Empirical Financial Economics

The Efficient Markets Hypothesis - Generalized Method of Moments
Random Walk Hypothesis

Random Walk hypothesis a special case of EMH

\[ E(r_{t+\tau}) = \tau \mu \]
\[ E(r_{t+\tau}^2) = \tau \sigma^2 + \tau^2 \mu^2 \]
\[ \tau = 1, 2, \ldots \]

Overidentification of model

- Provides a test of model (variance ratio criterion)
- Allows for estimation of parameters (GMM paradigm)
Variance ratio tests

\[ VR(\tau) = \frac{\text{Var}(r_{t+\tau})}{\tau \text{Var}(r_t)} = 1 + 2 \sum_{k=1}^{\tau-1} (1 - \frac{k}{\tau}) \rho(k) \]

using sample quantities

\[ \hat{\sigma}^2 = \frac{1}{n\tau} \sum_{\ell=1}^{n\tau} (r_{\ell+1} - \hat{\mu})^2 \]

\[ \hat{\sigma}^2(\tau) = \frac{1}{n} \sum_{\ell=1}^{n} (r_{\ell+\tau} - \tau \hat{\mu})^2 \]

The variance ratio \( \hat{VR}(\tau) = \frac{\hat{\sigma}^2(\tau)}{\tau \hat{\sigma}^2} \) is asymptotically Normal

\[ \sqrt{n\tau} (\hat{VR}(\tau) - 1) \xrightarrow{a} N[0, 2(\tau - 1)] \]
Overlapping observations

Non-overlapping observations

Overlapping observations

\[ \hat{\sigma}^2 = \frac{1}{n\tau - 1} \sum_{t=1}^{n\tau} (r_{t+1} - \hat{\mu})^2 \]

\[ \hat{\sigma}^2(\tau) = \frac{1}{m} \sum_{t=\tau}^{n\tau} (r_{t+\tau} - \tau\hat{\mu})^2; \quad m = \tau(n\tau - \tau + 1) \left(1 - \frac{1}{n}\right) \]

Variance ratio is asymptotically Normal

\[ \sqrt{n\tau}(V\hat{R}(\tau) - 1) \overset{a}{\sim} N[0, 2\frac{(2\tau - 1)(\tau - 1)}{3\tau}] \]

Random walk model and GMM

\[ r_{t+1} = \mu + v_{1t} \]
\[ r_{t+j}^2 = j\sigma^2 + j^2\mu^2 + v_{2t} \]
\[ r_{t+k}^2 = k\sigma^2 + k^2\mu^2 + v_{3t} \]

aggregate into moment conditions:

\[ \frac{1}{T} \sum r_{t+1} = \mu + \frac{1}{T} \sum v_{1t} \]
\[ \frac{1}{T} \sum r_{t+j}^2 = j\sigma^2 + j^2\mu^2 + \frac{1}{T} \sum v_{2t} \]
\[ \frac{1}{T} \sum r_{t+k}^2 = k\sigma^2 + k^2\mu^2 + \frac{1}{T} \sum v_{3t} \]

and express as three observations of a nonlinear regression model:

\[ y_1 = X_{11}\mu + X_{21}\sigma^2 + X_{31}\mu^2 + w_1 \]
\[ y_2 = X_{12}\mu + X_{22}\sigma^2 + X_{32}\mu^2 + w_2 \]
\[ y_3 = X_{13}\mu + X_{23}\sigma^2 + X_{33}\mu^2 + w_3 \]
An Aside on Linear Least Squares

\[ y = X \beta + u \Rightarrow \hat{u} = y - X \hat{\beta} \]

\[
\min_{\hat{\beta}} \hat{u}'A\hat{u}
\]

\[
\text{FOC: } \frac{\partial \hat{u}'}{\partial \hat{\beta}} A\hat{u} = 0 \Rightarrow X' A\hat{u} = 0
\]

Choose \( \hat{\beta} : \)
\[
X' A \left[ y - X \hat{\beta} \right] = 0
\]

\[
\hat{\beta} = \left[ X' A X \right]^{-1} X' A y = \beta + \left[ X' A X \right]^{-1} X' A u
\]

\[
\text{Var } \hat{\beta} = \left[ X' A X \right]^{-1} X' A \left( E u u' \right) A X \left[ X' A X \right]^{-1}
\]
An Aside on Nonlinear Least Squares

\[ y = f(X, \beta) + u \Rightarrow \hat{u} = y - f(X, \hat{\beta}) \]

\[ \approx (y - y_0) - D \left[ \hat{\beta} - \beta_0 \right] \]

\[ \text{Min } \hat{u}'A\hat{u} \]

\[ FOC : \quad \frac{\partial \hat{u}'}{\partial \beta} A\hat{u} = 0 \Rightarrow D'A\hat{u} = 0 \]

\[ \text{Choose } \hat{\beta} : \quad D'A \left[ (y - y_0) - D \left[ \hat{\beta} - \beta_0 \right] \right] = 0 \]

\[ \hat{\beta} = \left( D'AD \right)^{-1} D'A(y - y_0) + \beta_0 = \beta + \left( D'AD \right)^{-1} D'Au \]

\[ \text{Var } \hat{\beta} = \left( D'AD \right)^{-1} D'A(Euu')AD\left( D'AD \right)^{-1} \]
Generalized method of moment estimators

Choose \( \mu, \sigma \) to minimize \( w'Aw \). \( A \) is referred to as the optimal weighting matrix, equal to the inverse covariance matrix of \( w \)

- Estimators are asymptotically Normal and efficient
- Minimand is distributed as Chi-square with d.f. number of overidentifying information

Methods of obtaining \( A \)

1. Set \( A = I \) (Ordinary Least Squares). Estimate model.
   Set \( A = \Sigma^{-1} \) (Generalized Least Squares). Reestimate.

2. Use analytic methods to infer \( \Sigma \)
GMM and the Efficient Market Hypothesis

1 asset and 1 instrument: \[ E \left\{ (r_{j,t+\tau} - E[r_{j,t+\tau} | x_t, \theta_j]) \times z_t \right\} = 0 \]
... 1 equation and k unknowns:

m assets and 1 instrument: \[ E \left\{ (r_{i+\tau} - E[r_{t+\tau} | x_t, \theta]) \times z_t \right\} = 0 \]
... m equations and >k unknowns:

m assets and n instrument: \[ E \left\{ (r_{i+\tau} - E[r_{t+\tau} | x_t, \theta]) \times Z_t \right\} = 0 \]
... mn equations and >k unknowns:

\[ f_t(\theta) = E \left\{ (r_{i+\tau} - E[r_{t+\tau} | x_t, \theta]) \times Z_t \right\}; \quad g_t(\theta) = T^{-1} \sum_{i=1}^{T} f_i(\theta) \]

Autocovariances and cross autocovariances

\( \gamma_{xx}(k) \)

\( \gamma_{xy}(-k) = \gamma_{yx}(k) \)

\( \gamma_{xx}(k) \)

\( \gamma_{xy}(k) \)

\( \gamma_{yy}(k) \)

\( x_t \)

\( y_t \)

\( t-k \)

\( t \)

\( t+k \)
Cross autocovariances are not symmetrical!

Autocovariances are given by:

\[
\gamma_{xx}(k) = E[(x_t - \mu_x)(x_{t+k} - \mu_x)] = E[(x_t - \mu_x)(x_{t-k} - \mu_x)] = \gamma_{xx}(-k)
\]

\[
\gamma_{yy}(k) = E[(y_t - \mu_y)(y_{t+k} - \mu_y)] = E[(y_t - \mu_y)(y_{t-k} - \mu_y)] = \gamma_{yy}(-k)
\]

Cross autocovariances are given by:

\[
\gamma_{xy}(k) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)] = E[(y_t - \mu_y)(x_{t-k} - \mu_x)] = \gamma_{yx}(-k)
\]
Cross autocovariances and the weighting function

\[
\text{For } w = \left[ \frac{1}{T} \sum_{t=1}^{T} x_t \right] \\
\frac{1}{T} \sum_{t=1}^{T} x_t
\]

\[
\text{Cov}(w) = \frac{1}{T^2} \left[ \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{x_t x_{\tau}} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{x_t y_{\tau}} \right] \\
\sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{y_t x_{\tau}} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{y_t y_{\tau}}
\]
Assuming stationarity

\[
\text{Cov}(x) = \begin{bmatrix}
\gamma_{xx}(0) & \gamma_{xx}(1) & \gamma_{xx}(2) & \gamma_{xx}(3) & \gamma_{xx}(4) \\
\gamma_{xx}(1) & \gamma_{xx}(0) & \gamma_{xx}(1) & \gamma_{xx}(2) & \gamma_{xx}(3) \\
\gamma_{xx}(2) & \gamma_{xx}(1) & \gamma_{xx}(0) & \gamma_{xx}(1) & \gamma_{xx}(2) \\
\gamma_{xx}(3) & \gamma_{xx}(2) & \gamma_{xx}(1) & \gamma_{xx}(0) & \gamma_{xx}(1) \\
\gamma_{xx}(4) & \gamma_{xx}(3) & \gamma_{xx}(2) & \gamma_{xx}(1) & \gamma_{xx}(0) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

and so

\[
\frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{x_{t}x_{\tau}} = \gamma_{xx}(0) + 2 \sum_{k=1}^{T-1} \frac{T - k}{T} \gamma_{xx}(k) \approx \gamma_{xx}(0) + 2 \sum_{k=1}^{\infty} \gamma_{xx}(k)
\]
Apply this to cross covariances

\[ \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{x_t y_\tau} = \gamma_{xy} (0) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xy} (k) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xy} (-k) \]

\[ = \gamma_{xy} (0) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xy} (k) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yx} (k) \]

and

\[ \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \sigma_{y_t x_\tau} = \gamma_{yx} (0) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yx} (k) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yx} (-k) \]

\[ = \gamma_{yx} (0) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yx} (k) + \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xy} (k) \]
A simple expression for the inverse weighting matrix

\[
\text{Cov}(w) = \frac{1}{T^2} \left[ \sum_{t=1}^{T} \sum_{\tau=1}^{\tau} \sigma_{x_t x_{\tau}} \sum_{t=1}^{T} \sum_{\tau=1}^{\tau} \sigma_{x_t y_{\tau}} + \sum_{t=1}^{T} \sum_{\tau=1}^{\tau} \sigma_{y_t x_{\tau}} \sum_{t=1}^{T} \sum_{\tau=1}^{\tau} \sigma_{y_t y_{\tau}} \right] = \frac{A + A'}{2}
\]

where

\[
A = \begin{bmatrix}
\gamma_{xx} (0) + 2 \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xx} (k) & \gamma_{xy} (0) + 2 \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{xy} (k) \\
\gamma_{yx} (0) + 2 \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yx} (k) & \gamma_{yy} (0) + 2 \sum_{k=1}^{T-1} \frac{T-k}{T} \gamma_{yy} (k)
\end{bmatrix}
\]

\[
A' = \begin{bmatrix}
\gamma_{xx} (0) + \sum_{k=1}^{\infty} \gamma_{xx} (k) & \gamma_{xy} (0) + \sum_{k=1}^{\infty} \gamma_{xy} (k) \\
\gamma_{yx} (0) + \sum_{k=1}^{\infty} \gamma_{yx} (k) & \gamma_{yy} (0) + \sum_{k=1}^{\infty} \gamma_{yy} (k)
\end{bmatrix}
\]
Some applications of GMM

- **Fixed income securities**
  
  \[ \ln p_{t+\tau} = A(\theta, \tau) + B(\theta, \tau)i_{t+\tau} \]
  
  - Construct moments of returns based on distribution of \( i_{t+\tau} \)
  - Estimate \( \theta \) by comparing to sample moments

- **Derivative securities**
  
  - Construct moments of returns by simulating PDE given \( \theta \)
  - Estimate \( \theta \) by comparing to sample moments

- **Asset pricing with time-varying risk premia**
CEV Example

\[ dS = \left[ A S^{-(1-\gamma)} + BS \right] dt + \sigma_s S^{\gamma/2} dz \]

**Special cases:**

\( \gamma = 1: \)  \( dS \) is a mean reverting process

\[ dS = K(\theta - S_t)dt + \sigma_s \sqrt{S_t} dz \quad (K = -B, \theta = -A / B) \]

\( \gamma = 2: \)  \( dS / S_t \) is a standard Gaussian diffusion

\[ dS = \mu S_t dt + \sigma_s S_t dz \quad (\mu = A + B) \]

**Method 1: Solve for** \( S_t \)

Define moments of \( S_t, \quad \mu_S^i = f^i(A, B, \gamma) \)

Compare to sample moments ...
CEV Example

\[ dS = \left[ AS^{-(1-\gamma)} + BS \right] dt + \sigma_s S^{\gamma/2} dz \]

Special cases:

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\[ dS = K(\theta - S_t) dt + \sigma_s \sqrt{S_t} dz \quad (K = -B, \theta = -A/B) \]

\( \gamma = 2: \) \( dS / S_t \) is a standard Gaussian diffusion

\[ dS = \mu S_t dt + \sigma_s S_t dz \quad (\mu = A + B) \]

Method 2: Simulate \( dS \) given starting value \( S_0 \) and \((A, B, \gamma)\)

Estimate moments of \( S_t \), \( \hat{\mu}_s^i = \hat{f}^i(A, B, \gamma) \)

Compare to sample moments ...