the previous example, the Total Alpha Fund bought protection to eliminate the default risk on XYZ Corp. bonds, to produce a portfolio with credit quality better than AAA.

Alternatively, Total Alpha could hold a portfolio of Treasury bonds and sell protection on XYZ. It would then receive regular premium payments from its counterparty, but would bear the risk of loss if a default occurs. This way Total Alpha earns the return, including risk premium, on risky XYZ bonds (and bears the default risk) without actually owning them.
Credit Risk: Default as an Option and Credit Derivatives

CDS Pricing

The equilibrium CDS spread is the value of the spread $S$ that sets the (risk neutral) expected values of the "premium leg" and the "protection leg" equal. For a $T$ period CDS,

Let

$\pi_i = \text{probability of default in period } i$, for a bond that starts at time 0

$D_i = \text{the discount factor for premium payment } i$

$\theta_i = \text{the length of the payment period}$

$R = \text{fraction of face value recovered in case of default}$

$V_0 = \text{upfront payment by protection buyer (negative if seller pays upfront)}$

**Premium Leg**

$E[\text{PV(future stream of premiums)}] = V_0 + S \sum_{i=1}^{T} D_i \theta_i \left( 1 - \sum_{j=1}^{i-1} \pi_j \right)$

**Protection Leg**

$E[\text{PV(payoff in case of default)}] = \left(1 - R\right) \sum_{i=1}^{T} D_i \pi_i$

Note: Under the new procedures, $S$ is always set at 100 (or 500). The two legs will not have the same value. The difference between them is the upfront payment that will be needed.
The Equilibrium CDS Spread $S$ versus the Credit Spread in Bonds

The equilibrium CDS spread should be close to the credit spread on the firm's bonds in the bond market. But they are not equal in practice, and there are some practical reasons why they should not be equal.

- There is an option of which of several bonds to deliver against the CDS
- At the time of default, accrued premium since the previous payment date must be paid by the protection buyer, but the bondholder doesn't get any accrued interest
- Both corporate bonds and CDS are subject to market noise that reduces measured correlation between them (corporate bonds are pretty illiquid)
The Recovery Rate Assumption

• The recovery rate assumption has a very significant impact on CDS pricing.

• The recovery rate is hard to predict accurately. It is commonly simply set to the long run average recovery fraction, approximately 40%.

• Average recovery rates have been found to vary over a wide range across different defaults and over time

• recoveries are negatively correlated with (physical) default probabilities (the more likely default is, the less will probably be recovered if default happens)
Collateralized Debt Obligation (CDO)

CDS (Credit Default Swaps) are like a combination of futures and insurance. The primary purpose is to transfer exposure to risk.

CDOs (Collateralized Debt Obligations) are a different class of derivatives, like mortgage-backed securities. The primary purpose is to repackage risk exposure.

A CDO is a securitization of debt securities:

- risky bonds, or loans, are pooled and new securities similar to CMOs are issued, with different priorities over the cash flows
- CDOs can concentrate the default risk of the underlying bonds into a few high-risk securities, leaving the others essentially risk free
- "synthetic CDOs" are pools not of risky bonds, but of CDS
- one of the most important properties of the pool is the correlation in default risk across issuers (which determines the risk that a lot of bonds will go bad together)

Originally, most mortgages in the pools supporting mortgage pass-throughs and CMOs were guaranteed by the government, so there was no credit risk. In the years following 2000, a large volume of CDOs based on pools of "subprime" and "Alt-A" mortgage loans were created. These are not guaranteed, so the CDOs were exposed to both prepayment risk and default risk. In a number of cases the underlying mortgages were so bad that the tranching failed to protect the senior tranches.
Securitization and tranching used in creating mortgage-backed securities can be applied to other kinds of loans (auto loans, credit card receivables, etc., etc.).

Some of the most exciting new securitized instruments are Collateralized Debt Obligations and Collateralized Loan Obligations (CLOs)

- pools of credit-risky bonds or loans are bundled together ("cash flow CDO")
- a common alternative structure is the "synthetic CDO," in which the pool of securities exposed to credit risk consists of Credit Default Swaps, rather than actual bonds
Credit Risk: Default as an Option and Credit Derivatives

Allocation of Default Losses in CDOs

In a CDO, default losses are allocated first to the lowest tranche

- the "senior" and "super-senior" tranches are typically more than 80% of the total principal value
- these can be rated AAA, even though the underlying bonds or loans are rated much lower.

---

Pool of Bonds or Loans

- Single Loan
- Single Loan
- Single Loan
- Single Loan
- Single Loan

- Single Loan
- Single Loan

- Single Loan

- Super-senior
- Senior Tranche
- Mezzanine Tranche
- Equity Tranche
Allocation of Default Losses in CDOs

The price for the equity tranche is quoted differently from the other tranches in the market because it has a relatively high probability of being completely wiped out before the pool matures.
Credit Risk: Default as an Option and Credit Derivatives

CDOs

The essential features of a CDO are securitization and tranching.

Typical structure

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Range of Losses Covered</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;super senior&quot;</td>
<td>above 12%</td>
</tr>
<tr>
<td>&quot;senior&quot;</td>
<td>7 - 12%</td>
</tr>
<tr>
<td>&quot;mezzanine&quot;</td>
<td>3 - 7%</td>
</tr>
<tr>
<td>&quot;equity&quot; or &quot;first loss&quot;</td>
<td>0 - 3%</td>
</tr>
</tbody>
</table>

Just like with a CMO, the cash flows into the portfolio, either from bond coupon and principal payments (or from CDS premium received in a synthetic structure) are passed through and allocated to the tranches, in a "waterfall" pattern.

If there is a default, the principal backing the equity tranche is reduced by the amount of the loss. All default losses are allocated to the equity tranche until the total loss is greater than 3% of the initial principal and the tranche is totally wiped out. Any defaults after that will be borne by the mezzanine tranche, until 7% of the face value has been wiped out. (Note that because there are recoveries, this would mean that a lot more than 7% of the bonds have defaulted.)

The super senior tranche is insulated against all defaults on the portfolio that do not total more than 12% of initial principal. This is extremely unlikely unless defaults are highly correlated.
Correlation and CDO Valuation

One of the major determinants of default risk exposure in a CDO is correlation in defaults.

A simple example:

- The "portfolio" = 50% in bonds from issuer A and 50% in bonds from issuer B.
- Both A and B have 10% probability of defaulting within the next year.
- There are two CDO tranches issued, each covering 50% of face value.

The equity tranche (0 - 50%) takes the first loss. It has a total loss if either A or B (or both) defaults. The senior tranche only takes the second loss, so there is no loss to the senior tranche unless both A and B default.
Impact of correlation

If $\rho = 0$, defaults are independent

- $\text{prob(loss to equity tranche)} = p(A) + p(B) - p(A \text{ and } B) = 19\%$.
- $\text{prob(loss to senior tranche)} = p(A) \times p(B) = 1\%$.

If $\rho = 1.0$, defaults for both A and B occur together

- $\text{prob(loss to equity tranche)} = p(A) = p(B) = 10\%$.
- $\text{prob(loss to senior tranche)} = p(A) = p(B) = 10\%$.

The basic result: Higher default correlation increases the value of the equity tranche and reduces the value of the senior tranche.
Credit Risk: Default as an Option and Credit Derivatives

Other Credit Derivatives

First-to-default, nth-to-default basket credit default swap:

- The option payoff is based on default experience of a portfolio of reference entities (risky issuers). Like a single-name CDS, it pays off when there is a default, but only after there have been n-1 previous defaults in the group (n may be 1, making the first default trigger the payoff).

Total return swap:

- Counterparty A commits to pay Counterparty B the total return on some asset (in this case, a risky bond, or portfolio of risky bonds), including the loss of principal in case of a default
- B commits to pay A the total return on some default-free security (e.g., 10-year Treasuries)

CDO-squared, CDO-cubed (nearly extinct):

- A Collateralized Debt Obligation for which the underlying pool that is being tranched is made up of CDO tranches. For example, a CDO-squared could be based on a pool of the 3-7 mezzanine tranches of CDOs. The use of the term has evolved over time, so that "CDO" is often used for the CDO-squared structure, while a CDO of whole loans, not tranches, is called an ABS (asset backed security). The experience of 2008 showed that CDO-squared and cubed structures don't work.
Conclusions on Credit Risk and Credit Derivatives

Default risk is important. And interesting.

Credit derivatives are a major innovation for our financial system.

Implied default probabilities and correlations measure risk neutral values. We are a long way from fully understanding them, or from extracting dependable information about true probabilities from them.

It is very hard to test our models of credit risk, because defaults are very rare events. The world provides us with data on them very slowly. Fortunately!

We have had relatively little experience with how these instruments actually work when there are substantial numbers of defaults. Early credit events with CDS led to significant revision of the contract terms. The sub-prime mortgage crisis has led to changes in CDO structures.

One major effect of the 2008 crisis has been that the securitization industry remains essentially dead, except for securities issued by the quasi-government agencies Fannie Mae and Freddie Mac.)
In this final session, we look at options with more complex payoffs than the "plain vanilla" calls and puts we have considered so far. Some of these are nothing more than packages of standard options, while others are quite different, presenting both new payoff patterns and also new valuation problems, some of which become quite intractable.

We then consider two of the important "frontier" areas in applying derivatives concepts in the real world: Real Options and Structured Products.

One direction in which option theory has been extended is to aid in real investment decisions. Given enough assumptions, contingent claims theory can place a specific dollar value on the element of choice in an investment project, such as the ability to shut it down prematurely if it is unsuccessful, or to expand its scale if it does better than anticipated.

Another direction is toward the use of "financial engineering" to create tailor-made financial instruments designed for a specific purpose. For more complicated structures, this may involve creating a "Special Purpose Vehicle," which is a separate financial entity set up to hold some kind of existing security and to issue various types of derivatives against it.
Exotic Options

There is an infinite range of possibilities in designing option payoffs. This has led to a remarkable proliferation of "exotic" options. Most serve a real need, despite appearing very strange to an outside observer, in some cases.

We may distinguish four families of exotic options, based on the techniques needed to value them:

Path-independent exotics: These can be valued easily using the standard methodology, such as the Binomial model. Some have closed form valuation equations.

Path-dependent exotics with path-independent valuation: Like American options, these securities have payoffs that depend on the path followed by the underlying asset, but they can still be priced with standard methods.

Path-dependent instruments without path-independent valuation: For these, the path followed by the underlying determines the payoff in a way that requires different valuation techniques, such as Monte Carlo simulation.

Multivariate options: The payoff is a function of more than one random variable.
Path-Independent Exotic Options

The key characteristic is that they can be valued knowing only the current price of the underlying, not the price path up to this point in time.

Examples

**Package:** Complex payoff structures created by packages of simple options include spreads, straddles, collars and range forward contracts. Valuation is easy: just sum up the values of the component options.

**Binary or Digital Option:** Pays off a fixed amount if the underlying price at expiration is above (call) or below (put) the option strike, no matter how far the option is in-the-money. Valuation is easy with standard methods, but hedging is hard when the underlying is close to the option's strike price near expiration.

**Compound Option:** An "option on an option," is the right to buy (call) or sell (put) an underlying option. This is a useful conceptual device for modeling certain kinds of compound contingencies, such as the value of an option on a stock in a firm subject to bankruptcy risk. Valuation is not hard with standard methods.
Exotic Options, Real Options, and Structured Products

Path-Dependent Exotic Options with Path-Independent Valuation

Some path-dependent contracts can still be valued with standard methods, if the value only depends on the paths the underlying asset price might follow in the future. In other cases, like barrier options and lookbacks, a valuation equation exists because under risk-neutral valuation there is a closed-form expression for the expected value of the path statistic that determines the payoff (for example, there may be a formula for the expected value of the maximum price for the underlying over the option's lifetime).

American Options: An American option is exercised early the first time the price of the underlying asset reaches the early exercise boundary. This makes the option path-dependent, but it can be valued using standard tools like the Binomial model.

Barrier ("Knock-in" and "Knock-out") Options: The payoff depends on whether the price of the underlying hits a specified barrier level at some point during its lifetime. "In" options, such as a "down and in call," must reach the barrier to become activated. If the asset price does hit the barrier (called the "in strike"), the payoff at maturity will be the same as a European option; otherwise the option expires worthless regardless of the asset price at expiration. "Out" options must not hit the barrier ("out strike"); if the barrier is breached, the option is knocked out and becomes worthless.

Lookback Options: A lookback pays off at maturity based on the highest or lowest price the underlying asset reached during the option's life. For example, a lookback call pays the difference between the final asset price and the lowest price observed over the option's whole lifetime.
Asian Options: Payoff is based on the average price of the underlying over its lifetime (or over some specified portion of its lifetime). This payoff pattern is useful to manage risk exposures that are themselves averages, such as the average monthly cost of natural gas during the wintertime. It also reduces the incentive to try to manipulate the market price of the underlying to affect the payoff at option expiration.

Valuation is not possible with standard methodology because the average price of the underlying as of option expiration day is a function of the whole path followed by the asset price up to that point, and not just the final price (and the arithmetic average of lognormal variables is not lognormal). A variety of approximation formulas exist, or valuation may also be done using Monte Carlo simulation of paths.

Mortgage-backed Securities: Prepayments on the underlying mortgages are path-dependent, so a mortgage-backed security cannot be valued without taking into account the entire previous history of interest rates. Valuation is done using Monte Carlo simulation of interest rate paths and prepayments.
Some exotic contracts depend on the price of more than one asset. In some cases, the problem can be redefined to involve just one state variable, but usually not.

**Exchange Options**: The option to exchange asset X for asset Y clearly depends on the values of both X and Y, but only on their difference (payoff = \( \text{Max}(Y - X, 0) \)). For this case, a simple Black-Scholes-type formula exists (Margrabe, 1978).

**Rainbow Options**: These are often called an "option on the max" (or on the "min"). A call on the max gives the holder the right to pay the strike price and acquire the best performing among a set of underlyings. For example a call on the max might pay the realized return on the S&P 500 stock index or on a 30 year U.S. Treasury bond, whichever was greater. Closed-form valuation equations may exist, but are too complicated to be usable except for 2-color, or at most 3-color, rainbow options.

**Quantos**: The underlying is denominated in one currency, but the payoff is in a different currency. These are surprisingly common. An example is a call option based on the change in the level (in Yen) of the Japanese Nikkei stock index, with the payoff being made in U.S. dollars.
Real Options

An important extension of option pricing theory is toward "real" options. Real options represent options--choices--that may be available in the future, for example, in a real investment project. Optionality can contribute significant value to a project. Option theory provides a framework for analyzing those choices and placing an economic value on them in an investment decision.

Examples of real options include:

- the option to initiate a new project or to abandon an existing one (e.g., the option of when and how extensively to begin drilling in a new oil field; the option to shut down an investment project early if it turns out to be unprofitable)
- flexibility to change the scale of a project (e.g., the value of a factory design that easily allows production to be expanded or reduced)
- timing options (e.g., the option to speed up or slow down production; the option to suspend production temporarily)
- flexibility in product or input mix (e.g., the ability to switch a power plant from coal to natural gas)
- etc.
Real Options

The concept of real options helps frame consideration of future choice possibilities in an investment project, simply by recognizing some of the portfolio dominance properties that must hold. For example,

- as a kind of option, the ability to make a choice in the future has positive value
- the greater the volatility (uncertainty), the more the real option is worth
- the longer the period before the choice has to be made, the more the option is worth
- etc.

Placing a specific dollar figure on a real option requires assumptions (whose validity may be questionable, such as that the "underlying" follows a lognormal diffusion) and also a pricing model.

One big issue is that the "underlying" (e.g., expanded production capacity) is seldom an investible instrument, so there is no arbitrage between the underlying and the option. Pricing real options must be by equilibrium principles, rather than arbitrage, which requires putting a "price of risk" on every stochastic factor in the problem.

Real option theory is still very much under development. But just recognizing that the ability to make a choice in the future has tangible economic value which should be weighed in any investment decision, and a conceptual framework to estimate that value, are already major innovations in practical financial decision making.
Structured Products

Structured products are tailor-made financial instruments, often containing a variety of derivative features, and designed for a specific purpose.

Structured products are created for a number of reasons, including:

- to manage the incidence of undesirable types of risk or other characteristics of the underlying asset (e.g., prepayments, credit risk, ...)
- to create liquidity for an illiquid class of assets
- to secure more favorable tax or regulatory treatment
- to avoid unfavorable accounting treatment

Often, a new financial firm (a "special purpose vehicle" or "SPV") will be set up to hold the underlying collateral and to issue the derivative securities.
A Sophisticated Structured Deal: Morgan Stanley's "PLUS I" Program

The Problem: Banamex, a major Mexican bank was holding a large amount of an inflation-linked bond issued by the Mexican government, known as "Ajustabonos." There were a number of interlinked difficulties with these:

- They were illiquid. Inflation had gone down, so there was little demand for them.
- Prices had fallen, but they were being carried on the bank's books at face value. Selling them would require recognizing a large loss.
- They were denominated in pesos, so a US or other non-Mexican investor would have a sizable exchange rate risk in owning them.
- Mexican government bonds denominated in pesos had very little default risk, but dollar-denominated government debt had a low bond rating.

Banamex wanted to "sell" these bonds to use the funds tied up in them for more productive purposes, but

- There was no demand for them from Mexican investors,
- US investors wouldn't buy them unless they were highly rated and paid off in dollars, and
- Actually selling them would entail booking an unacceptable loss on the bank's accounting statements.
Morgan Stanley's "PLUS I" Program

The Solution: Securitization.

Use the Ajustabonos as the collateral to support the issuance of new bonds. A large tranche of the new bonds would be designed to have the features that US investors needed. And just like the individual mortgage loans in a mortgage pool that support a set of collateralized mortgage obligations, the Ajustabonos would provide the cash flow for the coupon interest and principal payment of the new bonds.

How to accomplish all of this?
Morgan Stanley's "PLUS I" Program

1. To do the securitization, it is common to set up a new financial firm--a "Special Purpose Vehicle" (SPV), wholly owned by the firm doing the deal. In this case, it was a Bermuda-based corporation. Legal aspects of doing this were amusing but not very important.

2. Transfer the Ajustabonos to the SPV. The SPV then issued two classes of new bonds. 80% of the new bonds were denominated in US dollars and had higher priority than the remaining 20% (i.e., they got paid first).

3. The lower priority bonds were retained by Banamex. These served as a cushion to improve the credit quality of the senior bonds. If the peso/dollar exchange rate changed adversely, the impact on the dollar-denominated bonds would be offset by reduction in the payments to the bonds held by Banamex. The PLUS I bonds were therefore rated AA- by Standard and Poor's, a high investment grade rating.

4. The bonds held by Banamex were counted as paid-in capital invested in the SPV. Since the SPV was just a wholly-owned subsidiary of Banamex, transferring the Ajustabonos to it did not count as a sale for accounting purposes, so no loss was recorded on Banamex's books.
Structured products are created for a number of reasons. This deal illustrates all of them:

- "to reduce the impact of undesirable types of risk or other characteristics of the underlying asset (e.g., prepayments, credit risk, ...)"
  
  In this case, it was exchange rate risk that was the problem.

- "to create liquidity for an illiquid class of assets"
  
  The illiquid Ajustabonos were turned into highly marketable PLUS I bonds.

- "to secure more favorable tax or regulatory treatment"
  
  Here the special treatment needed was an investment grade bond rating.

- "to avoid unfavorable accounting treatment"
  
  Banamex was able to bring in cash without actually selling the bonds and booking an accounting loss.
Partnoy is highly scornful of this deal. Let us think about the various parties involved and ask whether they were hurt by it, helped by it, or not affected one way or the other.

How about:

- Morgan Stanley
- Banamex
- Bermuda school girls
- US buyers of PLUS I notes, like the Wisconsin pension fund
- The Mexican government, issuer of the Ajustabonos
- Mexican borrowers who are now able to get loans from Banamex
- other Mexican holders of Ajustabonos

Bottom line: It seems as if almost everyone benefited from this deal and no one was hurt. The derivatives concept and some creative financial engineering produced a new structure in the financial market that made it possible for capital to flow more freely and more efficiently.
Concluding Thoughts

A Few General Principles of Options

Buying options reduces overall risk exposure; writing options increases it

You pay for what you get.

- If one position is better than another under one scenario, it will be worse under other scenarios, and the market considers the tradeoff to be fair.
- Writing out of the money options naked is extremely dangerous, even though they nearly always end up out of the money. Writing out of the money options and delta hedging them is also risky, but not quite as bad.

End users should use options to achieve desirable payoff patterns, not to exploit "mispricing."

- Options allow payoffs to be precisely tailored to match the investor's preferences and market view.
- Trading costs and risk are too large for a non-professional to profit from mispricing relative to a theoretical model.

Many option strategies "work" for a particular investor because they alter a position's tax or accounting treatment, or its liquidity.
Concluding Thoughts on the Derivatives Course

A Few General Principles to Carry Forward

Good luck on the final exam, and especially

BEST WISHES FOR THE SUMMER
AND FOR YOUR FUTURE CAREERS!!