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An American Call IS Worth More than a European Call: The Value of American Exercise When the Market is Not Perfectly Liquid

by

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An American Call IS Worth More than a European Call

Abstract

An easily proved theoretical principle in option pricing is that an American call option should never be exercised early, except possibly just before an ex-dividend date. This result depends on the ability to liquidate an option position, European or American, for its theoretical value (the European call price) in the market. But this is often impossible in a real world option market, where the best bid for an in-the-money short maturity option is frequently below intrinsic value even for an option on a liquid stock. The payoffs at expiration are the same, but if the investor does not hold to maturity, an American option can always be exercised to recover intrinsic value while a European contract must be liquidated at the best available bid in the market which can be considerably lower. This liquidity difference imparts a positive value to the early exercise option. Using a replication argument, I derive the value of liquidity-based early exercise in closed-form, as a function of the bid-ask spread.

I then explore how important this phenomenon is in the real world using option bid prices on eight active option stocks and find that the value of Americanness can be substantial, even for many of the most actively traded options whose spreads are very tight. For the stocks in the sample, the value of the early exercise option can of comparable magnitude to the theoretical value of American exercise to collect a dividend.

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One of the easiest demonstrations of the principles of option valuation in textbooks and classrooms is the proof that an American call on a non-dividend paying stock should not be exercised before maturity. At any time prior to expiration, the value of a European call option is a function of three factors: its intrinsic value (the payoff if it could be exercised immediately) plus the value of its "optionality" (the right to wait until maturity to decide whether to exercise) plus the value of delaying payment of the exercise price until option expiration day. The latter two components are the option's time value. Both are always positive (so long as interest rates are positive), so exercising an American call early is a mistake because it gives up the remaining time value. If an investor should have to liquidate a long call position before maturity, she should sell it for its European value in the market. If this is not possible, theory says an option position can be unwound synthetically by delta hedging the contract until maturity. Theoretically, that hedging strategy replicates the option's European value and allows the investor to capture the remaining time value along with the intrinsic value.[1]

These strategies for unwinding a call option position before expiration cannot be done in the real world. As Battalio et al. (2017) have shown, when an American equity option is in the money within a couple of months to expiration, the best available bid across all option trading venues is frequently well below the option's intrinsic value. For calls that are in the money by 10% or more within a month of expiration, they found this was true of more than 92% of the best bid quotes, even when option prices were observed intraday at one minute intervals. The alternative strategy of delta-hedging the option through expiration rather than selling it would be subject to very large transactions costs, as well as other problems like constraints on short sales. Jensen and Pedersen (2016) explored this strategy and showed that it is prohibitively expensive in practice and often dominated by exercise.

The theoretical proscription against early exercise of American calls holds only in frictionless markets. In the real world, when a position must be liquidated before expiration, an in-the-money American call will commonly have a higher possible payoff than the equivalent European option. This should lead to rational early exercise of American calls in the real world, and a liquidity-based premium for the Americanness feature that permits it. The literature on option exercise has largely focused on testing how well the theoretical no-early exercise result holds in the market.[2] Poteshman and Serbin (2003), Pool et al. (2008), Barraclough and Whaley (2012) and many others have found evidence they interpreted as irrational exercise, and failure to exercise, of American options. On the other side, Valkanov et al. (2011) look at the relative market prices for

[1] Non-early exercise was proved as a general principle by Merton (1973). Textbook demonstrations include Hull (2015, p. 245), Sundaram and Das (2011, p. 217), Kolb and Overdahl (2007, p. 514), Brooks and Chance (2007, p.68), and many others.

[2] See, for example, Bhattacharya (1983), Diz and Finucane(1993), Overdahl and Martin (1994), Finucane(1997), and others cited below.

American versus European calls and find the differences to be larger than would be justified if only the option to exercise early to collect a dividend payout was valuable.

This paper presents a model for valuing the early exercise feature of an American call option in a less-than-perfectly liquid market, when the investor may need to liquidate her position prior to option maturity. Using a replication approach similar to the Roll-Geske-Whaley model for pricing an American call on a dividend-paying stock, we derive a valuation model in closed-form.[3] Although today's fast computers can make approximation techniques like lattice models nearly as accurate as closed-form solutions within reasonable execution time, an analytic formula has important advantages over an approximation. One is that even if it is very complicated, an analytic formula is an exact solution, meaning there is no model error in implementation. A valid numerical technique gives an approximation to the exact solution, but it is only reached asymptotically. Also, the Greek letter risk exposures are available in closed form, again without approximation error, and the analytic expressions can be analyzed theoretically in ways that are not possible for a numerical approximation.

The next section presents a model in which the investor may need to liquidate early and is facing a fixed bid-ask spread. The best bid in the market will fall below intrinsic value when the option is deep enough in the money that its remaining time value becomes smaller than half the market's bid-ask spread. Section 3 presents a numerical example that shows the premium for early exercise can be meaningfully large and illustrates the effects of option maturity and moneyness. In Section 4, we extend the model to cover more realistic bid-ask spread behavior.

Section 5 applies the model to observed bid-ask spread behavior for calls written on twenty-four major stocks. We find considerable variability in spread behavior across stocks and among the options on each individual firm. The Americanness premium is non-negligible even for the most actively traded contracts, and for less active option stocks the liquidity-based premium can easily exceed the theoretical value of early exercise for a dividend. In Section 6, we present a more detailed comparison of the value of Americanness for liquidity versus for dividend in our sample of stocks.

The final section concludes. An extension of the model to a version that can handle complex bid-ask spreads is presented in the Appendix.

2. The Model

We consider a market in which the date t European option value for a call with strike price X and maturity T, $C_{Eur}(S_t, X, T-t)$, is common knowledge. S_t is the date t price of a non-dividend paying underlying stock.

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[3] See Whaley (1981).

Options held to maturity pay off in full with no transactions costs:

$$C(S_T, X, 0) = Max(S_T - X, 0). \quad (1)$$

Options sold prior to expiration must be sold to a market maker. The bid price at some liquidation date t < T will be the theoretical option value minus half the option's bid-ask spread. Assume the bid-ask spread in the options market is a fixed value 2B. The half-spread B is the discount below the option's fair (i.e., European call) value that an investor suffers when liquidating an option position in the market prematurely. B is the cost of liquidity.

An investor (option holder) who sells a European call in the market at date t receives:

$$C_{Eur}(S_t, X, T-t) - B. \quad (2)$$

An American call is subject to the same bid-ask spread. But in liquidating on date t, the holder has the choice of selling to the market maker or exercising. Behaving optimally, the holder of an American call receives:

$$Max(C_{Eur}(S_t, X, T-t) - B,\ S_t - X) \quad (3)$$

There are many reasons an investor may want or need to liquidate an existing option position prior to maturity. Her trading strategy may have been based on an expected price target and the option has now reached it. Or perhaps her expectations have changed with new information and the trade is no longer attractive. Maybe the option is part of a risk management strategy for a deal that is now completed and the hedge is no longer needed. It could be the investor needs the capital that is currently tied up as a margin deposit required to carry the position. Or any of a myriad of other possible reasons that exist in the real world but lie outside the model.

We will treat the decision to liquidate an existing option position as purely exogenous, the result of factors that are independent of the current price of the option or the stock. The option holder does not try to optimize the timing of early exercise to take account of expected future stock prices and bid-ask spreads. Instead, we will model the early liquidation decision for a call option as the first jump in a Poisson process, similar to the way default is modeled in a reduced-form credit risk model. Many alternative processes for liquidation probabilities can readily be accommodated in the model.

Exposure to the "risk" of early exercise leads to a survival function G(t). For an option purchased at time 0, G(t) is the probability that the position has not been liquidated before date t. So G(1) = 1 and G(T) is the probability that the option will be held to maturity. The survival function for a real world option position may well depend on the stock price path from time 0 to t, as well as other observable exogenous variables, but here we model it as only a function of time to maturity.

We assume liquidation is possible only once a day. L(t) is the unconditional probability, as of time 0, that liquidation will occur on date t:

$$L(t) = G(t) - G(t+1) \tag{4}$$

The probability of liquidation at maturity is the probability of no early liquidation:

$$L(T) = G(T) \tag{5}$$

Let D(s,t) be the date s price of a zero-coupon bond paying $1on date t > s, i.e., D(. , .) is the discount function.

The European option value at time 0 for an investor subject to early liquidation is:

$$C_0^E = E_0[\sum_{\tau=1}^{T-1} D(0,\tau)\, L(\tau)(C_{Eur}(S_\tau, X, T-\tau) - B) + D(0,T)L(T)Max(S_T - X, 0)] \tag{6}$$

where $E_t[.]$ denotes the date t expected value under the risk neutral distribution. The American option value at time 0 for an investor subject to early liquidation takes into account that the option will be exercised if that would produce a higher value than the price in the market.

$$C_0^A = E_0[\sum_{\tau=1}^{T-1} D(0,\tau)L(\tau)Max(C_{Eur}(S_\tau, X, T-\tau) - B, S_\tau - X) + D(0,T)L(T)Max(S_T - X, 0)] \tag{7}$$

We have assumed that the liquidation function and the stock price are independent.[4] Taking the expectation inside the summation in (7), we have:

$$C_0^A = \sum_{\tau=1}^{T-1} D(0,\tau)L(\tau)E_0[Max(C_{Eur}(S_\tau, X, T-\tau) - B, S_\tau - X)] + D(0,T)L(T)E_0[Max(S_T - X, 0)] \tag{8}$$

We now show that under Black-Scholes model assumptions, with the addition of the bid-ask spread and the liquidation function as described above, the value of early exercise can be computed in closed-form.

Additional assumptions to place the problem in a Black-Scholes world

The continuously compounded riskless interest rate is a known constant r:

[4] Independence between the option value and the probability of liquidation means that an out of the money option is equally likely to be liquidated early as one that is deep in the money. If liquidation and moneyness are correlated, the equivalent expression to (8) must include covariance terms. This more complicated modeling problem, and possible replication of the payoffs by a portfolio of claims that can be valued at time t = 0 is left for future research.

$$D(s,t) = e^{-r(t-s)} \quad (9)$$

Stock price dynamics under risk neutrality are given by:

$$dS/S = r\,dt + \sigma\,dz \quad (10)$$

Volatility of the log stock price is a known constant σ.

With strike price X, maturity T and no transactions costs, the well-known theoretical result is that the option value for both European and American calls on any date $t \leq T$ is the Black-Scholes model value, $C_{BS}(S_t, X, T-t)$.

Liquidation is modeled as a Poisson event with intensity parameter λ. The survival function *G(t),* giving the ex ante probability the option position will not be liquidated before date t is

$$G(t) = e^{-\lambda(t-1)} \qquad \text{for } t \leq T$$

$$G(t) = 0 \qquad \text{for } t > T \quad (11)$$

The unconditional probability that an option purchased at time 0 will be liquidated on date t, assuming liquidation is possible only once a day, is

$$L(t) = G(t) - G(t+1) = e^{-\lambda(t-1)} - e^{-\lambda t} \quad (12)$$

This assumption leads to a constant conditional probability of liquidation $(1 - e^{-\lambda})$ on any date $t < T$, given that the position has not been liquidated before t.

The bid-ask spread may be a function of the option's moneyness, maturity, volatility, and time. In the first analysis, we simply assume the half-spread is a fixed constant B. We will consider more realistic bid-ask spread behavior later.

The American call valuation equation becomes

$$C_0^A = \sum_{\tau=1}^{T-1} e^{-r\tau}\left(e^{-\lambda(t-1)} - e^{-\lambda t}\right) E_0[Max(C_{BS}(S_\tau, X, T-\tau) - B, S_\tau - X)] +$$

$$e^{-\lambda(T-1)} C_{BS}(S_0, X, T) \quad (13)$$

The last term comes from the fact that under risk neutrality, the Black-Scholes value of the call at time 0 is the discounted value of its expected payoff at date T. This payoff at expiration is multiplied in (13) by the probability that there is no liquidation prior to T. The summation term adds up the discounted payoffs from possible early liquidation on each date t < T, assuming the option holder chooses either to sell the option to the market maker or exercise it, whichever yields more, weighted by the probability that liquidation will occur at t.

We will derive a closed-form solution to the valuation problem in (13) by constructing a portfolio of derivative instruments that can all be priced at time 0, which replicates the payoffs to the optimally exercised American call.

As a first step, we consider the simpler problem of replicating the payoff on an option that must be liquidated on a single future date t, t < T. The American call should be exercised if

$$C_{BS}(S_t, X, T-t) - B < S_t - X \qquad (14)$$

The investor will sell the option to the market maker if the inequality is reversed. If the remaining time value on the option is less than the half spread B and the option is in the money, it is better to exercise. Note that if there is a bid-ask spread on the stock, S_t must refer to the market's bid price.

Similar to the case of dividend-related early exercise, on date t there is a stock price S_t^* for which the two sides of equation (14) are equal. S_t^* marks the early exercise boundary. If the date t stock price is below S_t^*, the option's remaining time value is more than B and it is better to sell to the market maker for the European call value minus B. At a stock price higher than S_t^*, the option's time value is less than B and early exercise becomes optimal.

Figure 1 provides an illustration. The highest curve is the European call value and the thinner curve below it is the price that would be obtained by selling the option in the market. The broken gray line is the option's intrinsic value. The stock price at which the two are equal is S*. In the example shown here, the option strike is 90, time to maturity T – t is one month, volatility is 25%, r = 5%, and the half-spread B = 1.00. With these values, the critical boundary price S* = 95.86.

Proposition 1: Under the assumptions made above, for an American call option with strike price X and maturity T, the payoff to optimal liquidation on date t < T can be replicated by the following portfolio of securities, all of which can be priced at time 0:

1. European call with strike X and maturity T. $C_{BS}(S, X, T)$

2. European call with strike S_t^* and maturity t. $C_{BS}(S, S_t^*, t)$

3. Short sale of a compound Call on the Call in 1., with strike $S_t^* - X + B$ and maturity t. $-C_{Call}(C_{BS}(S, X, T), S_t^* - X + B, t)$

4. Borrow the present value of *B* dollars, with repayment at t. $-B\, e^{-rt}$

Proof: The following table shows the payoffs on date t to the four components of the replicating portfolio for the two cases: $S_t \le S_t^*$ and $S_t^* < S_t$.

Position		Liquidation value if $S_t \leq S_t^*$ (Sell the American call)	Liquidation value if $S_t > S_t^*$ (Exercise the American call)
1	$C_{BS}(S, X, T)$	$C_{BS}(S_t, X, T-t)$	called by Option 3 exercise
2	$C_{BS}(S, S_t^*, t)$	0	$S_t - S_t^*$
3	$-C_{Call}(C_{BS}(S, X, T), S_t^* - X + B, t)$	0	$S_t^* - X + B$
4	*Borrow* $B\ e^{-rt}$	-B	-B
Total		$\boldsymbol{C_{BS}(S_t, X, T-t)}$ **- *B***	$\boldsymbol{S_t - X}$

If $S_t \leq S_t^*$, the optimal strategy is not to exercise, but to sell the call in the market for its European value less the half-spread B. In this case, the first option will be sold to the market maker. The second call is out of the money and will expire worthless.

The third component is a call on call number 1. The strike price on this compound call is $S_t^* - X + B$, but from (14), S_t^* is the date t stock price such that the European call is just equal to this amount. Call number 1 is worth less than this at any lower stock price, so the compound option will be out of the money and will expire worthless. The final component is the repayment -B on the initial loan of PV(B). Combining the four payoffs shows that the portfolio replicates the American option payoff at stock prices where the option will be liquidated by selling it to the market maker.

In the case where $S_t^* < S_t$, the American option should be exercised. Option number 2, which matures at t, will be exercised. The investor pays the strike of S* and acquires the stock. Figure 1 and the discussion in the previous paragraph indicate that option number 1 will be worth more than $S_t^* - X + B$. The compound call will therefore be in the money and will be exercised. This removes option 1 from the position and adds option 3's strike price $S_t^* - X + B$ to the portfolio's payoff. The loan repayment of -B completes the replication, which is seen to be just $S_t - X$, the exercised value of the American call. □

Let V(t) be the time 0 cost of this portfolio, which replicates the payoff to liquidation of the American call on a single future date t. The closed-form equation for V(t) is given by

$$V(t) = V_1(t) + V_2(t) - V_3(t) - V_4(t) \tag{15}$$

where

$$V_1(t) = SN[d_1] - Xe^{-rT}N\left[d_1 - \sigma\sqrt{T}\right] \tag{16}$$

with $$d_1 = \frac{\ln(S/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$V_2(t) = SN[d_2] - S_t^* e^{-rt}N\left[d_2 - \sigma\sqrt{t}\right] \tag{17}$$

with $$d_2 = \frac{\ln(S/S_t^*) + (r+\sigma^2/2)t}{\sigma\sqrt{t}}$$

$$V_3(t) = SN_2\left[d_1, d_2; \sqrt{t/T}\,\right] - Xe^{-rT}N_2\left[d_1 - \sigma\sqrt{T}\,, d_2 - \sigma\sqrt{t}\,; \sqrt{t/T}\right]$$

$$-(S_t^* - X + B)e^{-rt}N[d_2 - \sigma\sqrt{t}] \tag{18}$$

and $$V_4(t) = Be^{-rt} \tag{19}$$

Here $N_2[\,.\,]$ indicates the bivariate normal distribution function and $\sqrt{t/T}$ is the correlation coefficient between the stock's return to date t and its return to date T.

Proposition 2: Under the assumptions made in Proposition 1, the time 0 value of an American call on a non-dividend paying stock with strike price X and maturity T, assuming that it may need to be liquidated prior to expiration with liquidation probabilities given by (12), is

$$C_0^A = \sum_{\tau=1}^{T-1} L(t)V(t) + e^{-\lambda(T-1)}C_{BS}(S_0, X, T) \tag{20}$$

The proof is immediate, given the replication strategy for each possible liquidation date t and the assumption that L(t) and V(t) are independent for all t. □

3. An Illustration

Equations (15) - (20) give the value of an American call in a Black-Scholes world with a fixed bid-ask spread, when the probability of early liquidation is given by a Poisson process with a fixed intensity. We will use these functions to explore the difference in value between the

American call in (20) and the comparable European call that offers no way to avoid low bid prices in the market.

Assume the current stock price is $S_0 = 100$ and the interest rate is 5.00%. The value of the early liquidation intensity λ is chosen to be a function of the time to expiration in such a way as to produce a probability the position will be held to maturity of 25%. That is,

$$G(T) = e^{-\lambda(T-1)} = 0.25 \quad \Longleftrightarrow \quad \lambda = -\frac{1}{(T-1)}\ln(0.25)$$

This leads to a higher exercise probability per day for a short maturity contract than when expiration day is far off. For example, when there are only three days to maturity, early exercise will occur with 75% probability within the next two days, giving $\lambda = 0.693$. If there are 10 days to maturity, that 75% probability is spread out over 9 days and $\lambda = 0.154$.

Figure 2 plots the premium for American exercise relative to a European call with the same terms, as a function of the time to expiration. We show four different degrees of moneyness with strike prices of 80, 90, 100 and 105, and two levels of underlying stock volatility, 0.25 and 0.75. The bid-ask spread is set at 2.00, making the half-spread B = 1.00. Solid (dotted) curves are for the low (high) volatility regime.

Consider a call with a strike price of 90 and relatively low volatility of 25%. The option is 10% in the money, so if expiration is within the next 10 days, it is highly unlikely that the stock price will end up falling from 100 to below 90. The optionality component of time value is therefore very small, and the interest component is also small: 10 days of 5% interest on a strike of 90 for a total of $0.12. With a 75% chance of early exercise, the Americanness feature on a 10-day 90-strike call is worth about $0.70.

The premium goes down as the time to expiration becomes longer. This reflects the behavior of the two components of time value. The further off expiration is, the larger is the value of waiting until expiration to pay the strike price. Also, optionality is greater with longer maturity. That is, with more time to expiration there is a greater chance that the stock price will drift down below 90 and the option will be out of the money, making the right <u>not</u> to exercise valuable. Since volatility to expiration goes up with the square root of T, at 25% volatility 10% out of the money is about 2.4 standard deviations for a 10-day option, but only 1.4 and 0.8 standard deviations for 30-day and 90-day contracts, respectively. Even so, with exercise price of 90 and 25% volatility, American exercise in this example is worth $0.51 for a 30-day call and $0.28 for a 60-day contract.

It is interesting to note that an at the money call, and even one that is a little out of the money with X = 105, still have non-negligible value for American exercise.

Comparing these results with those for options on high volatility stocks with σ = 0.75, we see that for a deep in the money 80-strike contract, the early exercise premium is over $0.70 under both high and low volatility. But it drops off faster for longer maturities when volatility is high because optionality grows much larger with σ = 0.75. For the out of the money 105-strike contracts, higher volatility and longer maturity increase the value of liquidity-related American exercise because they increase the probability that future stock prices will exceed future S* values, making early exercise the better choice for liquidation.

For illustration, we have assumed an initial stock price of 100, so a call with a strike price of 80 may seem very deep in the money. But consider a more typical stock price of 30. A 20% in the money contract would refer to a 25-strike call, a configuration that does not seem so improbably deep in the money.

4. More Realistic Bid-Ask Spreads

The value of the early exercise option is quite substantial under the assumptions of the previous section. But the assumption about bid-ask behavior implies the same spread for a $1 call and a $20 call, which would be highly unusual in a real world options market. In this section we consider more realistic bid-ask behavior, and derive an extended replication strategy.

Options that are well out of the money trade for very low prices, at which even a very narrow bid-ask spread can be large relative to the option's value, while deep in the money contracts are typically illiquid and have spreads that are much wider in dollar terms, although not necessarily as a percent of option value.

The bid-ask spread in the options market is typically positively related to an option's moneyness, maturity, underlying stock volatility, and liquidity. We now model the more realistic case in which the date t bid-ask half-spread is a time-stationary linear function of moneyness:

$$B(S_t) \equiv B_0 + B_1 Max(S_t - (X + M), 0) \tag{21}$$

where B_0 is (one half of) the minimum bid-ask spread for any option and B_1 determines the rate at which the spread widens out for in the money options. The term M in (21) allows the spread to begin widening at a level of moneyness different from (S – X). M must be small enough that S_t* - (X+M) is positive, i.e., the spread at S_t* is greater than the minimum B_0. If this is not the case, the kink in the spread function occurs above S_t* and M does not affect the replication, because it does not affect the spread at stock price S_t* so the problem would revert to the previous case with a fixed spread equal to B_0..

With the new definition of B, for an option with required liquidation on a single future date t < T, equation (21) now leads to a modified formula for S_t*, the date t stock price at which the investor is indifferent between exercising an American call and selling it to the market maker for its European call value minus B. S_t* is the solution to

$$C_{BS}(S_t^*, X, T - t) = S_t^* - X + B(S_t^*) \tag{22}$$

The payoff to optimal exercise of the American call can also be replicated in this case by a portfolio of securities at time 0.

Proposition 3: Under the assumptions made in Propositions 1 and 2, with the exception that the bid-ask half-spread is now given by equation (21), the time 0 value of an American call on a non-dividend paying stock with strike price X and maturity T under the assumption that it may need to be sold or exercised prior to expiration is given by equation (20) in which V(t) is defined as follows:

$$V(t) = V_1(t) + (1 + B_1)\, V_2(t) - V_3(t) - V_4(t) - V_5(t) \qquad (23)$$

where

$V_1(t)$ is given by equation (16)

$V_2(t)$ is given by equation (17)

$$V_3(t) = SN_2\left[d_1, d_2; \sqrt{t/T}\,\right] - Xe^{-rT}N_2\left[d_1 - \sigma\sqrt{T}\,, d_2 - \sigma\sqrt{t}\,; \sqrt{t/T}\right]$$

$$-(S_t^* - X + B(S_t^*)e^{-rt}N[d_2 - \sigma\sqrt{t}] \qquad (24)$$

with $B(S_t^*)$ given by equation (21) and d_1 and d_2 as defined in (16) and (17),

$$V_4(t) = B_0 e^{-rt} \qquad (25)$$

and $V_5(t) = B_1(\,SN[d_3] - (X + M)e^{-rt}N\left[d_3 - \sigma\sqrt{t}\right])$

with $$d_3 = \frac{\ln(S/(X+M) + (r+\sigma^2/2)t}{\sigma\sqrt{t}} \qquad (26)$$

Proof: The payoff to optimal exercise on a single date t < T can be replicated by the following portfolio of securities, all of which can be priced at time 0:

1. European call with strike X and maturity T. $C_{BS}(S, X, T)$

2. (1 + B_1) European calls with strike S_t^* and maturity t. $(1 + B_1)C_{BS}(S, S_t^*, t)$

3. Short sale of a compound call on the Call in 1., with strike $S_t^* - X + B(S_t^*)$ and maturity t. $-C_{Call}(C_{BS}(S, X, T), S_t^* - X + B(S_t^*), t)$

4. Borrow the present value of B_0 dollars with repayment at t. $-B_0\, e^{-rt}$

5. Write B_1 European calls with maturity t and strike (X+M) $-B_1 C_{BS}(S, X + M, t)$

Proof: The following table shows that the payoffs on date t to the five elements of the replicating portfolio do replicate the return to optimal exercise of the American call on a future date t, for the two cases: $S_t \leq S_t^*$ and $S_t > S_t^*$.

Position		Liquidation value if $S_t \leq S_t^*$ Sell the American call	Liquidation value if $S_t > S_t^*$ Exercise the American call
1	$C_{BS}(S, X, T)$	$C_{BS}(S_t, X, T-t)$	called by Option 3 exercise
2	$(1+B_1)C_{BS}(S, S_t^*, t)$	0	$S_t - S_t^* + B_1 S_t - B_1 S_t^*$
3	$-C_{Call}(C_{BS}(S, X, T-t),$ $S_t^* - X + B(S_t^*), t)$	0	$S_t^* - X$ $+ B_0 + B_1 S_t^* - B_1(X+M)$
4	Borrow $B_0\, e^{-rt}$	$-B_0$	$-B_0$
5	$-B_1 C_{BS}(S, X+M, t)$	$-B_1 Max(S_t - (X+M), 0)$	$-B_1 S_t + B_1(X+M)$
Total		$\boldsymbol{C_{BS}(S_t, X, T-t) - B(S_t)}$	$\boldsymbol{S_t - X}$

The time 0 values of these components 1-5 from (16), (17), (24), (25, and (26) define $V_1(t)$ through $V_5(t)$, respectively, and V(t) is given by (23). Weighting each V(t) by the probability that liquidation will occur on that date gives the total expected payoff from early exercise. Combining with the probability weighted value of liquidation at maturity gives the theoretical value of the American call liquidated optimally in the presence of bid-ask spreads in the form of (23). □

The new strategy just adds one more component to the replicating portfolio. The only adjustments required to the first four components are to increase the quantity of the second call in the position and to modify the calculation of the strike on the compound option 3. The first adjustment offsets the price-related portion of the bid-ask spread for stock prices above S*, which is introduced by V_5. The general formula to replicate an American call when the bid-ask spread function has an arbitrary number of break points is provided in the Appendix. In the empirical exercise discussed in the next section, it was found that adding a second break in the bid-ask function made virtually no difference to the results, so we used the formulas from Proposition 3.

5. Valuing Liquidity-Related American Exercise in the Real World

The model in Proposition 3 allows us to estimate the real world liquidity-based value of American call exercise. In this section we look at the bid-ask spread behavior in the market for exchange-traded equity options and estimate the liquidity-related value of American exercise for the calls on a set of twenty-four underlying stocks. The following section will compare the liquidity value of Americanness to the theoretical value of American exercise to capture the dividends on those stocks.

The Options Clearing Corporation publishes data by exchange on the number of contracts traded for each of more than 4000 underlying equity securities, including ETFs.[5] I combined the year to date figures from December 2016 and 2017 for trading at the top seven option exchanges, and sorted by volume. This captured more than 80 percent of trading volume across fifteen options exchanges.

Twenty-four optionable stocks within the 500 most active were selected to represent the behavior of options on highly liquid versus less liquid names among those with a meaningful level of trading activity in the market. Table 1 shows the selected stocks and their total option volume and relative ranking within the population. There are three subsets that we consider representative of the Most Active (combined 2-year volume more than 15 million contracts), Medium Active (roughly 5 – 10 million contracts), and Less Active (1 – 3 million contracts) segments of the options market. All are well-known firms.

Data on closing Highest Bid and Lowest Offer quotes for all calls with 3 to 65 days to maturity were downloaded from OptionMetrics. Only regular third-Friday-of-the-month expirations were used. Several additional constraints were applied in forming the actual sample to remove clearly erroneous points from the downloaded data. These are summarized in Table 2.

Qualitative requirements that the bid price and spread should be positive are obvious. We will examine the case of liquidation probabilities that decay exponentially, as in equation (11). The probability of holding the option to maturity is set at 25%. When we compare the premia for liquidity against premia for dividend in the next section, we consider the case of an ex-dividend date that is halfway between $t = 0$ and option expiration.

To facilitate comparisons among options on stocks trading at much different price levels, all prices and spreads were converted to a common value of $S = 100$. Option strikes and price quotes were also converted by the same ratio, taking advantage of the property that option values are homogeneous of degree 1 in S and X.. Thus a 20-strike call on a stock trading at 25, with a bid-ask spread of 0.20 would be converted to an 80-strike option on a stock priced at 100, with a spread of 0.80. This allows us to think of moneyness equally easily in terms of dollar values for

[5] https://www.theocc.com/webapps/volbyclass-reports.

options on a $100 stock, or as percentages of the stock price: the 80-strike call is in the money by $20 or by 20%.

The second section of Table 2 displays quantitative constraints. We limit the option maturities considered to be between 3 and 65 calendar days. The average maturity in the sample was 34 days. Moneyness is limited to a range from X = 50 (50% in the money) to X = 150 (50% out of the money). The average X was 98.61, making the average call about 1.4% in the money. To make sure that we are looking at pricing for options that could potentially be liquidated, we required open interest of at least 50 contracts. The sample average open interest was over 8000 contracts.

Option price quotes that are outliers, far out of line with pricing for similar contracts, can often be flagged by looking at their implied volatilities. Moreover, for options that are far away from at the money, a little pricing noise can create a big error in implied volatility. This is especially true for the illiquid deep in the money calls that are the focus of the study. Volatilities are needed to price options, so to deal with this problem while still extracting usable volatility estimates from the market, on each day for each stock I took the average implied volatility from all of its options that were between 5% in the money and 5% out of the money and had maturities more than 5 days and less than 45 days. That at the money implied volatility was used for pricing all of that stock's options for that date. By using only at the money options, there was no need to screen out large outliers. The average implied volatility in the sample was about 26%.

Finally, there were a handful of clearly erroneous quotes that were screened out by requiring the spread on an out of the money option to be no more than 5 points, and eliminating cases where the bid was more than 10 percent below intrinsic value. The combined effect of the various screens removed a relatively small number of highly questionable data points, but the quantitative effect on the overall results turned out to be very small, given the large number of good observations.

For option pricing and replication, we used 3 month LIBOR downloaded from the St. Louis FRED database and converted to a 365-day year. Dividend amounts and dates used in the next section were downloaded from Compustat, through the WRDS system.

Table 3 provides summary information on the bid-ask spreads for the options in the sample. The first line aggregates across all of the stocks, the next three lines show averages for the stocks in each of the three subsets, and the rest of the table reports on the individual stocks. Median spreads are quite narrow for the two more active groups but widen out to a median value of 0.508 for the Less Active group. Note that these figures are whole spreads, i.e., 2 B in the model. The distribution of spreads is skewed, so that the 75th percentile is 3 to 4 times larger than the median in the more active groups. Median (at the money) implied volatilities are also quite reasonable in size, but those for the Less Active subset are substantially larger than for the Most Active subset.

There are about 210,000 observations that fit our criteria and each group and individual stock is well-represented. The last three columns focus on cases in which the best bid in the market was below intrinsic value, so that it would be better to exercise a call than to sell it in the market. The data show that this is not at all an uncommon occurrence. In our sample, which includes options that are out of the money and those far from maturity, nearly 15% of all bid quotes that satisfy the constraints in Table 2 were below intrinsic value.[6] Among those observations, the median shortfall was \$0.188, but the last column shows that for a quarter of them, the bid was more than \$0.50 too low. For the Less Active group the 75th percentile for the underbid expanded to \$0.702 and it was substantially worse for some of the individual stocks. An illiquidity penalty for selling an option in the market rather than exercising it is pervasive in the real world and of substantial size.

Table 4 reports the parameter estimates from fitting the bid-ask spread model in (11) to the full data set, the three activity subsets, and the 24 individual firms. Figure 3 displays the spreads and fitted spread model for the full sample of stocks and for the three subsets reflecting Most Active, Medium Active, and Less Active options trading volume. Figures 4A, 4B, and 4C show the spreads and fitted spread models for the individual stocks.

Across the board, the median spread was only \$0.141. It was less than half of the median for the Most Active stocks, but a full \$0.508 for the Less Active group. Not surprisingly, the median bid-ask spreads tend to increase for options with less trading activity. Recall that these figures are the full bid-ask spreads, twice the half-spreads B that go into the model.

Despite the obvious noise in the data seen in Figures 3 and 4, the spread model has an R^2 of .228 overall and about twice that for other than the Most Active subset, which displays considerable variability in spread behavior. On average, the level of moneyness at which spreads begin to widen out from the minimum are all within a few points of at the money. The rate at which the spread increases with moneyness above this is only about \$0.02 per dollar for the Most Active group, but about three times higher for the other stocks. Note also that the R^2 statistics are much higher for the individual stock equations than for the combined groups, meaning that constraining all firms within a group to have the same coefficients substantially worsens the fit.

Table 5 presents the estimates of the liquidity-based premium for American exercise for All Stocks and the three subsets at four maturities: 1, 2, 4, and 8 weeks (7, 14, 28, and 60 calendar days, respectively). Five strike prices are shown, ranging from 80 (deep in the money) to 105 (5% out of the money).

For example, consider the 90-strike call across All Stocks in the sample. From Table 4, the minimum spread is \$0.141 and it begins to widen at moneyness $M = S – X = -2.378$. That is, the spread begins to increase when an option is 2.378 out of the money, given the standardization to

[6] It is sometimes claimed that OptionMetrics data are suspect, because option quotes are reported at the close of trading, when it is believed that spreads are wider than during the trading day. That may be true, but it does not come close to eliminating the problem of bids below intrinsic value, as Battalio, Figlewski and Neal (2017) show. Using intraday data at one-minute intervals, they find that for the great majority of options that are in the money by more than a little and do not have long maturities, the best bid is below intrinsic value in 90 percent or more of the cases.

S = 100. The spread widens as moneyness increases (i.e., for lower X) at the rate of \$0.036 per dollar, so that a 90-strike call is estimated to have a spread of

$$0.143 + 0.036 \left((100 - 90) - 2.378 \right) = 0.589.$$

The half-spread B = 0.589 / 2 = \$0.294. With one week to maturity, assuming the median volatility level of 0.223 from Table 3 and the riskless interest rate equal to the average (adjusted) 3-month LIBOR of 0.0118 percent during this period, American exercise was worth \$0.178. For the less liquid Medium and Less Active subsets, these Americanness premia increase to \$0.303 and \$0.371, respectively. They are reduced for longer maturities, but even 8 weeks from expiration, the values are still substantial.

The time pattern of the Americanness premium across All Stocks is shown in Figure 5. As we saw in the earlier example, longer maturity for a call that is already fairly deep in the money tends to decrease the value of the early exercise option. Both the optionality and the time value of money components become larger so more time value is lost in an early exercise. But for at the money and out of the money contracts, longer maturity provides a greater chance that they will go into the money, and the further they go, the wider the bid-ask spread will become. It is interesting to note that even a 105-strike call on a Less Active option stock has \$0.039 of liquidity-related Americanness value with 8 weeks to expiration.

These results strongly suggest that the value of American exercise to obtain a better payoff upon early liquidation than the best bid available in the market is non-negligible in the U.S. options market. This is in addition to the value of early exercise to obtain a dividend, which in conventional theory is the only reason an American call should be worth more than a European call. The next section will compare the estimates of the liquidity value of Americanness for our stocks against the theoretical premium related to their dividends.

6. Early Exercise for Liquidity versus for Dividend

The theoretical value of American exercise for a call on a dividend-paying stock depends on several factors, including option moneyness and maturity, dividend amount, volatility, and the interest rate. The dividend yield on the S&P 500 index in recent years has been about 2.0% annually, so the average index stock pays about 0.50% each quarter. This is very close to the average dividend among the stocks in our sample.

To compare the values of liquidity- and dividend-related American exercise, we took each dividend paid by a dividend-paying firm during 2016-17 (6 out of 24 did not pay dividends) and computed the theoretical premium for an American call with 2, 4, or 8 weeks to maturity when the ex-dividend date occurred halfway to expiration.

For example, Apple paid a dividend of \$0.52 a share in the first quarter of 2016. Ex-dividend date was Feb. 4. The stock price at the time was 96.60 so adjusting to S = 100, the dividend was also adjusted, to \$0.538. We then considered how a two-week AAPL call option maturing 1 week after the ex-dividend date would have been valued on Jan. 28, given the stock price, interest rate, and volatility (which was set equal to the average at the money IV, computed as described above). The same calculation was repeated for 4- and 8-week maturities, using valuation dates of Jan, 21 and Jan. 7 and assumed expirations 2 and 4 weeks after the ex-dividend date, respectively.

Using the Whaley (1981) formula for the value of an American call on a dividend-paying stock, for a 90-strike (10% in the money) option as of 1, 2, or 4 weeks prior to ex-dividend day the theoretical early exercise premia to obtain Apple's $0.538 dividend on Feb. 4, 2016 were, respectively, $0.468 (2-week call as of Jan. 28), $0.233 (4-week call as of Jan. 21), and $0.108 (8-week call as of Jan. 7).

Table 6A displays the results of these calculations for 90-strike options on the stocks in our sample averaged over the dividends they paid during 2016-17, and the liquidity-based premia for the same stocks, calculated as if they did not pay dividends. Apple, as the third most actively traded among all options, has an extremely narrow bid-ask spread, with the median half-spread B being just 2.3 pennies on the adjusted basis of S = 100, so the liquidity-related Americanness premium was only 0.013 for a two week call and 0.003 for an eight week option, very much less than the theoretical premia to get the dividends. Like Apple, calls on the other Most Active stocks also showed less value to early exercise for liquidity than for dividend but the discrepancy was smaller for them. ATT, Exxon-Mobil, and Disney had relatively wider spreads than the others and the liquidity value of Americanness for them was around 40% of the premium for dividend. By contrast, four stocks in the Less Active group paid no dividend and for three of the four others, liquidity was as important or more important than the dividend.

A 90-strike 2-week call is deep in the money. At the Most Active stocks median implied volatility, a fall of the stock price from 100 to 90 in 14 days would be a 2.8 standard deviation change. But at the median IV for Less Active stocks, that price drop would only be 1.5 standard deviations, and at the 60-day horizon it would be 0.74 standard deviations, implying a meaningful optionality component to time value. Table 6B looks at deeper in the money 80-strike contracts. These have almost no chance of being out of the money after 2 or 4 weeks, and even at 8 weeks, the Less Active group still has only about 5% chance that the underlying stock would fall below 80.

Because of their narrow bid-ask spreads, the Most Active options still show the liquidity value of American exercise to be only $0.146 on average at two weeks and smaller values for longer maturities. Even so, options on T, XOM, and DIS have liquidity-exercise premia over $0.70. The less liquid stocks show considerable value for liquidity-related exercise, averaging around half a dollar at all maturities. It is interesting to note that of the 24 stocks, depending on the maturity the options for between 10 and 12 of them have greater estimated value for liquidity- than for dividend-related American exercise.

These results provide strong evidence that liquidity-based American exercise is likely to be quantitatively important to investors in the real world.

7. Generalizing the Bid-Ask Spread Model

The model for the bid-ask spread as a function of an option's moneyness shown in (21) can be generalized to allow multiple discontinuities where the sensitivity of the spread to an increase in the option's moneyness changes. This allows a more flexible bid-ask spread relationship to be handled.

Suppose there are N prices K_i, $1 \le i \le N$ where the slope of the relationship changes. The half-spread is now given by:

$$B(S_t) \equiv B_0 + \sum_{i=1}^{N} B_i \, Max(S_t - K_i, 0) \qquad (27)$$

The K_i need to be taken into account for all values of the date t stock price for which the optimal strategy will be to sell at the market price and give up the half-spread. That is, for $S_t < S_t^*$. Assume the K_i are arranged in increasing order and define m_t as the index of the highest discontinuity below S_t^*, i.e.,

$$m_t = max\,(i \mid K_i < S_t^*) \qquad (28)$$

and define

$$B^-(S_t^*) = \sum_{i=1}^{M_t} B_i \qquad (29)$$

Proposition 4: Under the assumptions made in Propositions 1 and 2, with the exception that the bid-ask half-spread is now given by equation (27), the time 0 value of an American call on a non-dividend paying stock with strike price X and maturity T that must be liquidated with probability L(t) on some date t, $1 \le t \le T$, is given by equation (20) in which V(t) is defined as follows:

$$V(t) = V_1(t) + (1 + B^-(S_t^*))\, V_2(t) - V_3(t) - V_4(t) - V_5(t) \qquad (30)$$

where

$V_1(t)$ is given by equation (16),

$V_2(t)$ is given by equation (17),

$$V_3(t) = SN_2\left[d_1, d_2; \sqrt{t/T}\right] - Xe^{-rT}N_2\left[d_1 - \sigma\sqrt{T}\,, d_2 - \sigma\sqrt{t}\,; \sqrt{t/T}\right]$$

$$-(S_t^* - X + B^-(S_t^*)e^{-rt}N[d_2 - \sigma\sqrt{t}] \qquad (31)$$

with $B^-(S_t^*)$ given by equation (29) and d_1 and d_2 as defined in (16) and (17),

$V_4(t)$ is given by equation (25)

$$V_5(t) = \sum_{i=1}^{M} B_i(SN[d_{4i}] - K_i e^{-rt}N\left[d_{4i} - \sigma\sqrt{t}\right]) \qquad (32)$$

with $$d_{4i} = \frac{\ln(S/K_i) + (r+\sigma^2/2)t}{\sigma\sqrt{t}}$$

The proof follows from a demonstration that the payoff to optimal exercise of the American option on each possible date t can be replicated in every subinterval bounded by S_t* and the K_i , for $i \leq m$ by a package of securities that can all be valued at time 0. The full proof is provided in the Appendix. □

7. Concluding Comments

The theoretical principle that one should not exercise an American call on a non-dividend paying stock before expiration is easily proved in the classroom, where transactions costs and issues of market liquidity can be assumed away. In real world markets, however, Battalio, Figlewski, and Neal (2017) have found that options that are in the money and approaching expiration can very rarely be sold in the market for their theoretical European option values, or even for their intrinsic values. When an American option position must be liquidated before maturity, this creates an incentive to exercise, rather to sell at the best available bid in the market. American exercise has value in the real world even though it is worthless in theory, except for dividends.

This paper has derived a closed-form theoretical model in the Black-Scholes framework for the value of the early exercise option of an American call when the market's bid price is the Black-Scholes European call price minus half of the bid-ask spread. An analytic formula is particularly useful because it yields an exact solution, normally much faster than approximation techniques, and analytic expressions for its "Greek letter" derivatives are also available to be explored.

We first considered the case in which the spread was simply a constant, unrelated to option moneyness and showed how a replicating portfolio can be set up at time 0 that reproduces the payoff to optimal exercise of an American call on a specific future date t, prior to expiration. Assuming early liquidation is governed by probabilities that are independent of the stock price leads to an expression for the risk neutral value of the early exercise option. We then extended the model to allow more realistic bid-ask spread behavior. The half-spread on an option was modeled as a minimum amount plus a multiple of its intrinsic value plus a constant, and a slightly more complicated replication strategy led to a modified valuation formula.

To get an idea of the order of magnitude for the value of liquidity-based American exercise in U.S. option markets, the bid-ask spread model was fitted to option quotes for 24 active option stocks, whose trading volume in 2016-17 was among the top 10% of those reported by the Options Clearing Corporation. The results showed that the liquidity value of the American exercise feature is non-negligible, and in many cases it is as large or larger than the theoretical value of early exercise for American exercise to capture the stock's dividend.

Appendix

Proof of Proposition 4

Proposition 4 states that the payoff to an American call with strike price X and maturity T, liquidated optimally on date t prior to expiration, can be replicated by a package of optional securities that can be priced at time 0. The bid ask spread is assumed to be a function of the option's intrinsic value, with multiple prices K_i at which the sensitivity of the spread to option moneyness changes, as shown in equation (27).

Under the paper's assumptions that place the problem in the Black-Scholes framework, with M_t defined as in (28) and $B^-(S_t^*)$ defined as in (29), that replicating portfolio is the sum of five components:

1. European call with strike X and maturity T.	$V_1 = C_{BS}(S, X, T)$
2. $(1 + B^-(S_t^*))$ European calls with strike S_t^* and maturity t.	$V_2 = \left(1 + B^-(S_t^*)\right) C_{BS}(S, S_t^*, t)$
3. Short sale of a compound call on the Call in 1., with strike $S_t^* - X + B^-(S_t^*)$ and maturity t.	$-V_3 = -C_{Call}(C_{BS}(S, X, T), S_t^* - X + B^-(S_t^*), t)$
4. Borrow the present value of B_0 dollars with repayment at t.	$-V_4 = -B_0\, e^{-rt}$
5. Write B_i European calls with maturity t and strike K_i, for i = 1,...,m_t	$-V_5 = -\sum_i^{M_t} B_i C_{BS}(S, K_i, t)$

The value of the American call optimally liquidated on date t is the sum of these five components: $V(t) = V_1(t) + V_2(t) - V_3(t) - V_4(t) - V_5(t)$.

Appendix Table A1 demonstrates how this portfolio replicates the payoffs to optimal liquidation for the case with two discontinuities K_1 and K_2 below the option's strike price X. This can be extended to any number of discontinuities in an obvious way.

The table breaks down the date t state space into four price ranges. In the first three, optimal liquidation is to sell the call in the market, with the bid-ask half-spread equal to, respectively: the minimum value B_0, for $S_t < K_1$; $B_0 + B_1(S_t - K_1)$, for $K_1 \leq S_t < K_2$; and $B_0 + B_1(S_t - K_1) + B_2(S_t - K_2)$, for $K_2 \leq S_t < S_t^*$. In the fourth range, $S_t^* \leq S_t$, the American call is exercised, the payoff is $S_t - X$ and the half-spread is not paid. Note that the option will be sold in the market only for stock prices $S_t < S_t^*$ so any discontinuities that might exist in the bid-ask function above the exercise boundary, S_t^* will not affect the net payoff on date t.

The first component in the replicating portfolio is the European call with the same terms as the American call. This option provides the payoff on expiration date T in the case where the optimal strategy on date t is not to exercise. Its value V_1 is the Black-Scholes model value given by equation (16).

The second component V_2 is the value of a European call with strike price S_t* that matures on the early liquidation date t multiplied by 1 plus the sum of the B_i slope coefficients in the bid ask spread function eq. (27) for all discontinuities below S_t*. The Black-Scholes value of the call is given by eq. (17). This second component has two functions. One unit of call 2 provides the portion of the payoff above S_t* when the American call is exercised. The extra (B_1 + B_2) of these calls serves to offset the part of the bid-ask spread that depends on the stock price, which enters from component 5, for the cases where the American call is exercised. To make it easier to see how these payoffs offset, the payoff to the V_2 component for the case $S_t \leq S_t$* is broken out in detail.

The third component of the replicating portfolio, -V_3, is the compound call on the call in step 1. It has maturity t and strike $S_t^* - X + B^-(S_t^*)$. As discussed in the text and illustrated in Figure 1 this strike is the value of the call option 1 at stock price S_t* on date t. At any lower stock price, option 1 is worth less than this amount and the compound call will expire worthless. Above S_t* it will be exercised, which will remove option 1 from the portfolio and add the strike price to the payoff. The $B_0 + B^-(S_t^*)$ term in the strike price offsets the bid-ask half-spread that enters from components 4 and 5 at S_t*.

Component 4 produces a negative payoff of the minimum bid-ask spread on date t at every stock price and component 5 adds the portion of the half-spread that depends on the option's intrinsic value. V_4 and the option payoffs in V_5 at S_t* are offset for the case where the call is exercised, by receipt of the strike price from the compound call option 3. Above S_t* the stock price dependent V_5 is offset by the payoff on (B_1 + B_2) of the second call.

Thus in each range for the date t stock price, this replicating portfolio whose time 0 value is given by V(t) produces the same return as optimal liquidation of the American call. Multiplying each V(t) by the risk neutral probability of liquidation on date future date t ≤ T gives the date 0 value of the American call, equation (30).

Table A1: Replicating Portfolio for Maturity T American Call with Strike Price X

Exercised Early on Date t When the Bid-Ask Spread has Two Discontinuities Below S_t*

Position		$S_t < K_1$	$K_1 \leq S_t < K_2$	$K_2 \leq S_t < S_t$*	S_t* $\leq S_t$
1	$C_{BS}(S,X,T)$	$C_{BS}(S,X,T)$	$C_{BS}(S,X,T)$	$C_{BS}(S,X,T)$	*Called by exercise of Option 3*
2	$(1+B_1+B_2)C_{BS}(S,S_t^*,t)$	0	0	0	S_t - S_t* + B_1S_t - B_1S_t* + B_2S_t - B_2S_t*
3	$-C_{Call}(C_{BS}(S,X,T-t),$ $S_t^* - X + B(S_t^*),t)$	0	0	0	S_t*- X + B_0 + B_1 (S_t* - K_1) + B_2 (S_t* - K_2)
4	Borrow $B_0\, e^{-rt}$	-B_0	-B_0	-B_0	-B_0
5	$-B_1C_{BS}(S,K_1,t)$ $-B_2C_{BS}(S,K_2,t)$	0	- B_1 (S_t - K_1)	- B_1 (S_t - K_1) - B_2 (S_t - K_2)	- B_1 (S_t - K_1) - B_2 (S_t – K_2)
TOTAL		$C_{BS}(S,X,T)$ $-B(S_t)$	$C_{BS}(S,X,T)$ $-B(S_t)$	$C_{BS}(S,X,T)$ $-B(S_t)$	S_t - X

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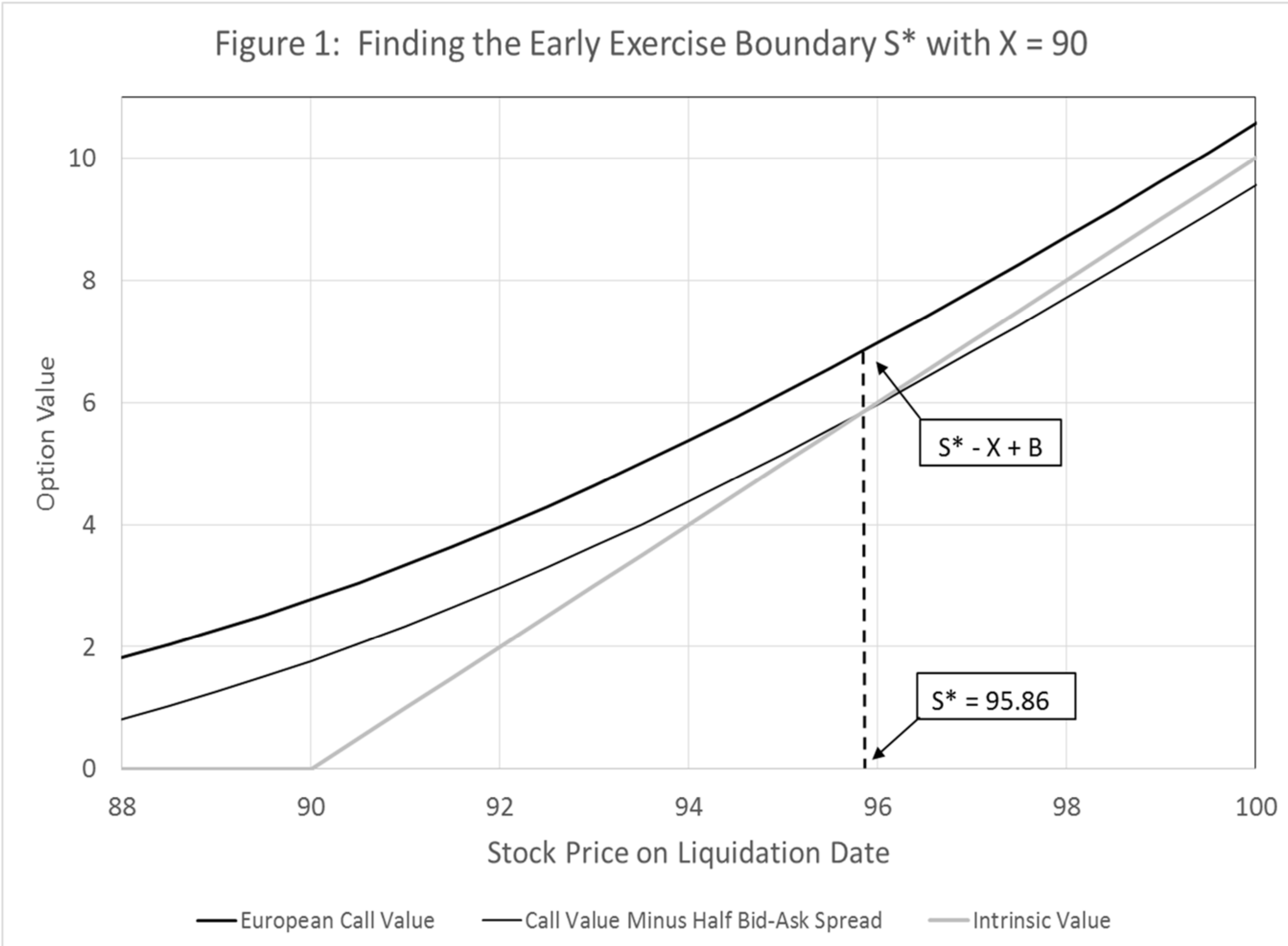
Figure 1: Finding the Early Exercise Boundary S* with X = 90
Option Value
0
2
4
6
8
10
88
90
92
94
96
98
100
Stock Price on Liquidation Date
S* - X + B
S* = 95.86
European Call Value
Call Value Minus Half Bid-Ask Spread
Intrinsic Value

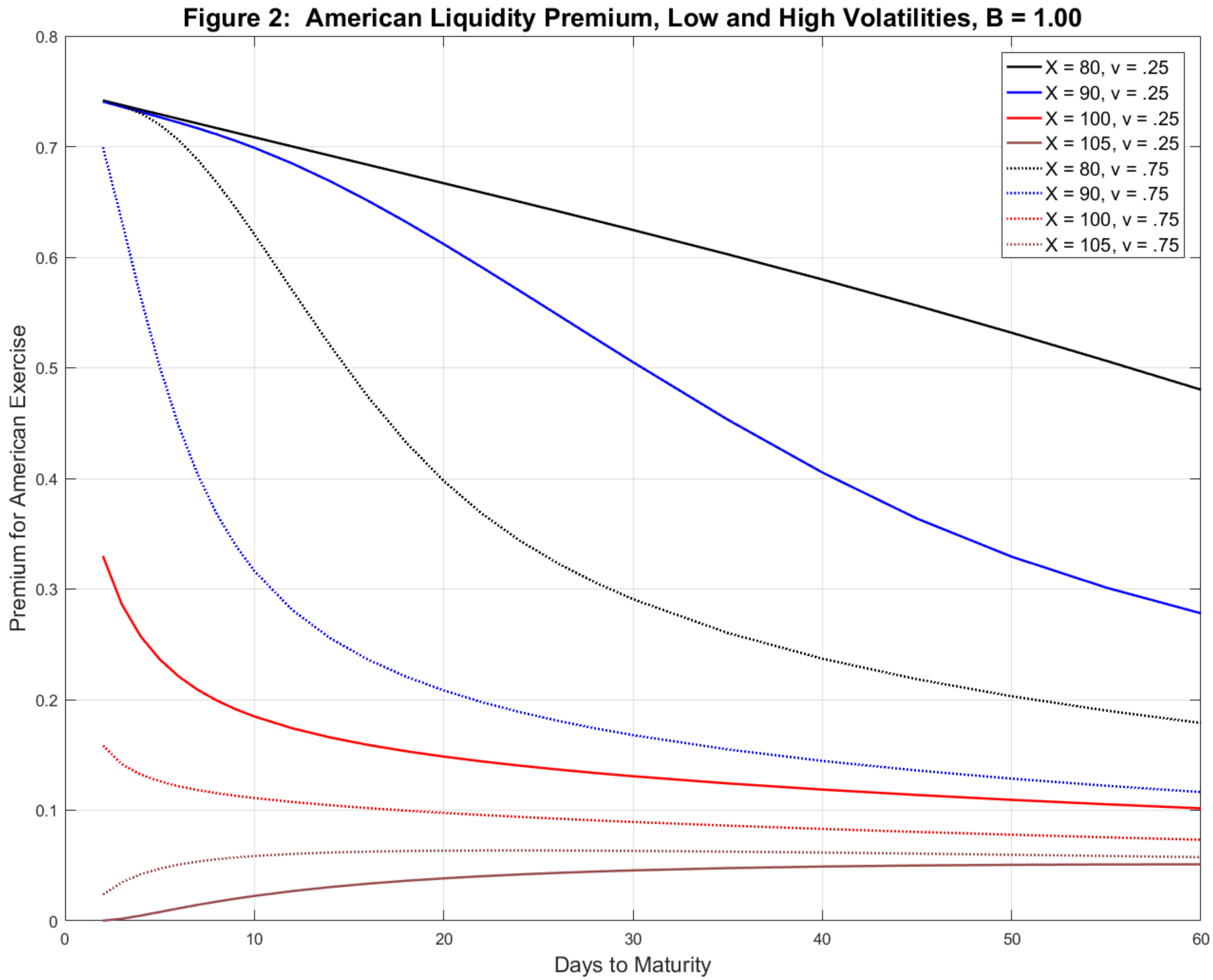

Figure 2: American Liquidity Premium, Low and High Volatilities, B = 1.00
X = 80, v = .25
X = 90, v = .25
X = 100, v = .25
X = 105, v = .25
X = 80, v = .75
X = 90, v = .75
X = 100, v = .75
X = 105, v = .75
Premium for American Exercise
Days to Maturity
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
10
20
30
40
50
60

Figure 3: Option Bid-Ask Spreads and Fitted Spread Model for All Stocks and Subsets by Trading Activity

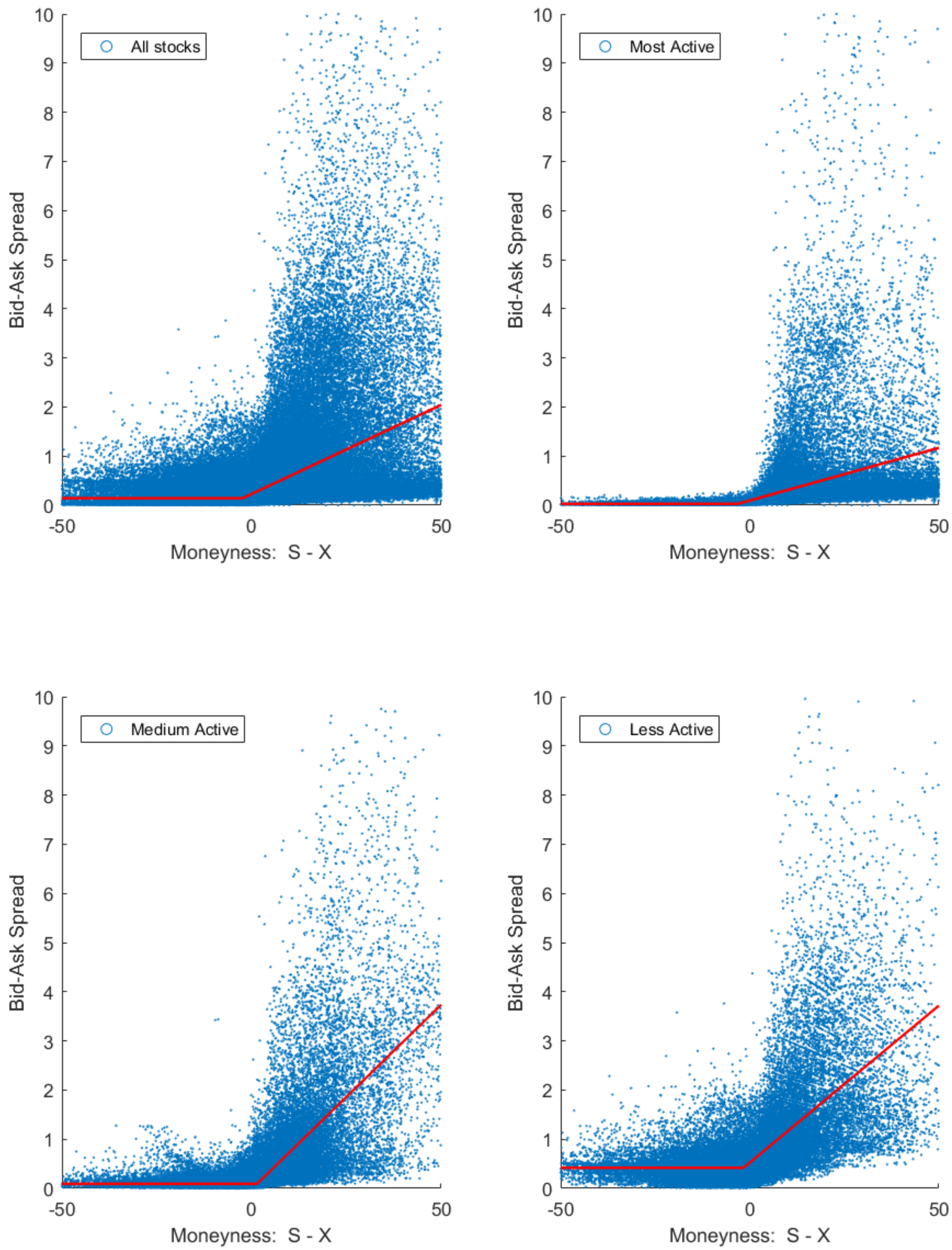

Figure 4A: Option Bid-Ask Spreads and Fitted Spread Model for Most Active Firms

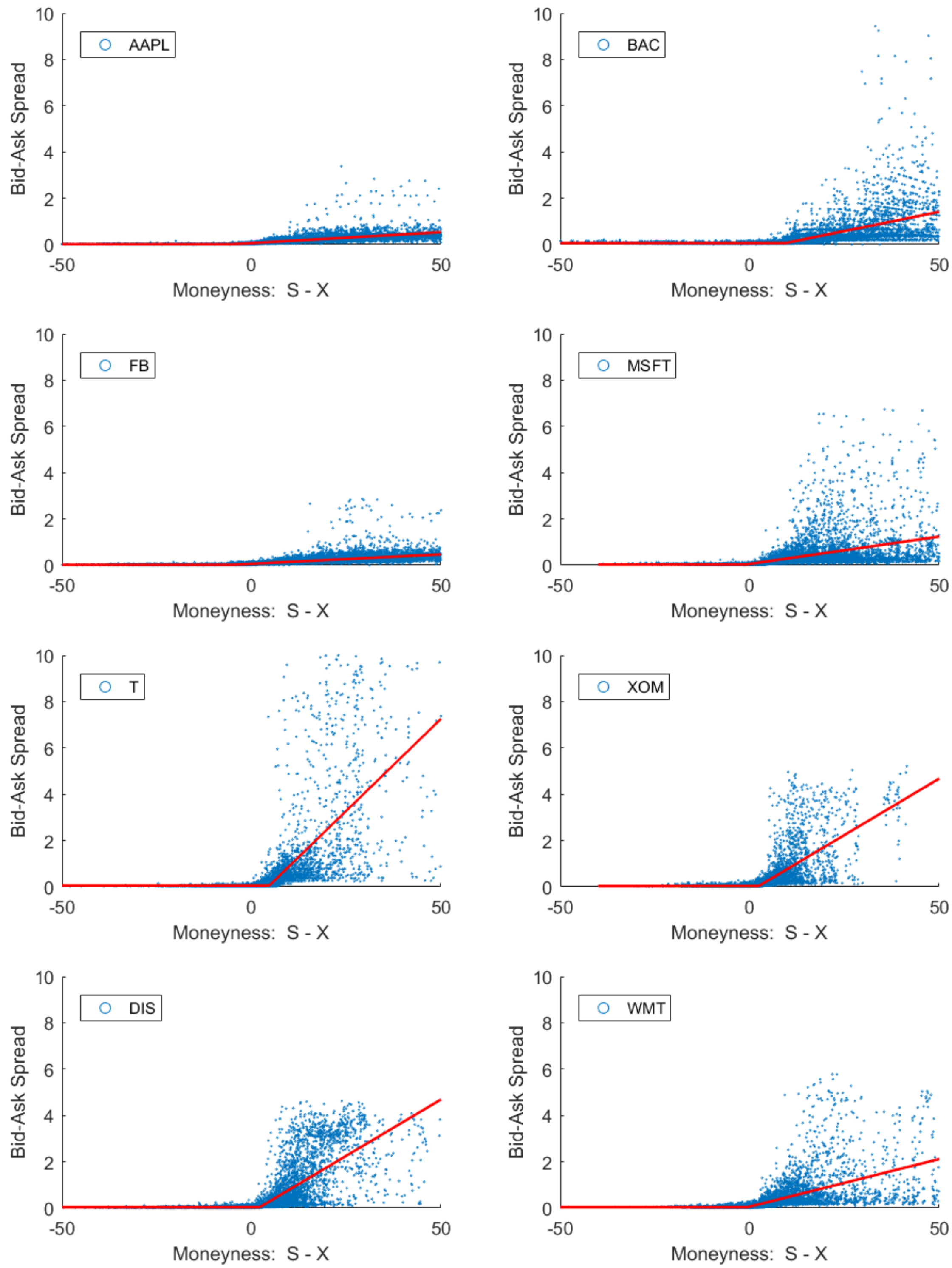

Figure 4B: Option Bid-Ask Spreads and Fitted Spread Model for Medium Active Firms

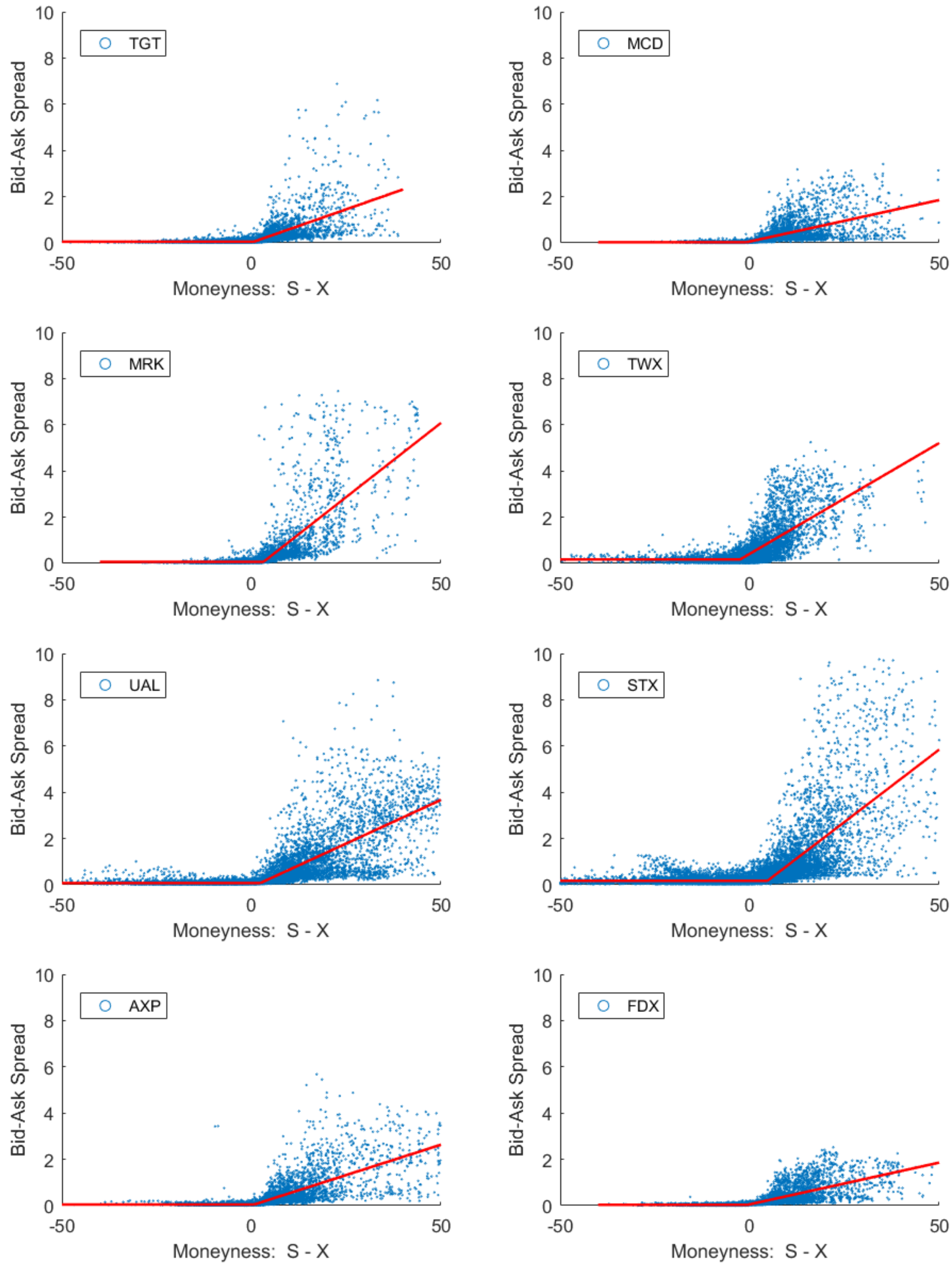

Figure 4C: Option Bid-Ask Spreads and Fitted Spread Model for Less Active Firms

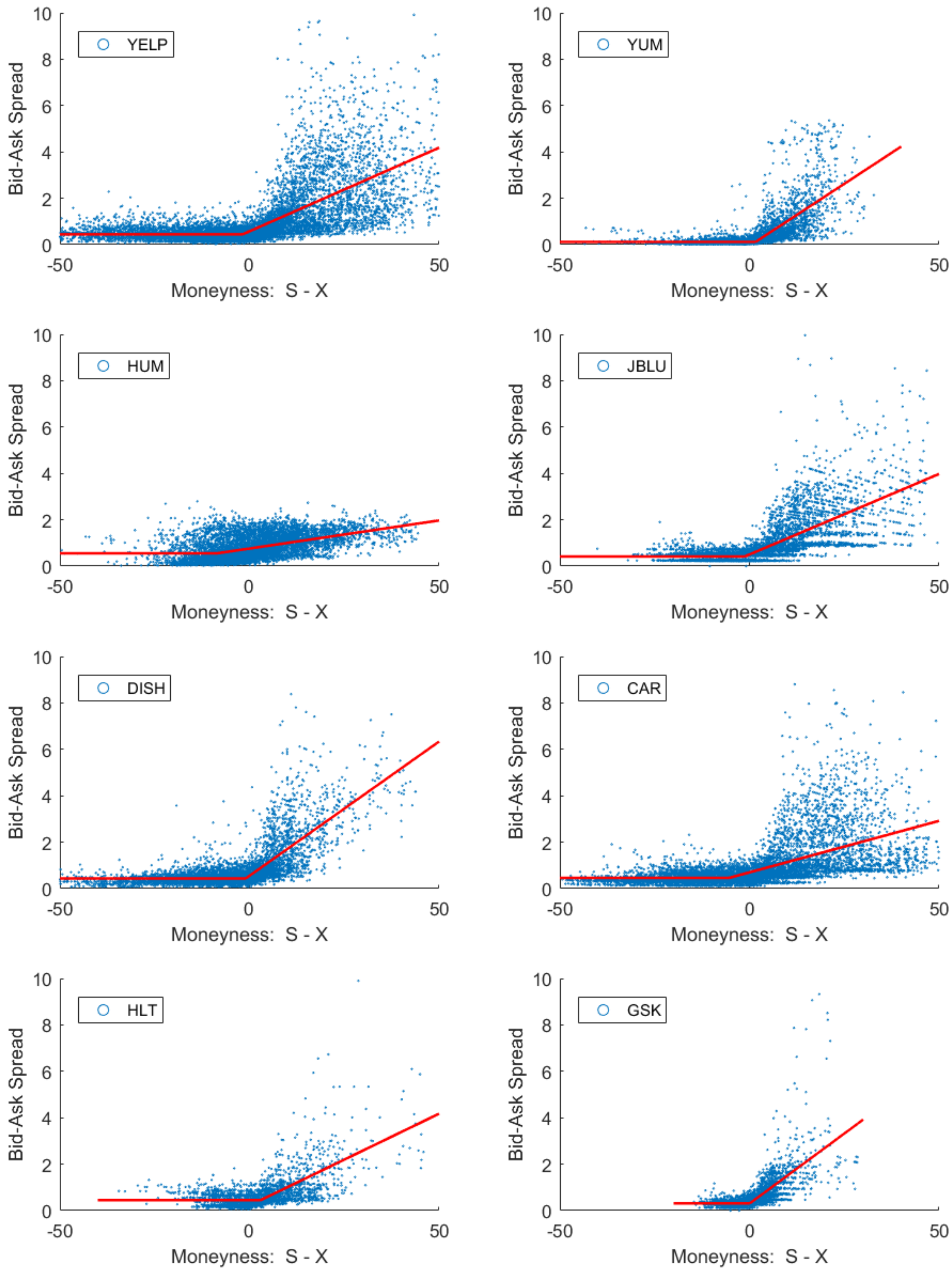

Figure 5: All Stocks--American Liquidity Premium by Moneyness and Maturity

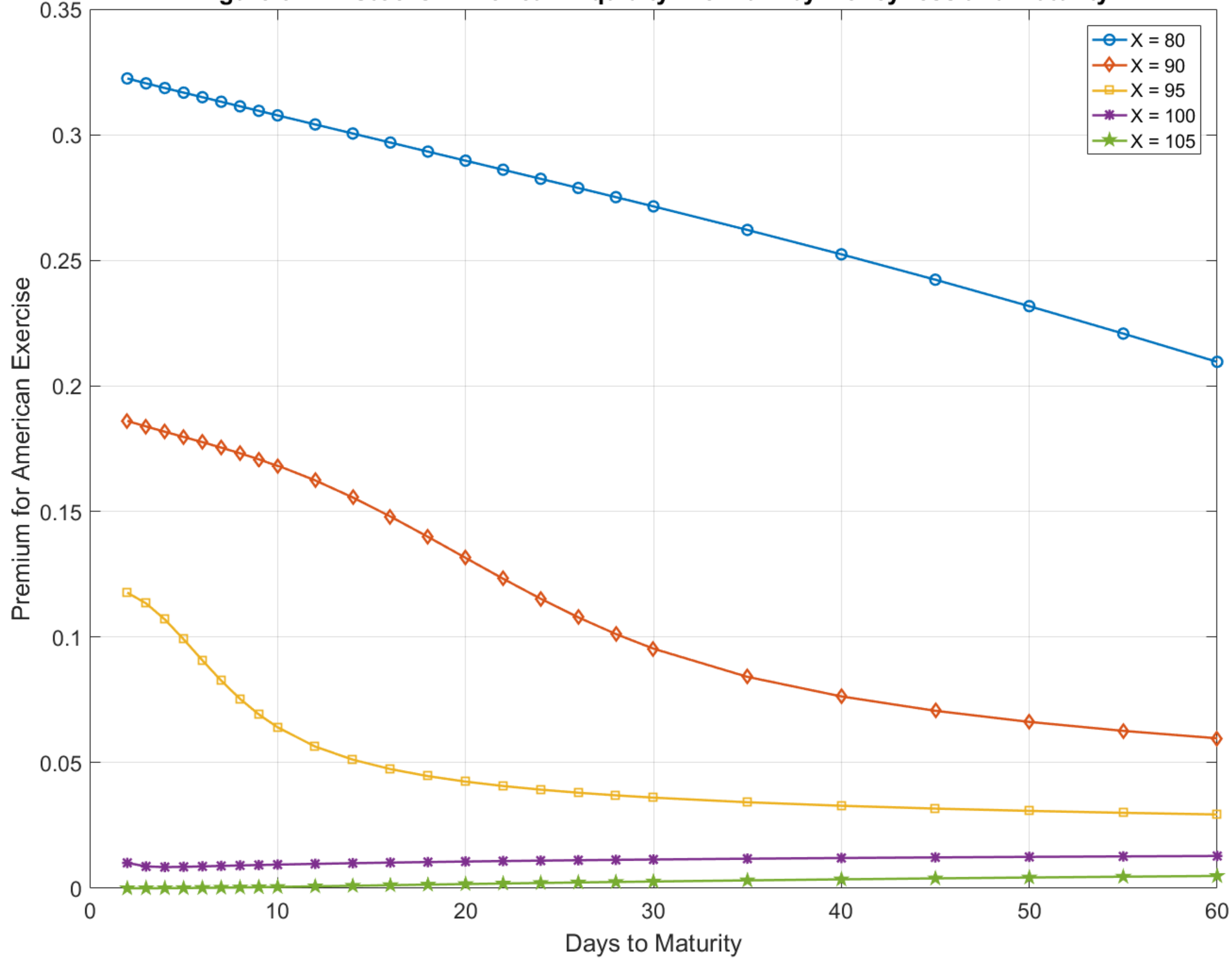

Table 1: Sample Firms Ranked by Option Volume 2016-17

Firms	Contracts traded 2016-7	Rank	Name
ALL	1,300,890,699		
AAPL	204,508,556	3	APPLE INC
BAC	146,861,688	6	BANK AMER CORP
FB	102,251,275	8	FACEBOOK INC
MSFT	42,659,552	20	MICROSOFT CORP
T	33,767,289	32	AT&T INC
XOM	21,314,995	42	EXXON MOBIL CORP
DIS	18,516,832	48	DISNEY WALT CO
WMT	15,210,314	53	WAL-MART STORES INC
MOST ACTIVE	585,090,501		
TGT	10,130,729	85	TARGET CORP
MCD	9,059,196	96	MCDONALDS CORP
MRK	8,226,955	107	MERCK & CO INC
TWX	7,369,121	122	TIME WARNER INC
UAL	6,807,250	135	UNITED CONTL HLDGS INC
STX	5,962,300	148	SEAGATE TECHNOLOGY PLC
AXP	5,058,378	171	AMERICAN EXPRESS CO
FDX	4,518,700	185	FEDEX CORP
MEDIUM ACTIVE	57,132,629		
YELP	3,091,431	240	YELP INC
YUM	2,831,669	250	YUM BRANDS INC
HUM	2,511,313	272	HUMANA INC
JBLU	2,278,091	293	JETBLUE AIRWAYS CORP
DISH	1,879,835	338	DISH NETWORK CORP
CAR	1,508,926	383	AVIS BUDGET GROUP
HLT	1,256,888	431	HILTON WORLDWIDE HLDGS INC
GSK	1,086,286	473	GLAXOSMITHKLINE PLC
LESS ACTIVE	16,444,439		

Table 2: Sample Construction Criteria

Qualitative Constraints

Calls / Puts	Calls only
Bid price minimum	> 0
Bid-Ask spread minimum	> 0
Liquidation probability decay	exponential
Liquidation at expiration	25%
Ex-dividend horizon	50% of option lifetime

Quantitative Constraints

	Min	Max	In-sample average
Maturity range	3	65	34.0
Exercise price range	50	150	98.61
Open interest	50	no upper limit	8162
ATM implied volatility	0	no upper limit	0.263
Bid minus intrinsic value	-10	no upper limit	5.57
OTM bid-ask spread too wide	0	5	0.143

Table 3: Characteristics of Bid-Ask Spreads in the Sample

	Bid-Ask Spread		Median IV	NOBS	Best Bid < Intrinsic Value		
	median	**75th percentile**			**Nobs**	**Median size**	**75th percentile**
ALL FIRMS	0.141	0.430	0.223	209997	30351	0.188	0.515
MOST ACTIVE	0.062	0.203	0.193	96656	13812	0.104	0.257
MEDIUM ACTIVE	0.112	0.339	0.253	65372	9158	0.259	0.654
LESS ACTIVE	0.508	0.920	0.352	47969	7381	0.374	0.702
Most active firms							
AAPL	0.047	0.187	0.199	18619	2600	0.065	0.105
BAC	0.076	0.322	0.253	11495	1957	0.138	0.250
FB	0.078	0.177	0.215	18842	2203	0.062	0.116
MSFT	0.080	0.240	0.186	11775	1858	0.116	0.266
T	0.052	0.204	0.153	8495	1406	0.179	0.483
XOM	0.038	0.165	0.161	8546	888	0.239	0.687
DIS	0.047	0.174	0.172	10045	1433	0.474	1.239
WMT	0.060	0.261	0.174	8839	1467	0.123	0.294
Medium active							
TGT	0.066	0.182	0.209	6378	535	0.160	0.392
MCD	0.064	0.210	0.148	7426	1170	0.100	0.256
MRK	0.079	0.317	0.179	6399	895	0.264	0.676
TWX	0.320	0.802	0.189	5563	1177	0.421	0.865
UAL	0.102	0.390	0.338	11516	1711	0.388	1.009
STX	0.179	0.463	0.363	13990	1670	0.393	0.996
AXP	0.063	0.239	0.220	7280	1078	0.171	0.430
FDX	0.068	0.202	0.192	6820	922	0.181	0.401
Less active							
YELP	0.535	1.004	0.446	9706	1341	0.511	0.990
YUM	0.147	0.360	0.253	5556	785	0.249	0.591
HUM	0.731	1.225	0.250	6691	1260	0.321	0.503
JBLU	0.553	0.987	0.333	5864	1259	0.396	0.758
DISH	0.492	0.787	0.355	6113	568	0.600	1.146
CAR	0.583	0.970	0.501	7913	1094	0.415	0.832
HLT	0.519	0.836	0.245	2689	444	0.202	0.479
GSK	0.387	0.637	0.178	3437	630	0.274	0.506

Table 4: Bid-Ask Spread Model Parameters

	Median B/A in sample	Spread Model Parameters			
		Minimum B/A	**M where B/A widens**	**B/A slope**	**R-squared**
ALL FIRMS	0.141	0.143	-2.378	0.036	0.228
MOST ACTIVE	0.062	0.027	-3.207	0.021	0.170
MEDIUM ACTIVE	0.112	0.091	1.411	0.075	0.476
LESS ACTIVE	0.508	0.417	-1.678	0.064	0.410
Most active firms					
AAPL	0.047	0.012	-4.984	0.009	0.602
BAC	0.076	0.061	9.434	0.033	0.267
FB	0.078	0.019	-4.909	0.008	0.477
MSFT	0.080	0.027	-1.101	0.024	0.250
T	0.052	0.055	4.696	0.159	0.510
XOM	0.038	0.034	2.365	0.097	0.536
DIS	0.047	0.032	2.134	0.097	0.580
WMT	0.060	0.033	-0.710	0.041	0.381
Medium active					
TGT	0.066	0.054	0.640	0.057	0.439
MCD	0.064	0.028	-0.860	0.036	0.451
MRK	0.079	0.068	3.156	0.128	0.580
TWX	0.320	0.172	-2.531	0.096	0.543
UAL	0.102	0.075	2.339	0.075	0.614
STX	0.179	0.172	4.680	0.125	0.553
AXP	0.063	0.044	0.662	0.052	0.570
FDX	0.068	0.034	-0.566	0.036	0.569
Less active					
YELP	0.535	0.445	-1.808	0.072	0.462
YUM	0.147	0.118	1.623	0.107	0.542
HUM	0.731	0.551	-8.585	0.024	0.238
JBLU	0.553	0.416	-1.226	0.070	0.509
DISH	0.492	0.440	-0.979	0.116	0.599
CAR	0.583	0.459	-5.443	0.044	0.334
HLT	0.519	0.448	2.911	0.079	0.344
GSK	0.387	0.310	0.077	0.121	0.479

Table 5: Liquidity-Based Premia for American Exercise

by Moneyness and Maturity

Strike Price	Category	Option Maturity			
		1 week	2 weeks	4 weeks	8 weeks
80	All Stocks	0.315	0.305	0.284	0.231
	Most Active	0.161	0.150	0.130	0.083
	Medium Active	0.587	0.577	0.556	0.493
	Less Active	0.625	0.615	0.579	0.454
90	All Stocks	0.178	0.161	0.118	0.082
	Most Active	0.079	0.066	0.043	0.027
	Medium Active	0.303	0.281	0.224	0.173
	Less Active	0.371	0.307	0.229	0.188
95	All Stocks	0.089	0.063	0.050	0.043
	Most Active	0.032	0.023	0.018	0.014
	Medium Active	0.133	0.105	0.093	0.091
	Less Active	0.170	0.133	0.119	0.116
100	All Stocks	0.013	0.015	0.017	0.020
	Most Active	0.004	0.005	0.005	0.006
	Medium Active	0.019	0.025	0.032	0.044
	Less Active	0.041	0.047	0.056	0.069
105	All Stocks	0.000	0.002	0.004	0.008
	Most Active	0.000	0.000	0.001	0.002
	Medium Active	0.001	0.003	0.009	0.019
	Less Active	0.006	0.013	0.023	0.039

Table 6A: Liquidity and Dividend Value of American Exercise, X = 90

Stock	Average dividend	2 weeks		4 weeks		8 weeks	
		Dividend	Liquidity	Dividend	Liquidity	Dividend	Liquidity
AAPL	0.477	0.446	0.013	0.306	0.006	0.241	0.003
BAC	0.386	0.336	0.107	0.240	0.065	0.140	0.044
FB	0	0	0.011	0	0.005	0	0.003
MSFT	0.614	0.574	0.070	0.535	0.039	0.274	0.021
T	1.247	1.205	0.590	1.172	0.558	1.041	0.459
XOM	0.908	0.878	0.351	0.817	0.317	0.674	0.220
DIS	0.760	0.739	0.349	0.699	0.311	0.499	0.211
WMT	0.678	0.656	0.138	0.607	0.101	0.414	0.054
MOST ACTIVE	0.721	0.677	0.061	0.608	0.032	0.455	0.018
TGT	0.916	0.767	0.204	0.686	0.153	0.501	0.093
MCD	0.697	0.679	0.117	0.642	0.086	0.545	0.041
MRK	0.779	0.763	0.479	0.720	0.438	0.558	0.321
TWX	0.462	0.385	0.396	0.336	0.349	0.225	0.232
UAL	0	0	0.228	0	0.160	0	0.130
STX	1.875	1.481	0.421	1.142	0.310	0.864	0.253
AXP	0.439	0.403	0.181	0.335	0.127	0.225	0.080
FDX	0.221	0.191	0.119	0.157	0.079	0.072	0.044
MEDIUM ACTIVE	0.770	0.736	0.273	0.645	0.199	0.442	0.134
YELP	0	0	0.261	0	0.188	0	0.159
YUM	0.508	0.440	0.404	0.342	0.319	0.231	0.218
HUM	0.164	0.117	0.254	0.075	0.166	0.021	0.088
JBLU	0	0	0.323	0	0.215	0	0.156
DISH	0	0	0.483	0	0.347	0	0.265
CAR	0	0	0.158	0	0.119	0	0.102
HLT	0.275	0.235	0.424	0.156	0.334	0.067	0.207
GSK	1.502	1.477	0.541	1.405	0.499	1.099	0.371
LESS ACTIVE	0.612	0.548	0.288	0.511	0.190	0.453	0.142

Table 6B: Liquidity and Dividend Value of American Exercise, X = 80

Stock	Average dividend	2 weeks		4 weeks		8 weeks	
		Dividend	Liquidity	Dividend	Liquidity	Dividend	Liquidity
AAPL	0.477	0.461	0.050	0.439	0.025	0.387	0.010
BAC	0.386	0.369	0.247	0.347	0.221	0.286	0.149
FB	0	0	0.043	0	0.020	0	0.008
MSFT	0.614	0.598	0.162	0.583	0.137	0.512	0.079
T	1.247	1.208	1.189	1.192	1.165	1.162	1.109
XOM	0.908	0.893	0.720	0.876	0.695	0.836	0.638
DIS	0.760	0.742	0.717	0.725	0.693	0.685	0.636
WMT	0.678	0.662	0.296	0.647	0.270	0.589	0.212
MOST ACTIVE	0.721	0.692	0.146	0.674	0.121	0.623	0.064
TGT	0.916	0.899	0.426	0.880	0.401	0.822	0.339
MCD	0.697	0.681	0.255	0.666	0.230	0.635	0.172
MRK	0.779	0.767	0.964	0.751	0.940	0.714	0.883
TWX	0.462	0.388	0.760	0.404	0.735	0.354	0.677
UAL	0	0	0.569	0	0.532	0	0.397
STX	1.875	1.838	0.979	1.703	0.932	1.519	0.755
AXP	0.439	0.422	0.385	0.406	0.360	0.368	0.296
FDX	0.221	0.203	0.259	0.174	0.233	0.136	0.174
MEDIUM ACTIVE	0.770	0.827	0.572	0.807	0.547	0.674	0.468
YELP	0	0	0.675	0	0.582	0	0.383
YUM	0.508	0.493	0.824	0.449	0.798	0.396	0.719
HUM	0.164	0.143	0.363	0.120	0.337	0.069	0.258
JBLU	0	0	0.653	0	0.616	0	0.470
DISH	0	0	1.008	0	0.963	0	0.785
CAR	0	0	0.460	0	0.342	0	0.215
HLT	0.275	0.261	0.737	0.242	0.712	0.177	0.635
GSK	1.502	1.501	0.997	1.471	0.973	1.422	0.916
LESS ACTIVE	0.612	0.581	0.610	0.586	0.567	0.642	0.408