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An Integrated Model for Hybrid Securities

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We develop a model for pricing securities whose value may depend simultaneously on equity, interest-rate, and default risks. The framework may also be used to extract probabilities of default (PD) functions from market data. Our approach is entirely based on observables such as equity prices and interest rates, rather than on unobservable processes such as firm value. The model stitches together in an arbitrage-free setting a constant elasticity of variance (CEV) equity model (to represent the behavior of equity prices prior to default), a default intensity process, and a Heath-Jarrow-Morton (HJM) model for the evolution of riskless interest rates. The model captures several stylized features such as a negative relation between equity prices and equity volatility, a negative relation between default intensity and equity prices, and a positive relationship between default intensity and equity volatility. We embed the model on a discrete-time, recombining lattice, making implementation feasible with polynomial complexity. We demonstrate the simplicity of calibrating the model to market data and of using it to extract default information. The framework is extensible to handling correlated default risk and may be used to value distressed convertible bonds, debt-equity swaps, and credit portfolio products such as collateralized debt obligations (CDOs). Applied to the CDX INDU (credit default index–industrials) Index, we find the S&P 500 index explains credit premia.

Key words: securities; hybrid models; default; credit risk
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1. Introduction

Several financial securities depend on more than just one category of risk. Prominent among these are corporate bonds (which depend on interest-rate risk and on credit risk of the issuing firm) and convertible bonds (which depend, in addition, on equity risk). In this paper, we develop and implement a model for the pricing of securities whose values may depend on one or more of three sources of risk: equity risk, credit risk, and interest-rate risk.

Our framework is based on generalizing the reduced-form approach to credit risk (Duffie and Singleton 1999, Madan and Unal 2000) to include a process for equity. The typical reduced-form model involves two components, one describing the evolution of (riskless) interest rates, and the other, an intensity process that captures the likelihood of firm default; equity is not modeled explicitly. But any default process for a company's debt must obviously also apply to that company's equity. That is, when debt is in default, equity must also go into some post-default value. Motivated by this, we knit an equity process into a reduced-form model in an arbitrage-free manner; equity in the integrated model now follows a *jump-to-default process*, i.e., it gets absorbed at zero when a default happens.[1] The resulting framework captures simultaneously the three sources of risk mentioned above, and can be calibrated to market data to extract default probabilities or price hybrid securities.

Although our model is anchored in the reduced-form approach, the specifics draw on insights gained from the structural approach to credit risk (cf. Merton 1974, Black and Cox 1976, and others). Our starting point, the idea that default is associated with an absorbing value for equity, is itself borrowed from structural models. The process we posit for the evolution of equity prices prior to default—a constant elasticity of variance (CEV) process—is also motivated by structural models. An important characteristic of the Merton (1974) model is its generation of the so-called *leverage effect*, a negative relationship between equity prices and equity volatility. The leverage effect has also been documented empirically (e.g., Christie

[1] As the junior-most claim of the firm, it is natural to set the value of equity in default to zero; this is also consistent with the assumption that absolute priority holds in bankruptcy. However, our model is easily modified to allow a nonzero value for equity in the event of default.

1982). The CEV specification for equity prices generates a leverage effect in our reduced-form setting. Finally, we take the default intensity in our model to vary inversely with equity prices (and, therefore, directly with equity volatility). This specification is also motivated by the existence of a similar relation in the Merton (1974) model between default likelihood, equity prices, and equity volatility.

Our final framework, then, involves the following components. We have a CEV model describing the evolution of equity prices prior to default, an intensity process for default, and a riskless interest-rate model (for which purpose we use the Heath-Jarrow-Morton 1990 (HJM) model, although any other interest-rate model could be used). The result is a single parsimonious model accounting for correlations that combines the three major sources of risk.

We implement the model in a discrete-time setting, using the Nelson and Ramaswamy (1990) approach to discretize the CEV model. Rather than specifying an exogenous process for the default probability, we make it a dynamic function of both equity and interest-rate information. This enables us to derive default probabilities as *endogenous* functions of the information on the lattice, jointly calibrated to equity prices and default spreads. As a consequence, default information in the model is extracted from *both* equity- and debt-market information rather than from just debt-market information (as in reduced-form credit-risk models) or from just equity-market information (as in structural credit-risk models). This allows valuation, in a single consistent framework, of hybrid debt-equity securities such as convertible bonds that are vulnerable to default, as well as of derivatives on interest rates, equity, and credit. Our model can also serve as a basis for valuing credit portfolios where correlated default is an important source of risk. Finally, the model enables the extraction of credit-risk premia.

Our framework has several antecedents and points of reference in the literature. We have already mentioned the connection to both reduced-form and structural models. *Jump-to-default* equity models, in which equity gets absorbed at zero following a default, have also been examined in Davis and Lischka (1999), Carayannopoulos and Kalimipalli (2003), Campi et al. (2005), Carr and Linetsky (2006), and Le (2006).[2] The first two papers use the Black-Scholes (1973) model for the equity-price process prior to default which is a special case of the CEV model we use, and which does not admit the leverage effect; the other three, like ours, use the CEV process.[3] The specification of the default intensity process in Davis and Lischka (1999) is somewhat more restrictive than ours; their default intensity is perfectly correlated with the equity process, whereas we allow it to depend on both equity returns and interest rates and other information as well. Carayannopoulos and Kalimipalli (2003) use a default intensity specification similar to ours but their model does not allow for stochastic interest rates.

Campi et al. (2005) assume a constant intensity process for default; they do not allow for stochastic interest rates either. Le (2006) and Carr and Linetsky (2006) endogenize the default probability in a manner similar to our paper. Le calibrates the model to option prices to recover default probabilities in the model. Then he applies these default probabilities to credit spreads to identify implied recovery rates. Carr and Linetsky, working in a continuous-time setting but without interest-rate risk, are able to provide explicit closed-form solutions for survival probabilities, credit default swaps (CDSs) spreads, and European option prices.

Also related to our paper are the reduced-form models in Schönbucher (1998, 2002) and Das and Sundaram (2000), which study "defaultable HJM" models. These are HJM models with a default process tacked on. Our model generalizes these to include equity processes as well. In particular, the Das and Sundaram (2000) model results as a special case of our framework if the equity process is switched off. Our framework may also be viewed as a generalization of Amin and Bodurtha (1995) (see also Brenner et al. 1987). The Amin-Bodurtha model combines interest-rate risk and equity risk (in the form of a Black-Scholes model) but does not incorporate credit risk. Because there is no default, equity in their model is necessarily infinitely lived and never gets "absorbed" in a postdefault value. Other frameworks are nested within our model too. For example, if the equity and hazard-rate processes are switched off, we obtain the HJM model, whereas if the interest-rate and hazard-rate processes are switched off, we obtain a discrete-time CEV tree, as described by Nelson and Ramaswamy (1990).

Our lattice design allows recombination, making the implementation of the model simple and efficient; indeed, the model is fully implementable on a spreadsheet. Unlike many earlier models, we are able to

[2] See also Linetsky (2004) for solutions in continuous time. Carr and Wu (2005) show how a similar model may be calibrated with options data. Other related papers in the literature include Jarrow (2001) and Takahashi et al. (2001). Incorporation of equity risk into reduced-form models has also been examined in Jarrow (2001) and Mamaysky (2002), but using a different approach: Equity values are derived through a posited dividend process.

[3] An earlier version of our paper used the Black-Scholes model too. Our investigation of a more general framework was motivated by comments from the referee and editor concerning the shortcomings of the Black-Scholes framework, in particular the absence of a leverage effect.

(a) price derivatives on equity and interest rates with default risk; (b) extract probabilities of default (PD) endogenously in the model; (c) provide for the risk-neutral simulation of correlated default risk in a manner consistent with no arbitrage and consistent with equity correlations (which we believe, has not been undertaken in any model so far); and (d) extract credit risk premia.

The rest of this paper proceeds as follows. In §2 we develop the pricing lattice in the state variables of the model in a manner that allows for additional structure to accommodate default risk. Section 3 deals with implementation issues, including a discussion of how default swaps may be used to calibrate the model for subsequent use. We show that the model may generate a wide range of spread curve shapes. Empirical calibration to markets is undertaken to evidence the ease of implementation. This section also explores the impact of default risk on embedded options within classic bond structures. Section 4 applies the model to the extraction of credit risk premia and uses data from Dow Jones CDX index firms to examine the principal components of these premia. Finally, an analysis of the model application to correlated default products is provided. Section 5 concludes by summarizing the economic and technical benefits of the model.

2. The Model

As we have noted, the motivation for our model is simple. If the default process for a company's debt is described by a hazard rate λ (as in the standard reduced-form model approach), then λ must also apply to that company's equity. That is, when debt is in default, equity must also go into some default value. As the junior-most claim of the firm, it is natural to set the value of equity in default to zero, but our model is easily modified to allow a nonzero value for equity in the event of default.

An early model of *defaultable equity* was presented in Samuelson (1972) and is discussed in Merton (1976). We begin with a brief description of Samuelson's result, then discuss the directions in which we generalize it.

2.1. The Samuelson (1972) Model

Consider a continuous-time setting in which equity prices evolve according to a geometric Brownian motion

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),$$

but with the added twist that equity prices could suddenly jump to zero and get absorbed there. Suppose that the jump-to-default is governed by a constant intensity Poisson process with hazard rate $\xi > 0$. This is a simple example of a jump-to-default equity process. Samuelson (1972) shows that the price of a call option on such equity is given by

$$C = \exp(-\xi T) C^{\mathrm{BS}}[S e^{\xi T}, K, T, \sigma, r] = C^{\mathrm{BS}}[S, K, T, \sigma, r + \xi].$$

Here, $C^{\mathrm{BS}}(S, K, T, \sigma, r)$ is the standard Black-Scholes call option pricing function with current stock price S, option strike K, option maturity T, interest rate r, and stock volatility σ; and ξ is, of course, the default intensity. Note that the call is priced by the Black-Scholes model with an adjusted risk-neutral interest rate $(r + \xi)$.[4]

From the perspective of a credit-risk model, there are three weaknesses to this setting. One is the assumption of a constant equity volatility σ on the nondefault segment. The so-called leverage effect suggests that equity volatility should increase as equity prices fall. Empirical support for the leverage effect is provided in several papers, such as Christie (1982). Theoretical support comes from structural models such as Merton (1974) in which equity prices and equity volatility are inversely related. Ideally, we would like the posited equity price process to incorporate this feature. A second weakness is the assumption of constant interest rates, which makes the model inappropriate for studying hybrids such as convertible bonds that depend on both equity risk and interest-rate risk. The third is the assumption of a constant hazard rate. In general, one would expect the likelihood of a jump to default to be inversely related to firm value, increasing as firm value decreases. Equally, because equity is a strictly monotone function of firm value, we would expect hazard rates to move inversely to equity prices. Structural models exhibit the analogy of such a property with the likelihood of default and equity prices moving in opposite directions.

In the following sections, we describe a model that has these properties. We proceed in several steps, describing first the equity process we shall employ to capture the leverage effect, then the interest-rate model, and finally, the specification of the hazard rate function.

2.2. The Equity Model

The first step in our model is to identify a process for describing the movement of equity prices prior to default, in which equity prices and volatility move in opposite directions. A simple generalization of the Black-Scholes model, which possesses the desired

[4] Note that the option price is not just the price of a nondefaultable call option, i.e., $BMS[S_0, K, T, \sigma, r]$, multiplied by the risk-neutral probability of survival $\exp(-\xi T)$. The drift of the risk-neutral equity process is also affected by the jump-to-default compensator (ξ). For an excellent exposition of default jump compensators, see Giesecke (2001).

property, is the CEV model. In continuous time, the risk-neutral CEV equity process is described by

$$dS(t) = r(t)S(t)\,dt + \sigma S(t)^{\gamma}\,dZ(t), \tag{1}$$

where $\gamma \in (0, 1]$ is the CEV coefficient (sometimes called the *leverage coefficient*). The instantaneous volatility is $\sigma S^{\gamma-1}$, so this specification exhibits the leverage effect for $\gamma < 1$.[5] For $\gamma = 1$, the CEV process reduces to Black-Scholes geometric Brownian motion. Our objective is to modify this process in two directions. One is to discretize the model. The second is to append to it an intensity process for default.

As a parenthetical comment, we should note that for $\gamma < 1$, the CEV price process (1) can penetrate zero even in the absence of jumps (see, e.g., Davydov and Linetsky 2001 for details). Thus, when a default intensity process is appended to it, the resulting model will admit both *drift to default* as in structural credit-risk models, and *jump to default* as in reduced-form models.

Nelson and Ramaswamy (1990) show how to discretize the CEV process (1) in a recombining binomial tree. To achieve our object, we generalize their construction so as to allow for a third branch from each node of the tree representing a jump to default. We describe here the branching process at a generic node t. Let the stock price at t be denoted $S(t)$ and the length of one period on the tree be h years. Let $\lambda(t)$ denote the (risk-neutral) likelihood of a jump to default at node t; this probability may depend on the information at node t, but to keep notation simple, we suppress these arguments. Finally, let $R(t)$ denote the gross (i.e., 1 + net, continuously compounded) one-period interest rate at node t. Then, the risk-neutral evolution of stock prices on the tree is given by the following set of equations:

$$S[Y_s(t)] = \begin{cases} [\sigma_s(1-\gamma)Y_s(t)]^{1/(1-\gamma)} & \text{if } Y_s(t) > 0 \\ 0 & \text{if } Y_s(t) \le 0 \end{cases}$$

$$Y_s(t+h) = \begin{cases} Y_s(t) + \sqrt{h} \\ \quad \text{with probability } q(t)[1-\lambda(t)] \\ Y_s(t) - \sqrt{h} \\ \quad \text{with probability } [1-q(t)][1-\lambda(t)] \\ 0 \quad \text{with probability } \lambda(t) \end{cases}$$

$$q(t) = \begin{cases} \dfrac{[R(t)/(1-\lambda(t))] - b(t)}{a(t) - b(t)} & \text{if } Y_s(t) > 0 \\ 0 & \text{if } Y_s(t) \le 0, \end{cases} \tag{2}$$

$$a(t) = \frac{S[Y_s(t) + \sqrt{h}]}{S[Y_s(t)]}, \tag{3}$$

$$b(t) = \frac{S[Y_s(t) - \sqrt{h}]}{S[Y_s(t)]}. \tag{4}$$

Note again that the probabilities above are the risk-neutral probabilities. In the language of the usual binomial tree, $a(t)$ is the size of the "up move" in the binomial tree, and $b(t)$ the size of the "down move." On the nondefault part of the tree, we may write

$$Y_s(t+h) = Y_s(t) + X_s(t)\sqrt{h}, \quad X_s(t) \in \{+1, -1\}, \tag{5}$$

where X_s is a binomial random variable driving the equity process in the model prior to default. The representation (5) makes it easier to show how the desired correlation between the equity process in the model and the term structure of interest rates may be injected.

The next segment describes the interest-rate model. Following that, we stitch together the equity and interest-rate processes, and then, finally, take up the specification of the default probability $\lambda(t)$.

2.3. The Term-Structure Model

We adopt the discrete-time, recombining form of a one-factor HJM model. We provide a short review of the model here. Initially, we prepare the univariate HJM lattice for the evolution of the term structure, and subsequently stitch on the equity process defined above.

At any time t, we assume that zero-coupon bonds of all maturities are available. For any given pair of time points (t, T) with $0 \le t \le T \le T^* - h$, let $f(t, T)$ denote the time-t forward rate applicable to the period $(T, T+h)$. The short rate is $f(t, t) = r(t)$. Forward rates are taken to follow the stochastic process:

$$f(t+h, T) = f(t, T) + \alpha(t, T)h + \sigma(t, T)X_f(t)\sqrt{h}, \tag{6}$$

where α is the drift of the process and σ the volatility; $X_f(t)$ is a random variable taking values in the set $\{-1, +1\}$. Both α and σ are taken to be only functions of time, and not other state variables. This is done to preserve the computational tractability of the model.

We denote by $P(t, T)$ the time-t price of a default-free zero-coupon bond of maturity $T \ge t$. As usual,

$$P(t, T) = \exp\left\{-\sum_{k=t/h}^{T/h-1} f(t, kh) \cdot h\right\}. \tag{7}$$

The well-known recursive representation of the drift term α of the forward-rate and spread processes, is required to complete the risk-neutral lattice. Let $B(t)$ be the time-t value of a money-market account that

[5] As noted earlier, the Merton (1974) structural model also implies an equity process that admits the leverage effect. The relationship between the processes is discussed in online Appendix B (provided in the e-companion, which is part of the online version that can be found at http://mansci.journal.informs.org/).

uses an initial investment of \$1 and rolls the proceeds over at the default-free short rate:

$$B(t) = \exp\left\{\sum_{k=0}^{t/h-1} r(kh)\cdot h\right\}. \qquad (8)$$

The equivalent martingale measure Q is defined with respect to $B(t)$ as numeraire; thus, under Q all asset prices in the economy discounted by $B(t)$ will be martingales. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$: $Z(t,T) = P(t,T)/B(t)$. Z is a martingale under Q, i.e., $Z(t,T) = E_t[Z(t+h,T)]$ for all t, T. It follows that $Z(t+h,T)/Z(t,T) = (P(t+h,T)/P(t,T))\cdot(B(t)/B(t+h))$. Algebraically manipulating the martingale equation leads to a recursive expression relating the risk-neutral drifts α to the volatilities σ at each t:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) = \frac{1}{h^2}\ln\left(E_t\left[\exp\left\{-\sum_{k=t/h+1}^{T/h-1}\sigma(t,kh)X_f h^{3/2}\right\}\right]\right). \qquad (9)$$

This completes the description of the interest-rate process.

2.4. The Joint Process

We now connect the two processes for the term structure and the defaultable equity price together on a bivariate lattice. There are two goals here. First, we set up the probabilities of the joint process so as to achieve the correct correlation between equity returns and changes in the spot rate, which we denote as ρ. Second, our lattice is set up so as to be recombining, allowing for polynomial computational complexity, providing for fast computation of derivative security prices.

Specification of the joint process requires a probability measure over random shocks $[X_f(t), X_s(t)]$. This probability measure is chosen to (i) obtain the correct correlations, (ii) ensure that normalized equity prices and bond prices are martingales, and (iii) to make the lattice recombining. Our lattice model is hexanomial, i.e., from each node, there are six emanating branches or six states (of which two are absorbing states). Table 1 depicts the states.

Note that the table contains two free parameters m_1 and m_2 in the probability measure. We solve for the correct values of m_1 and m_2 to provide a default-consistent martingale measure, with the appropriate correlation between the equity and interest-rate processes, also ensuring, that the lattice recombines. The details of this derivation and the properties of the tree are presented in the online appendix (provided in the e-companion). As shown there, m_1 and m_2 have the form

Table 1 **Branching Process and Probability Measure**

X_f	X_s	Probability
1	1	$p_1 = \frac{1}{4}(1+m_1)[1-\lambda(t)]$
1	−1	$p_2 = \frac{1}{4}(1-m_1)[1-\lambda(t)]$
−1	1	$p_3 = \frac{1}{4}(1+m_2)[1-\lambda(t)]$
−1	−1	$p_4 = \frac{1}{4}(1-m_2)[1-\lambda(t)]$
1	def	$p_5 = \lambda(t)/2$
−1	def	$p_6 = \lambda(t)/2$

Notes. This tableau presents the six branches from each node of the pricing lattice, as well as the probabilities for each branch. "Def" stands for the default/absorbing state. The first four branches relate to the nondefaulted path and the last two branches lead to absorbing states.

$$m_1 = \frac{A+B}{2}, \qquad m_2 = \frac{A-B}{2}$$

$$A = \frac{4e^{r(t)h}(1-\lambda(t))^{-1} - 2[a(t)+b(t)]}{a(t)-b(t)}, \qquad B = \frac{2\rho}{1-\lambda(t)}.$$

These values may now be used in Table 1. Probability bounds are presented in the online appendix in Table EC.1. In the special case where $\gamma = 1$, i.e., we have the basic geometric Brownian motion, and the discrete time model is implemented with the usual Cox et al. (1979) approach. In this case, the expressions above for $a(t)$, $b(t)$ are given by $a(t) = \exp[\sigma_s\sqrt{h}]$, and $b(t) = \exp[-\sigma_s\sqrt{h}]$.

2.5. The Default Process

To close the model, we must specify a process for the default probability $[\lambda(t)]$. One way to do this is to embed an exogenous $\lambda(t)$ process, but this increases implementation complexity by adding an extra dimension to the lattice model. It will also make it harder to ensure the desired correlations between default likelihood and equity returns or equity volatility.

We take a different approach, therefore, and *endogenize* the default likelihood by assuming that $\lambda(t)$ at a given node is a function of equity prices and interest rates at that node. That is, with $\lambda(t) = 1 - e^{-\xi(t)h}$, we express the default intensity $\xi(t)$ as

$$\xi[\mathbf{f}(t), S(t), t; \theta] \in [0, \infty). \qquad (10)$$

As usual, $\mathbf{f}(t)$ denotes the forward rates at that node, and $S(t)$ the stock price; t indexes current time, and θ some set of parameters. We note that a similar endogenous default intensity extraction, but in a less general setting, has been implemented in Das and Sundaram (2000), Carayannopoulos and Kalimipalli (2003), and Acharya et al. (2002).

Endogenizing the default probability in this fashion creates a parsimonious lattice and facilitates implementation of the model. We note also that default probabilities have been shown to be connected to the term structure (see Duffie et al. 2005 who find that default probabilities are functions of the equity index and term structure).

Various possible parameterizations of the default intensity function may be used. For example, the following model (subsuming the parameterization of Carayannopoulos and Kalimipalli 2003) prescribes the relationship of the default intensity $\xi(t)$ to the stock price $S(t)$, short rate $r(t)$, and time on the lattice $(t-t_0)$.

$$\xi(t) = \exp[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t-t_0)] = \frac{\exp[a_0 + a_1 r(t) + a_3(t-t_0)]}{S(t)^{a_2}}. \tag{11}$$

For $a_2 \geq 0$, we get that as $S(t) \to 0$, $\xi(t) \to \infty$, and as $S(t) \to \infty$, $\xi(t) \to 0$.

In addition to the probability of default of the issuer, a recovery rate is required in the two states in which default occurs. The recovery rate may be treated as constant, or as a function of the state variables in this model. It may also be pragmatic to express recovery as a function of the default intensity, supported by the empirical analysis of Altman et al. (2002).

2.6. Example: Two-Period Tree

Here, we present a simple illustration of a pricing tree in three dimensions, one each for time (t), interest rate (i), and stock price (j)—a tree in (t, i, j) space. The initial node is denoted $(1, 1, 1)$ and after one period, we have four nodes (interest rates go up and down, and stock price goes up and down), denoted $\{(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. Because the tree is recombining, after two periods, we will have only nine nodes. In Table 2 are the results of calculations for two periods. At each node, we show the one-period probability of default. The table presents all the details of the inputs used in the example. This table should be useful to readers who wish to implement the model, and it also details all the inputs required for building the pricing lattice. We have assumed the Cox, Ross, and Rubinstein (1979) (CRR) process for the equity model, i.e., the CEV coefficient is $\gamma = 1$. The approach requires a parsimonious set of inputs, all of which are observable and may be accessed from standard sources.

Table 2 Two-Period Tree Example

Input values

Parameter	Value	Forward rates	Forward rate volatilities
a_0	0.1	0.060	0.0020
a_1	0.1	0.065	0.0019
a_2	1.0	0.070	0.0018
a_3	0.1		
S	100		
σ_s	0.40		
ρ	0.4		
h	0.5		
γ	1		

Output price lattice

[t	i	j]	r	S	λ
1	1	1	0.0600	100.0000	0.0058
2	1	1	0.0663	132.6896	0.0044
2	1	2	0.0663	75.3638	0.0077
2	2	1	0.0637	132.6896	0.0046
2	2	2	0.0637	75.3638	0.0081
3	1	1	0.0725	176.0654	0.0033
3	1	2	0.0725	100.0000	0.0058
3	1	3	0.0725	56.7971	0.0102
3	2	1	0.0700	176.0654	0.0035
3	2	2	0.0700	100.0000	0.0061
3	2	3	0.0700	56.7971	0.0108
3	3	1	0.0675	176.0654	0.0037
3	3	2	0.0675	100.0000	0.0064
3	3	3	0.0675	56.7971	0.0113

Notes. In this table, we present the results of a two-period tree based on given input parameters. The example here may be useful for anyone replicating our model to check their results. The input parameters are the default function values $\{a_0, a_1, a_2, a_3\}$, the stock price S, stock volatility σ_s, correlation of term structure with stock prices ρ, and the time step on the tree h. The initial forward rate term structure and corresponding volatilities are also given. The output price lattice is recombining, and therefore, there are $(n+1)^2$ nodes at the end of the nth period on the lattice. The lattice starts at node (1, 1, 1) and then moves to four nodes in the subsequent period, and then to nine nodes, etc. At each time step there are two axes (i, j) for interest rates and stock prices, respectively. The default probability (λ) for the next period is also stated at each node, and is a function of r, S, and time. The default function is $\lambda = 1 - \exp[-\xi h]$, where $\xi = \exp[a_0 + a_1 r + a_3 ih]/(S^{a_2})$, where i indexes nodes on the interest rate branch of the tree. Note that a_3 modulates the slope more severely when rates are low than high. Alternative specifications would be to replace i with t. It can be seen that the default probability declines as S increases, and increases in r.

3. Numerical Analysis

This section performs numerical analysis with the model developed above. We open in §3.1 with a discussion of CDSs and their pricing in the model. Section 3.2 presents examples showing how different parameters result in various default swap spread term structures. In §3.3, we examine the effect of the leverage coefficient γ on the spread curves generated by the model. Section 3.4 looks at the pricing of hybrid securities, in particular convertible bonds, and examines how default risk affects the prices of corporate bonds, with or without convertible features. Finally, §3.5 describes how the model may be used for pricing credit correlation products.

3.1. Calibrating the Model with Credit Default Swaps (CDSs)

A credit default swap (or CDS) is a bilateral contract in which one party (the *protection buyer*) makes a steady stream of payments to the other party (the *protection seller*) until the occurrence of a credit event on a reference credit (that we refer to as a *bond*).

The price of a default swap is quoted as a spread rate per annum. Therefore, if the default swap rate is 100 bps, and the protection buyer is to make quarterly payments, then the buyer would pay 25 bps per $100 of par each quarter to the seller. Pricing of default swaps has been described in detail in Duffie (1999). The increasing amount of trading in default swaps now offers a source of empirical data for calibrating the model. Other models, such as CreditGrades,[6] also use default swap data. A recent paper by Longstaff et al. (2005) undertakes an empirical comparison of default swap and bond premia in a parsimonious closed-form model.

Because the value of a default swap is zero at inception, the fair price of the swap can be identified by equating (under the risk-neutral measure) the present values of the expected payments made on the swap by the protection buyer to the expected receipts from the swap on default. We make the following assumptions: (a) in any period in which default occurs, recovery payoffs are realized at the end of the period; and (b) default is based on the default intensity at the beginning of the period.

We price a T maturity default swap on a unit face value reference bond (RB) of maturity $T' \geq T$, denoted $\mathrm{RB}(t)$ at time t, with corresponding cashflows of $\mathrm{CF}(t)$. The pricing recursion for the bond under the recovery of market value (RMV) condition at each node on the tree obeys the following condition:

$$\mathrm{RB}(t) = e^{-r(t)h}\left\{\sum_{k=1}^{4} \hat{p}_k(t)[\mathrm{RB}_k(t+h)+\mathrm{CF}_k(t+h)]\right\} \cdot [1-\lambda(t)(1-\phi)], \quad (12)$$

where ϕ is the recovery rate on the bond (here assumed constant), and $\hat{p}_k(t) = p_k(t)/[1-\lambda(t)]$, $k = 1, \ldots, 4$ are the four probabilities for the nondefault branches of the lattice, *conditional* on no default occurring, and k indexes the four states of nondefault. Therefore, $\sum_{k=1}^{4} \hat{p}_k(t) = 1, \forall t$.

Next, we compute the expected present value of all payments in the event of default of the zero-coupon bond, denoted $\mathrm{CDS}(t)$. Again, the lattice-based recursive expression at each node is:

$$\mathrm{CDS}(t) = e^{-r(t)h}\left\{\sum_{k=1}^{4} \hat{p}_k(t)\mathrm{CDS}_k(t+h)\right\}[1-\lambda(t)] + \lambda(t)\mathrm{RB}(t)(1-\phi), \quad \mathrm{CDS}(T) = 0. \quad (13)$$

[6] CreditGrades is a model developed by RiskMetrics, which uses default swaps to calibrate an extended Merton-type model to obtain PD.

The formula above has two components: (i) the first part is the present value of future possible losses on the default swap, given that default does not occur at time t. (ii) the second part is the present value of the loss (sustained at the end of the period). Note that the formula contains $\mathrm{RB}(t)(1-\phi)$, which is the present value of loss at the end of the period, $\mathrm{RB}(t+h)(1-\phi)$.

Finally, we calculate the expected present value of a $1 payment at each point in time, conditional on no default occurring. The recursion at each node is as follows:

$$G(t) = \left[e^{-r(t)h}\left\{\sum_{k=1}^{4} \hat{p}_k(t)G_k(t+h)+1\right\}\right] \cdot [1-\lambda(t)], \quad G(T) = 0. \quad (14)$$

$G(0)$ will then represent the present value of $1 in premiums paid at each time period, conditional on no default having occurred.

In order to get the annualized basis points spread (s) for the premium payments on the default swap, we equate the quantities $s \times h \times G(0) = \mathrm{CDS}(0)$, and the premium spread is

$$s = \frac{\mathrm{CDS}(0)}{h \times G(0)} \times 10{,}000 \text{ bps}. \quad (15)$$

In the equation above, we multiply by 10,000 and divide by the time interval h in order to convert the amount into annualized basis points. We use this calculation in the illustrative examples that are provided in the following section.

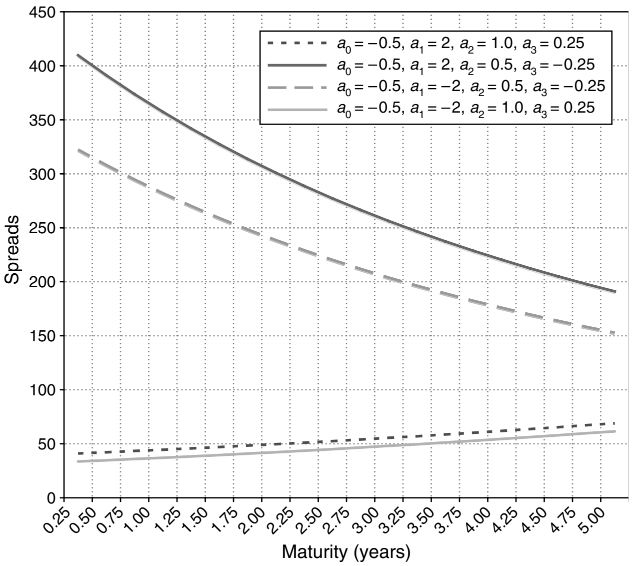
3.2. Credit Default Swap Spread Curves

In this section we demonstrate that the model is able to generate varied spread curve shapes. The reference instrument is taken to be a unit valued zero coupon bond with the same maturity as the default swap.

In the plots in Figure 1 we present the term structure of default swap spreads for maturities from one to five years. The default intensity is specified as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t-t_0)]/S(t)^{a_2}$. Keeping a_0 fixed, we varied parameters a_1 (impact of the short rate), a_2 (impact of the equity price), and a_3 (impact of the term structure of credit premia) over two values each. Four plots are the result. The other inputs to the model, such as the forward rates and volatilities, stock price and volatility, are provided in the description of the figure. Comparison of the plots provides an understanding of the impact of the parameters.

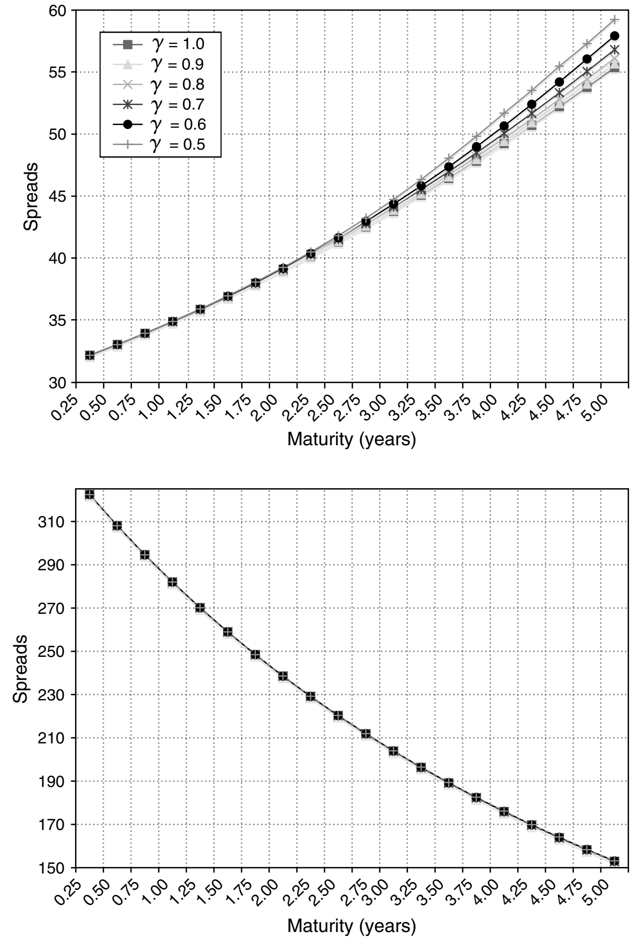
When $a_3 > 0$, the term structure of default swap spreads is upward sloping, as would be expected. When $a_3 < 0$, i.e., default spreads are declining, consistent with a reduction in premia over time. Hence, we may think of a_3 as the slope parameter in the model. Comparison of the plots also shows the effect of parameter a_2, the coefficient of the equity price $S(t)$.

Figure 1 **Term Structure of Default Swap Spreads for Varied Default Function Parameters** ($\gamma = 1$)



Notes. This figure presents the term structure of default swap spreads for maturities from one to five years. The figure has four plots. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we varied parameters a_1, a_2, and a_3. Hence, the four plots are the result. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 100, and the stock return volatility is 0.30, given $\gamma = 1$. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. The default function parameters are presented on the plots.

Figure 2 **Term Structure of Default Swap Spreads for Varying Leverage (Varying γ)**



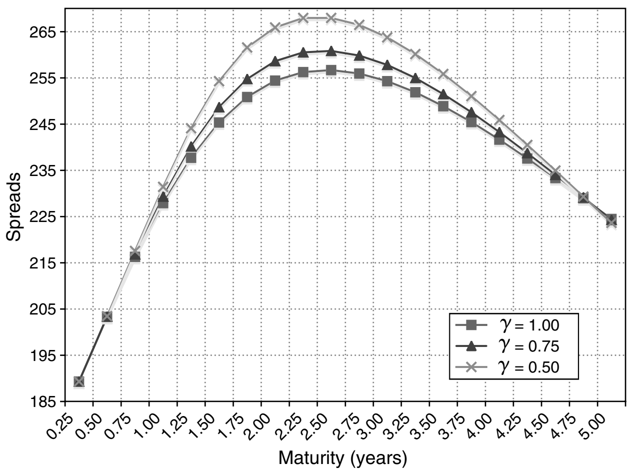
Notes. This figure presents the term structure of default swap spreads for maturities from one to five years. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. The figure has two graphs. Keeping all the other parameters fixed, we fixed parameters $a_0 = -0.5$, $a_1 = -2$, $a_2 = 1$, and $a_3 = 0.25$ for the first plot and $a_0 = -0.5$, $a_1 = -2$, $a_2 = 0.5$, and $a_3 = -0.25$ for the second. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 100, and the stock return volatility is 0.30, given $\gamma = 1$. Because the stock volatility when $\gamma = 1$ is 0.4, as we change the CEV coefficient γ, we also adjust the variable σ so as to keep the conditional total volatility of the diffusion roughly the same, by the following equation: $\sigma_s S_0^{\gamma} = \sigma_{CRR} S_0$. This ensures that the total diffusion volatility in the CEV model is approximately the same as in the CRR model, and is the same approach as used by Nelson and Ramaswamy (1990) for comparisons. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. We varied γ from 0.5 to 1.0. We note very minor changes in CDS spread curves.

As $a_2 > 0$ increases, default spreads decline as the stock price lies in the denominator of the default intensity function, as can be seen in the plots. A comparison of curves in Figure 1 shows that parameter a_1, the coefficient on interest rates, has a level effect on the spread curve. In sum, our four-parameter default function is flexible enough to capture a variety of economic phenomena as well as to generate a spectrum of curve shapes.

3.3. The Impact of the Leverage Parameter γ

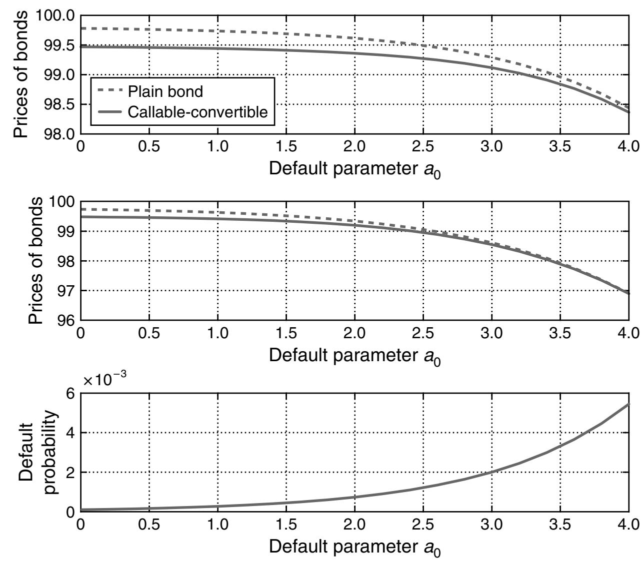
Because our model is based on a default-extended CEV process, varying the CEV coefficient γ enables the simulation of varied leverage effects. If we assume the CRR model ($\gamma = 1$) as a base case, then reductions in γ will increase the leverage effect. Figures 2 and 3 show how changes in γ impact the term structure of CDS spreads. The diffusion coefficient in the CEV model is set up so that the total volatility is roughly the same across varied choices of γ, which will result in a meaningful comparison. For this, we follow Nelson and Ramaswamy (1990) in setting σ_s to satisfy the following condition: $\sigma_s S(0)^{\gamma} = \sigma_{CRR} S(0)$ (total CEV volatility at inception equals total CRR volatility). From all three figures, we see that increases in the leverage effect result in an increase in spreads indicating that the direction of the impact conforms to theory. However, the change in leverage appears to have only a small quantitative effect on spreads, suggesting that the level of total volatility matters more in the pricing of CDS than the particular form of the volatility function. For simplicity, therefore, in the remainder of the paper, we set $\gamma = 1$ (i.e, use the defaultable CRR model) in examples and computa-

Figure 3 **Term Structure of Default Swap Spreads for Varying Leverage (Varying γ)**



Notes. This figure presents the term structure of default swap spreads for maturities from one to five years. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we fixed parameters $a_0 = -1.814681$, $a_1 = 58.55167$, $a_2 = 2.16029$, and $a_3 = -0.37119$. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 58.31, and the stock return volatility is 0.40, given $\gamma = 1$. Correlation between stock returns and forward rates is 0.0, and recovery rates are a constant 40%. We varied γ from 0.5 to 1.0. Since the stock volatility when $\gamma = 1$ is 0.4, as we change the CEV coefficient γ, we also adjust the variable σ so as to keep the conditional total volatility of the diffusion roughly the same, by the following equation: $\sigma_s S_0^{\gamma} = \sigma_{\text{CRR}} S_0$. This ensures that the total diffusion volatility in the CEV model is approximately the same as in the CRR model, and is the same approach as used by Nelson and Ramaswamy (1990) for comparisons. We note very minor changes in CDS spread curves.

Figure 4 **Comparison of Callable-Convertible Bonds and Plain Defaultable Bonds in Different Volatility Environments**



Notes. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be five years, and interest is assumed to be paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities that come from the model in Equation (11). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 2$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$. The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 100. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains three panels. The top panel presents a comparison of bond prices when equity volatility is set to 20% (for $\gamma = 1$), and the default probability parameter a_0 is varied on the x-axis. The middle panel shows the same comparison when the volatility is 40%. The bottom panel shows the corresponding default probability.

tions. Further, in online Appendix B we analyze the link between the Merton structural model and our CEV reduced-form model; as we show, the relationship is a complex one, relating the parameters for firm volatility and leverage in the Merton case to equity volatility and elasticity in the CEV model.

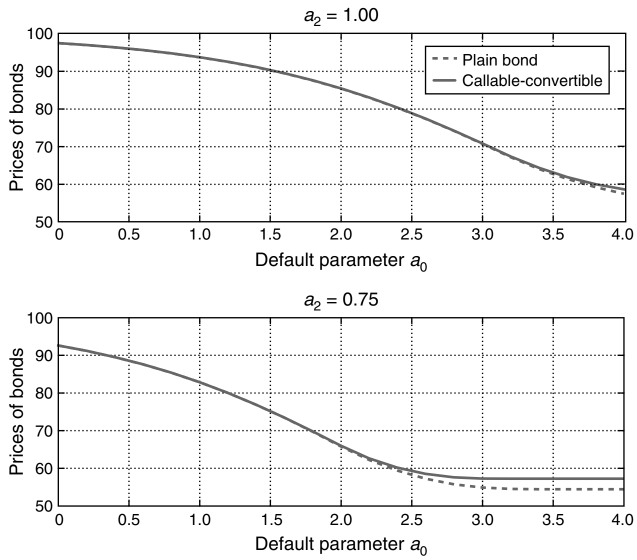
3.4. Impact of Default Risk on Embedded Options

The model may be easily used to price callable-convertible debt. One aspect of considerable interest is the extent to which default risk impacts the pricing of convertible debt through its effect on the values of the call feature (related to interest-rate risk) and the convertible feature (related to equity-price risk). We chose an initial set of parameters to price convertible debt and examined to what extent changing levels of default risk impacted a plain vanilla bond versus a convertible bond. The parameters and results are presented in Figure 4.

Given the base set of parameters, we varied a_0 from zero to four. As a_0 increases, the level of default risk increases too. For each increasing level of default risk, we plot the prices of a defaultable plain vanilla coupon bond with no call or convertible features. We also plot the prices of a callable-convertible bond. Note that this numerical experiment has been kept simple by setting $a_1 = a_3 = 0$, so that there are no interest-rate and term effects on the default probabilities.

The results comparing the plain coupon bond with a callable-convertible coupon bond are presented in Figure 4 (top panel). The value of a_0 is varied from zero (low default risk) to four (higher risk). Bond values decline as default risk (a_0) increases. As default risk increases, the difference in price between the callable-convertible and vanilla bonds rapidly declines and eventually goes to zero. Because default risk effectively shortens the duration of the bonds, it also reduces the value of the call option. Hence, the price difference between the vanilla bond and the callable-convertible bond declines as a_0 increases. Moving from the top to middle panel is based on one change, i.e., equity volatility was increased from 20% per year to 40% per year. The results are the same, but bond prices converge faster. Hence at high equity

Figure 5 **Comparison of Default Risk Effects on Callable-Convertible Bonds and Plain Defaultable Bonds for Different Equity Dependence**



Notes. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be five years, and interest is assumed to be paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities that come from the model in Equation (11). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$ and stock volatility is 20% (for $\gamma = 1$). The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 105. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains two panels. The upper panel presents a comparison of bond prices when $a_2 = 1$, and the default probability parameter a_0 is varied on the x-axis. The lower panel shows the same comparison when $a_2 = 0.75$, which is higher default risk.

volatility, default risk impacts the convertible value faster, as there are more regions in the state space on our pricing tree with greater PD. Therefore, default risk systematically impacts the commingled values of interest-rate calls and equity convertible features in debt contracts. By shortening the effective duration of the bond, both options decline in value, which is driving the price of the callable-convertible closer to that of the vanilla bond (see Buchan 1998 for early work on such effects in the pricing of convertible bonds).

In Figure 5 we vary the dependence of default risk on equity prices. The base case is presented in the upper panel of the figure when the coefficient $a_2 = 1$. In the lower panel, we changed $a_2 = 0.75$, resulting in higher default risk. Hence, the prices are lower in the lower panel. The values of parameters for the conversion feature and for the call feature were chosen so as to make the plain bond and the callable-convertible equal in price in the upper panel. Reducing the value of a_2 to inject more default risk in fact increases the price of the callable-convertible relative to that of the plain bond, and the effect is higher for greater levels of default risk. Here, increases in default risk tend to increase the difference between equity call option values and the bond callable feature, ceteris paribus, and this drives an increasing wedge between the convertible bond and the plain bond. Further, the level of parameter a_2 also determines whether defaultable bond prices are convex or concave in default risk—both possibilities are pictured in the two panels of Figure 5. At lower levels of default risk, the convertible bond is concave in a_0, and at higher levels it becomes convex.

Therefore, depending on market conditions and the level of default risk, increases in default risk may increase or decrease the price differential of two bonds that have embedded options. This highlights the need for careful consideration of default risk effects using an appropriate model that considers all forms of risk and their interactions.

3.5. Correlated Default Analysis

The model may be used to price credit baskets. There are many flavors of these securities, and some popular examples are nth to default options, and collateralized debt obligations (CDOs). These securities may be valued using Monte Carlo simulation, under the risk-neutral measure, based on the default functions fitted using the techniques developed in this paper.

The first step in modeling default correlations is to model the correlation of default intensity amongst issuers. Because our model calibrates default functions $\xi(\mathbf{f}(t), S(t), t)$ for each issuer, credit correlations are determined from the correlations of the forward curve, and issuer stock prices, which are observable. Suppose we are given the function for default intensity of issuer i, $i = 1, \ldots, n$, as $\xi_i(t) = \exp[a_0^i + a_1^i r(t) + a_3^i (t - t_0)]/S_i(t)^{a_2^i}$. Let the covariance matrix of $[r(t), S_1(t), S_2(t), \ldots, S_n(t)]'$ be Σ. Then, the covariance matrix of default intensities $\{\xi_i(t)\}_{i=1,\ldots,n}$ is $V(t) \approx J(t)\Sigma J(t)'$, where $J(t) \in R^{n\times(n+1)}$ is the Jacobian matrix whose ith row is as follows:

$$J_i(t) = \left[\frac{\partial \xi_i(t)}{\partial r(t)}, 0, \ldots, 0, \frac{\partial \xi_i(t)}{\partial S_i(t)}, 0, \ldots, 0\right]$$
$$= \left[a_1^i \xi_i(t), 0, \ldots, 0, \frac{-a_2^i}{S_i(t)} \xi_i(t), 0, \ldots, 0\right].$$

We may contrast this approach with the somewhat ad hoc practice of using equity correlations as a proxy for asset correlations, which are used in turn to drive default correlations in structural models. Our method is closer to the approach, also used in practice, of a factor structure that drives default correlations. However, our approach has a significant advantage over other factor models—i.e., we calibrate each default function in a manner that is based on observables, *and* is also consistent with a no-arbitrage model over default, equity, and interest-rate risks. A comprehensive examination of credit correlations in this framework is undertaken in Bandreddi et al. (2005).[7]

4. Default Risk Premia

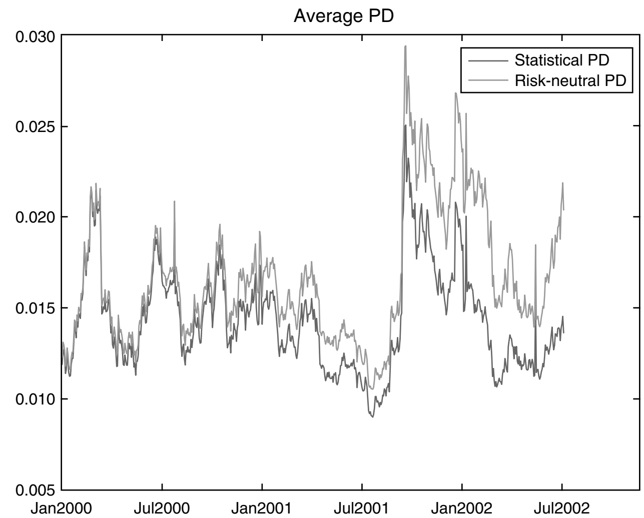
In this section, we present a simple empirical application of the model by fitting it to the 30 names in the CDX INDU Index. Our calibration will permit us to extract credit risk premia.

The model is calibrated as follows. The stock price is taken from CreditGrades. Stock volatility is based on a historical estimate from 1,000 days past returns.[8] To obtain the forward curve of interest rates each day, we extracted constant maturity yields for all maturities up to five years available from the Federal Reserve website (http://www.federalreserve.gov/releases/h15/data.htm), and through the standard bootstrapping approach, converted the yields into forward rates at quarterly intervals. Where necessary, linear interpolation is used. The interest-rate volatility is computed as the historical volatility of each forward rate over the sample period. The correlation between equity and interest rates was set equal to the historical correlation between the stock return and the three-month interest rate, computed on a rolling basis, with one-year histories. The CDS spreads for maturities from one to five years are taken from CreditGrades. The four parameters of the default function are fitted to these CDS spreads using Matlab. A least-squares difference of the CDS spreads to model spreads is undertaken to obtain best fit. Once calibrated, the one-year risk-neutral PD is calculated.

We compare the default probabilities from CreditGrades (that are under the physical measure, and represent the real-world PD), to the risk-neutral probabilities that we extract from the CDS spreads. The ratio of the risk-neutral probabilities to the real-world ones (usually greater than one), are a metric of the risk premium in the market for credit risk. See Berndt et al. (2005) for a comprehensive look at default risk premia extraction from CDS and expected default frequencies. For our analysis we use the one-year default probabilities.

Because our model develops a function for default intensities $\xi_i(t)$, for each issuer i, the one-year probability of default is a function of the expected integrated intensity for one year, taken under the risk-neutral measure, which is $Y = E^*[\int_0^1 \xi_i(t)\,dt]$, where E^* is the expectations operator. We undertook this integration using a tree. Given the integral, the one-year probability of default is $[1-\exp(-Y)]$.

Figure 6 **Average Risk-Neutral One-Year PD Plotted Against Those Under the Statistical Measure**



Notes. The data comes from 30 firms in the Dow Jones CDX Index. The period covered is from January 2000 to June 2002.

4.1. Empirical Analysis

Using data from CreditGrades for the period from January 2000 to June 2002, for the 30 issuers from the CDX INDU Index, we calibrated the model each day to CDS spreads. Using the fitted parameters, we computed the one-year risk-neutral default probabilities, and quantified the risk premia by dividing the risk-neutral default probability by the default probability from CreditGrades. Figure 6 shows the plot of the average default probabilities (equally weighted across 30 firms) over time, and Table 3 shows the average premia over time. This corresponds to the measure presented in Berndt et al. (2005).

A principal components decomposition of risk premia shows that there are two main components, as shown in Figure 7. We also compared the time series of the main principal component to the time series of the S&P 500 index, and found them to track closely, with a correlation of 50%; see Figure 8. This finding has connections to Duffie et al. (2005) where the S&P index is found to contain predictive value for defaults. We also compared the principal components to the time series of the VIX (volatility) index. In this case, the correlations were not significantly different from zero.

5. Concluding Comments

The following economic objectives are met by our model. First, we develop a pricing model covering multiple risks, which enables security pricing for hybrid derivatives with default risk. Second,

[7] This approach to credit correlations is a bottom-up model, similar to the popular class of copula models. In contrast are the top-down class of models, see for example, Giesecke and Goldberg (2005) and Longstaff and Rajan (2006).

[8] Recall that we are assuming for this illustration that $\gamma = 1$. In general we would have to calibrate the model to the value of γ also.

Table 3 Issuers with Respective Risk Premium Ratios and Date Ranges

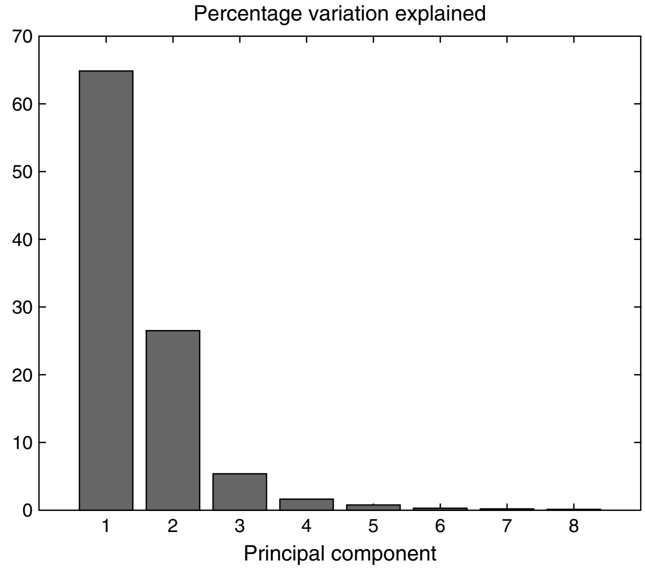
Ticker	No. of obs.	Start date	End date	Risk premium
AL	531	2000-04-28	2002-07-05	3.6935
AA	608	2000-01-03	2002-07-05	7.4249
CSX	608	2000-01-03	2002-07-05	1.0750
DE	608	2000-01-03	2002-07-05	1.0955
F	608	2000-01-03	2002-07-05	1.0463
BA	608	2000-01-03	2002-07-05	2.2341
CAT	608	2000-01-03	2002-07-05	1.1370
CTX	607	2000-01-04	2002-07-05	1.1144
GR	608	2000-01-03	2002-07-05	1.9886
IP	608	2000-01-03	2002-07-05	1.4127
DOW	608	2000-01-03	2002-07-05	3.4821
NSC	608	2000-01-03	2002-07-05	1.1005
LEN	608	2000-01-03	2002-07-05	1.3695
RTN	196	2001-09-17	2002-07-05	1.3161
TXT	608	2000-01-03	2002-07-05	1.1899
UNP	608	2000-01-03	2002-07-05	1.2366
ROH	608	2000-01-03	2002-07-05	1.9879
WY	608	2000-01-03	2002-07-05	1.6437
PHM	608	2000-01-03	2002-07-05	1.1663
MWV	608	2000-01-03	2002-07-05	2.0782
EMN	608	2000-01-03	2002-07-05	1.2730
NOC	608	2000-01-03	2002-07-05	1.4382
BNI	608	2000-01-03	2002-07-05	1.1417
LMT	608	2000-01-03	2002-07-05	1.4602
LEA	608	2000-01-03	2002-07-05	1.0968
HOM	608	2000-01-03	2002-07-05	9.0993
AXL	369	2000-12-29	2002-07-05	1.1679
GM	608	2000-01-03	2002-07-05	1.2862
IR	608	2000-01-03	2002-07-05	1.6958
DD	606	2000-01-03	2002-07-05	14.7494

Notes. The dates are in the form YYYY-MM-DD. The issuers are primarily from the Dow Jones CDX index set. The risk premium is the average ratio of the risk-neutral default probability to that under the physical measure.

the model enables the extraction of easy-to-calibrate default probability functions for state-dependent default. Third, using observable market inputs from equity and bond markets, we value complex securities via relative pricing in a no-arbitrage framework, e.g., debt-equity swaps, distressed convertibles. Fourth, the model is useful in managing credit portfolios and baskets, e.g., CDOs and basket default swaps. Finally, the extraction of credit risk premia is feasible in the model.

Technically, our hybrid defaultable model combines the ideas of both structural and reduced-form approaches. It is based on a risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, after accounting for default. We use a default-extended CEV equity model that allows volatility to vary in accordance with the leverage effect from structural models. The martingale measure in the paper is default consistent. The model is embedded on a recombining lattice, providing fast computation with polynomial complexity for run times. Cross-sectional spread data permits calibration of an

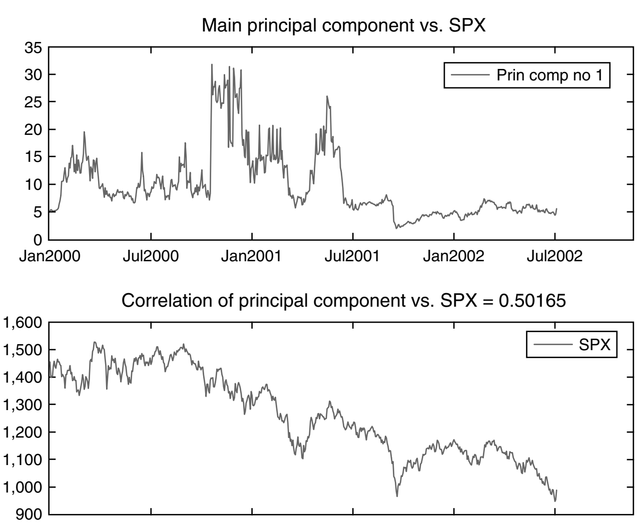
Figure 7 Principal Components Decomposition of Average Credit Risk Premia, Computed as the Ratio of Risk-Neutral One-Year PD to Those Under the Statistical Measure



Notes. The data comes from 30 firms in the Dow Jones CDX Index. The period covered is from January 2000 to June 2002. There are two main components.

implied default probability function, which dynamically changes on the state space defined by the pricing lattice. The model is parsimonious and we have been able to implement it on a spreadsheet. Further research, directed at parallelizing the algorithms in this paper and improving computational efficiency is

Figure 8 Comparison of the First Principal Component of Credit Risk Premia Against the S&P 500 Index



Notes. The main component comes from a principal components decomposition of average credit risk premia, computed as the ratio of risk-neutral one-year PD to those under the statistical measure. The data comes from 30 firms in the Dow Jones CDX index. The period covered is from January 2000 to June 2002. There are two main components. The correlation between the main component and the S&P 500 index is 50%.

under way. On the economic front, the model's efficacy augurs well for extensive empirical work.

6. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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Online Appendix

A. Deriving the Branching Process with Risk-Neutral Probabilities

In order for the normalized equity process to be a martingale, we require that at every node, we adjust the probability measure over $X_s(t)$ such that

$$E\left[\frac{S(t+h)}{S(t)}\right]=\exp[r(t)h].$$

In addition, under the Heath-Jarrow-Morton (1990) (HJM) model, the mean value of the random variable X_f must be zero, and its variance should be 1. These properties are verified as follows:

$$\begin{aligned}E(X_f)&=\frac{1}{4}[1+m_1+1-m_1-1-m_2-1+m_2](1-\lambda(t))+\frac{\lambda(t)}{2}[1-1]\\&=0,\\ \mathrm{Var}(X_f)&=\frac{1}{4}[1+m_1+1-m_1+1+m_2+1-m_2](1-\lambda(t))+\frac{\lambda(t)}{2}[1+1]\\&=1.\end{aligned}$$

Now, we compute the two conditions required to determine m_1 and m_2. We use the expectation of the equity process to determine one equation. We exploit the fact that under risk-neutrality the equity return must equal the risk free rate of interest. This leads to the following:

$$\begin{aligned}E\left[\frac{S(t+h)}{S(t)}\right]&=\frac{1}{4}(1-\lambda(t))[a(t)(1+m_1)+b(t)(1-m_1)+a(t)(1+m_2)+b(t)(1-m_2)]+\frac{\lambda(t)}{2}[0]\\&=\exp[r(t)h].\end{aligned}\tag{EC.1}$$

Hence the stock return is set equal to the risk-free return. This implies the following from a simplification of Equation (EC.1):

$$m_1+m_2=\frac{4e^{r(t)h}/(1-\lambda(t))-2[a(t)+b(t)]}{a(t)-b(t)}\equiv A.\tag{EC.2}$$

Our second condition comes from the correlation specification. Let the correlation (coincident with covariance for unit valued variables) between the shocks $[X_f(t), X_s(t)]$ be equal to ρ, where $-1\le\rho\le 1$. A simple calculation follows (ignoring the branches of default, since the correlation in that case is undefined):

$$\begin{aligned}\mathrm{Cov}[X_f(t),X_s(t)]&=\frac{1}{4}(1-\lambda(t))[1+m_1-1+m_1-1-m_2+1-m_2]\\&=\frac{m_1-m_2}{2}(1-\lambda(t)).\end{aligned}\tag{EC.3}$$

Setting this equal to ρ, we get the equation

$$m_1-m_2=\frac{2\rho}{1-\lambda(t)}\equiv B.\tag{EC.4}$$

Table EC.1 Bounds on Default Probabilities

Condition on m_i	Limit value of $\lambda(t)$
$0 \le \frac{1}{4}[1+m_1]$	$\frac{1}{2b(t)}[-2e^{r(t)h} - a(t)\rho + b(t)(\rho+2)]$
$\frac{1}{4}[1+m_1] \le 1$	$\frac{1}{4a(t)-2b(t)}[-2e^{r(t)h} - a(t)(\rho-4) + b(t)(\rho-2)]$
$0 \le \frac{1}{4}[1-m_1]$	$\frac{1}{2a(t)}[-2e^{r(t)h} - a(t)(\rho-2) + b(t)\rho]$
$\frac{1}{4}[1-m_1] \le 1$	$\frac{1}{2a(t)-4b(t)}[2e^{r(t)h} + a(t)(\rho+2) - b(t)(\rho+4)]$
$0 \le \frac{1}{4}[1+m_2]$	$\frac{1}{2b(t)}[-2e^{r(t)h} - b(t)(\rho-2) + a(t)\rho]$
$\frac{1}{4}[1+m_2] \le 1$	$\frac{1}{4a(t)-2b(t)}[-2e^{r(t)h} - b(t)(\rho+2) + a(t)(\rho+4)]$
$0 \le \frac{1}{4}[1-m_2]$	$\frac{1}{2a(t)}[-2e^{r(t)h} - b(t)\rho + a(t)(\rho+2)]$
$\frac{1}{4}[1-m_2] \le 1$	$\frac{1}{2a(t)-4b(t)}[2e^{r(t)h} + b(t)(\rho-4) - a(t)(\rho-2)]$

Notes. These eight conditions specify limit values on one period default probabilities $\lambda(t)$. Some of these conditions may not apply in the sense that they suggest negative limits for probabilities, which are superseded by the lower limit condition of zero value. The upper bound on $\lambda(t)$ will be the minimum of the positive limits in the table.

Solving the two Equations (EC.2) and (EC.4) leads to the following solution:

$$m_1 = \frac{A+B}{2}, \qquad m_2 = \frac{A-B}{2}. \tag{EC.5}$$

These values may now be substituted into the probability measure in Table 1. Notice that since the interest rate $r(t)$ only enters the probabilities and not the random shock $X_s(t)$, $\forall\, t$, the equity lattice will also be recombining, just as was the case with the HJM model for the term structure. Hence, the product space of defaultable equity and interest rates will also be recombining. As interest rates change, the probability measure will also change, but this will not impact the recombining property of the lattice. Finally, note that the analysis here is valid irrespective of the properties of the equity process chosen, and applied, in particular to any choice of constant elasticity of variance (CEV) coefficient in the models we consider.

A.1. Ensuring a Valid Probability Measure

It is also necessary that the solutions for m_1 and m_2 be such that the resultant probabilities do not become negative or greater than 1. From Table EC.1, we see that the necessary condition is $-1 \le m_i \le +1$, $i = 1, 2$. To see this, note that the greatest absolute value of the probabilities on the non-defaultable branches is when $\lambda = 0$. Given this, we require the following 2 conditions on m_1, so as to be valid probabilities:

$$0 \le \tfrac{1}{4}[1+m_1] \le 1, \qquad 0 \le \tfrac{1}{4}[1-m_1] \le 1,$$

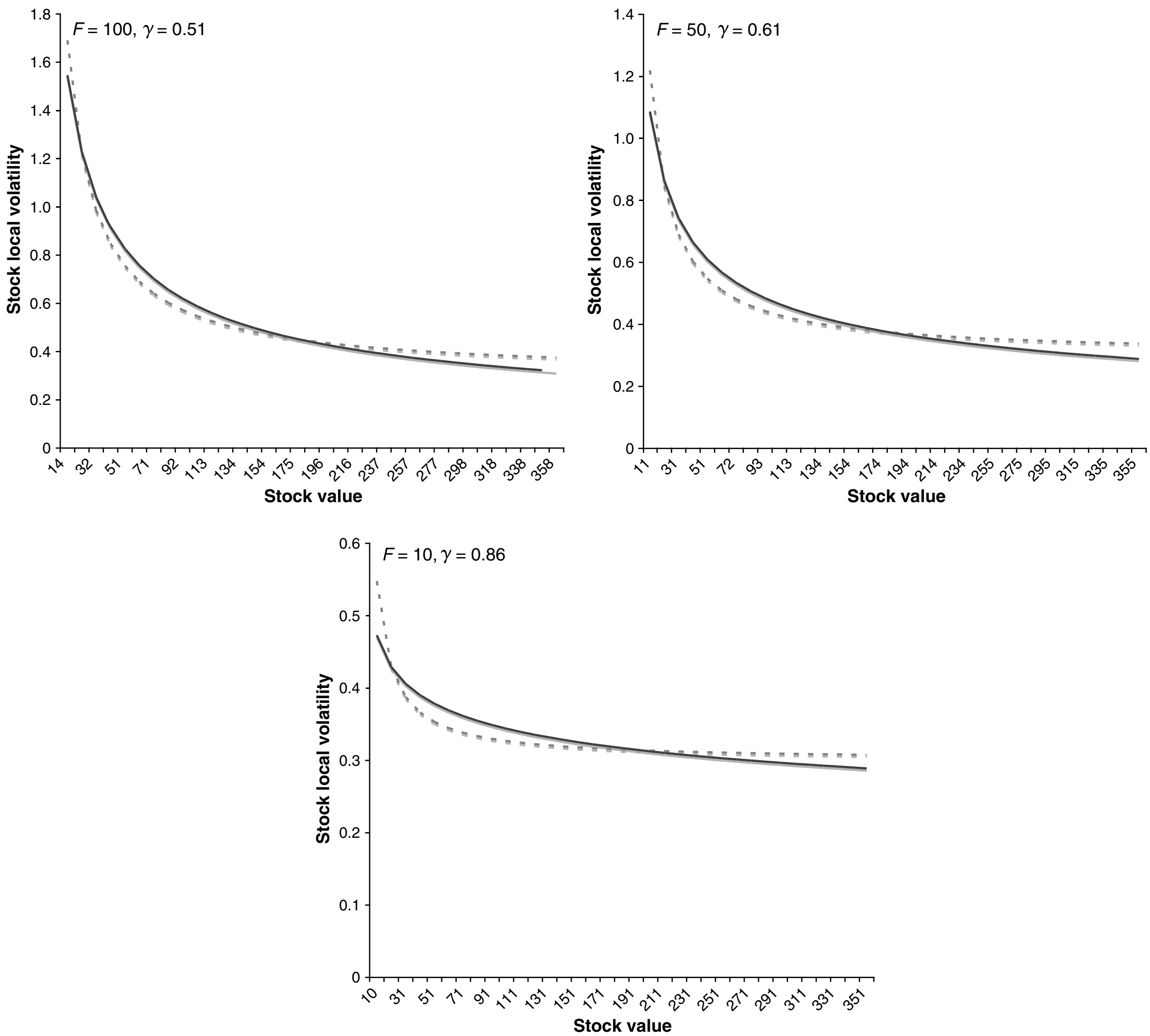
which implies that $-1 \le m_1 \le +1$. The same condition is derived for m_2.

Of course, the preceding analysis really implies that there is a range for the value of default probability $\lambda(t)$ which is consistent with the equity and term structure processes. Hence, we can derive the corresponding values of $\lambda(t)$ that correspond to the permissible ranges for m_1, m_2 above. This results in 8 bounds, which are presented in Table EC.1. After computing the value of the default probability we check that it satisfies these bounds, else we set it to be within the range values.

B. Leverage Effects in the CEV and Structural Models

In the classical structural model of Merton (1974), the stock value and its volatility are derived from assumptions concerning the firm value process. As the value of the firm falls in this model, equity

Figure EC.1 Matching the CEV Model to the Merton Model for Varying Debt Levels



Notes. We present the equity value-equity volatility plot for varying debt F. We plot the stock price on the x-axis (in increments of ten above the debt face value) and the volatility from the Merton (dotted line) and CEV (full line) models on the y-axis. We also report the γ coefficient that fits the two curves best. The input values used are $F = \{10, 50, 150\}$, $\sigma = 0.3$, $r_f = 0.02$, $T = 5$. We can see that γ ranges from 0.5 to 1 depending on the debt level F. The curves from the CEV model fit the Merton model rather well. When debt levels are low, the coefficient γ becomes closer to 1, as in the last plot. Reading from left to right and top to bottom, the fitted value of parameter σ_C is $\{5.72, 2.78, 0.65\}$.

value falls while the volatility of equity rises, which is, of course, the leverage effect. Formally, let V denotes the value of the firm's assets, σ the volatility of these asset returns, F the face value of the zero-coupon debt held by the firm, T the maturity date of the debt, and r the risk-free rate. Then, the value S and the volatility σ_S of equity in the Merton model are described by the following well-known equations:

$$S = VN(d_1) - e^{-rT} FN\big(d_1 - \sigma\sqrt{T}\big) \tag{EC.6}$$

$$\sigma_S = \sigma \frac{\partial S}{\partial V} \frac{V}{S} \equiv \sigma N(d_1) \frac{V}{S}. \tag{EC.7}$$

Equations (EC.6) and (EC.7) implies that equity volatility declines as S increases.

In the CEV model, the equity process is specified directly as $dS = rS\,dt + \sigma_C S^{\gamma}\,dZ$. Thus, the local volatility of equity returns is

$$\sigma_{\mathrm{CEV}} = \sigma_C S^{\gamma-1}, \quad \gamma \le 1. \tag{EC.8}$$

Equation (EC.8) too implies a negative relationship between equity volatility and equity values, and indeed, in both cases, equity volatility is a *convex* function of equity values. This raises an interesting question concerning the potential structural foundations of CEV models: To what extent can the equity volatility-equity value relationship (EC.7) be "mimicked" by the CEV process?

Figure EC.1 addresses this question. Each panel plots equity volatility against equity values. The dashed line in each panel represents outcomes in the Merton model. To generate these, we fixed the debt face value F of debt in the model, as also all the other parameters except for the initial firm value V. Then, we varied V from low to high values, and used (EC.6)–(EC.7) to obtain the curves. The solid line in the figures represents the CEV plot from (EC.8); in each case, σ_C and γ were chosen to minimize the sum of squared differences between the Merton-model implied curve and the CEV one.

The figure indicates that the leverage effect of CEV models can, for given ranges of equity values, imitate very closely that generated by structural models. Nonetheless, some caution is in order before interpreting the CEV model as an approximation of a structural model. For one thing, equity volatility in the CEV model goes to zero as $S \to \infty$ (except when $\gamma = 1$). On the other hand, equity volatility in the Merton model converges to the firm volatility $\sigma > 0$ as $V \to \infty$. Thus, the two curves cannot resemble each other "globally."

More generally, local equity volatility in the structural model is the result of an interplay of four parameters: the leverage ratio F/V, the time-to-maturity T, the risk-free rate r, and firm volatility σ, while in the CEV model, it depends on three parameters, γ, σ_C, and the *level* of current equity prices S. There is no obvious way to "map" the structural parameters into the CEV parameters. Thus, for example, while the leverage effect in the CEV model is regulated by the parameter γ, there is no direct interpretation of this parameter within the Merton framework.

References

See references list in the main paper.