Point Process Control

The following note is based on Chapters I, II and VII in Brémaud's book *Point Processes and Queues* (1981) .

1 Basic Definitions

Consider some probability space $(\Omega, \mathcal{F}, \mathcal{P})$. A real-valued mapping $X : \Omega \to \mathbb{R}$ is a *random variable* if for every $C \in \mathcal{B}(\mathbb{R})$ the pre-image $X^{-1}(C) \in \mathcal{F}$.

A *filtration* (or history) of a measurable space $(\Omega, \mathcal{F})$ is a collection $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ such that for all $0 \leq s \leq t$

$$\mathcal{F}_s \subseteq \mathcal{F}_t.$$

We denote by $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

A family $(X_t)_{t\geq 0}$ of real-valued random variables is called a *stochastic process*. The filtration generated by X_t is

$$\mathcal{F}_t^X := \sigma\left(X_s \,:\, s \in [0,t]\right).$$

For a fixed $\omega \in \Omega$, the function $X_t(\omega)$ is called a *path* of the stochastic process.

We say that the stochastic process is *adapted* to the filtration $\mathcal{F}_t$ if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \geq 0$. We say that X_t is $\mathcal{F}_t$*-progressive* if for all $t \geq 0$ the mapping $(t, \omega) \to X_t(\omega)$ from $[0,t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

Let $\mathcal{F}_t$ be a filtration. We define the $\mathcal{F}_t$*-predictable σ-field* $\mathcal{P}(\mathcal{F}_t)$ as follows:

$$\mathcal{P}(\mathcal{F}_t) := \sigma\left((s,t] \times A \,:\, s \in [0,t] \text{ and } A \in \mathcal{F}_s\right).$$

A stochastic process X_t is $\mathcal{F}_t$*-predictable* if X_t is $\mathcal{P}(\mathcal{F}_t)$-measurable.

Proposition 1 *A real-valued stochastic process X_t adapted to $\mathcal{F}_t$ and left-continuous is $\mathcal{F}_t$-predictable*

Given a filtration $\mathcal{F}_t$, a process $X_t$is called a $\mathcal{F}_t$*-martingale* over $[0,c]$ if the following three conditions are satisfied

1. X_t is adapted to $\mathcal{F}_t$.
2. $\mathrm{E}[|X_t|] < \infty$ for all $t \in [0,c]$.

3. $\mathrm{E}[X_t|\mathcal{F}_s] = X_s$ a.s., for all $0 \le s \le t \le c$.

If the equality in (3) is replaced by $\ge$ ($\le$) then X_t is called a *submartingale* (*supermartingale*).

Exercise 1:
Let X_t be a real-valued process with *independent increment*, that is, for all $0 \le s \le t$, $X_t - X_s$ is independent of $\mathcal{F}_s^X$. Suppose that X_t is integrable and $\mathsf{E}[|X_t|] = 0$. Show that X_t is a $\mathcal{F}_t^X$-martingale.
If in addition X_t^2 is integrable then X_t^2 is a $\mathcal{F}_t^X$-submartingale and $X_t^2 - \mathsf{E}[X_t^2]$ is a $\mathcal{F}_t$-martingale.

2 Counting Processes

Definition 1 *A sequence of random variables* $\{T_n : n \ge 0\}$ *is called a point process if for each* $n \ge 0$, $T_n \in \mathcal{F}$ *and*

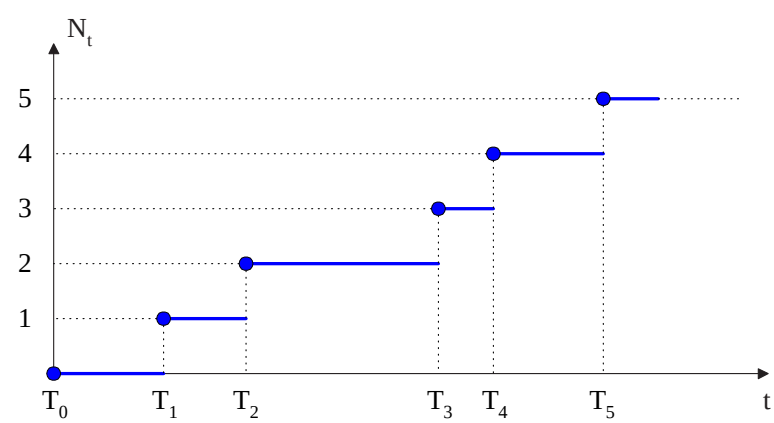
$$\forall \omega \in \Omega \quad T_0(\omega) = 0 \quad \textit{and} \quad T_n(\omega) < T_{n+1}(\omega) \ \textit{whenever} \ T_n < \infty.$$

We will only consider *nonexplosive* point process, that is, process for which $\lim_{n\to\infty} T_n = \infty$ $\mathcal{P}$-a.s.

Associated to a nonexplosive point process $\{T_n\}$, we define the corresponding *counting process* $\{N_t : t \ge 0\}$ as follows

$$N_t = n \quad \text{if } t \in [T_n, T_{n+1}).$$

Thus, N_t is a right-continuous step function starting at 0 (see figure). Since we are considering



nonexplosive point processes, $N_t < \infty$ for all $t \ge 0$ P-a.s. In addition, if $\mathrm{E}[N_t]$ is finite for all t then the point process is said to be integrable.

Example 1: Simple Renewal Process

Consider a sequence $\{X_n : n \geq 1\}$ of iid nonnegative random variables. We define the point process recursively as follows: $T_0 = 0$ and $T_n = T_{n-1} + X_n$, $n \geq 1$. A sufficient condition for the process T_n to be nonexplosive is $\mathsf{E}[X] > 0$.
One of the most famous renewal process is the Poisson process. In this case X, the inter-arrival interval, has an exponential distribution with time homogenous rate λ.

Example 2: Queueing Process
A simple queueing process Q_t is a nonnegative integer-valued process o of the form

$$Q_t = Q_0 + A_t - D_t,$$

where A_t (arrival process) and D_t (departure process) are two nonexplosive point processes without common jumps. Note that by definition $D_t \leq Q_0 + A_t$.

Exercise 2:
Let $N_t(1)$ and $N_t(2)$ be two nonexplosive point processes without common jumps and let Q_0 be a nonnegative integer-valued random variable. Define $X_t = Q_0 + N_t(1) - N_t(2)$ and $m_t = \min_{0 \leq s \leq t}\{X_s^-\}$. Show that $Q_t = X_t - m_t$ is a simple queueing process with arrival process $A_t = N_t(1)$, departure process $D_t = \int_0^t \mathbb{1}(Q_{s^-} > 0)\, \mathrm{d}N_s(2)$, and $m_t = -\int_0^t \mathbb{1}(Q_{s^-} = 0)\, \mathrm{d}N_s(2)$.

Poisson processes are commonly used in practice to represent point process. One possible explanation for this popularity is its inherent mathematical tractability. However, a well-known result in renewal theory says that the sum of a large number of independent renewal processes converges –as the number of summands goes to infinity– to a Poisson process. So, the Poisson process can be in fact a good approximation to many applications in practice. To make the Poisson process even more realistic we would like to have more flexibility on the arrival rate λ. We can achieve this generalization as follows.

Definition 2 (Doubly Stochastic or Conditional Poisson Process) *Let N_t be a point process adapted to a filtration $\mathcal{F}_t$, and let λ_t be a nonnegative measurable process. Suppose that*

$$\lambda_t \text{ is } \mathcal{F}_0 - \text{measurable for all } t \geq 0,$$

and that

$$\int_0^t \lambda s\, ds < \infty \quad \mathcal{P}\text{-a.s., for all } t \geq 0.$$

If for all $s \in [0, t]$ the increment $N_t - N_s$ is independent of $\mathcal{F}_s$ given $\mathcal{F}_0$ and

$$\mathcal{P}[N_t - N_s = k | \mathcal{F}_s] = \frac{1}{k!} \exp\left(\int_s^t \lambda_u\, du\right) \left(\int_s^t \lambda_u\, du\right)^k$$

*then N_t is called a $(\mathcal{P} - \mathcal{F}_t)$-*doubly stochastic *Poisson process with stochastic intensity λ_t.*

The process λ_t is referred as the *intensity* of the process. A special case of a doubly stochastic Poisson process occurs when $\lambda_t = \lambda \in \mathcal{F}_0$. Another example is the case where $\lambda_t = f(t, Y_t)$ for a measurable nonnegative function f and a process Y_t such that $\mathcal{F}_\infty^Y \subseteq \mathcal{F}_0$.

Exercise 3:
Show that for a doubly stochastic Poisson process N_t is such that $\mathsf{E}[\int_0^t \lambda_s \, \mathsf{d}s] < \infty$ for all $t \geq 0$ then

$$M_t = N_t - \int_0^t \lambda_s \, \mathsf{d}s$$

is a $\mathcal{F}_t$-martingale.

Based on this observation we have the following important result.

Proposition 2 *If N_t is an integrable doubly stochastic Poisson process with $\mathcal{F}_t$-intensity λ_t, then for all nonnegative $\mathcal{F}_t$-predictable processes C_t*

$$E\left[\int_0^\infty C_s \, dN_s\right] = E\left[\int_0^\infty C_s \, \lambda_s \, ds\right]$$

where $\int_0^t C_s \, \mathrm{d}N_s := \sum_{n\geq 1} C_{T_n} \, \mathbb{1}(T_n \leq t)$. It turns out that the converse is also true. This was first proved by Watanabe in a less general setting.

Proposition 3 (Watanabe (1964)) *Let N_t be a point process adapted to the filtration $\mathcal{F}_t$, and let $\lambda(t)$ be a locally integrable nonnegative function. Suppose that $N_t - \int_0^t \lambda(s) \, ds$ is an $\mathcal{F}_t$-martingale. Then N_t is Poisson process with intensity $\lambda(t)$, that is, for all $0 \leq s \leq t$, $N_t - N_s$ is a Poisson random variable with parameter $\int_s^t \lambda_u \, du$ independent of $\mathcal{F}_s$.*

Motivated by this result we define the notion of stochastic intensity for an arbitrary point process as follows.

Definition 3 (Stochastic Intensity)

Let N_t be a point process adapted to some filtration $\mathcal{F}_t$, and let λ_t be a nonnegative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$

$$\int_0^t \lambda_s \, ds < \infty \qquad \mathcal{P} - a.s.$$

If for all nonnegative $\mathcal{F}_t$ predictable processes C_t, the equality

$$E\left[\int_0^\infty C_s \, dN_s\right] = E\left[\int_0^\infty C_s \, \lambda_s \, ds\right]$$

is verified, then we say that N_t admits the $\mathcal{F}_t$-intensity λ_t.

Exercise 4: Let N_t be a point process with the $\mathcal{F}_t$-intensity λ_t. Show that if λ_t id $\mathcal{G}_t$-progressive for some filtration $\mathcal{G}_t$ such that $\mathcal{F}_t^N \subseteq \mathcal{G}_t \leq \mathcal{F}_t$ $t \geq 0$ then λ_t is also the $\mathcal{G}_t$-intensity N_t.

Similarly to the Poisson process, we can connect point processes with stochastic intensities to martingales.

Proposition 4 (Integration Theorem)

If N_t admits the $\mathcal{F}_t$-intensity λ_t (where $\int_0^t \lambda_s \, ds < \infty$ a.s.) then N_t is nonexplosive and

1. *$M_t = N_t - \int_0^t \lambda_s \, ds$ is an $\mathcal{F}_t$-local martingale.*
2. *if X_t is $\mathcal{F}_t$-predictable process such that $E[\int_0^t |X_s| \, \lambda_s \, ds] < \infty$ then $\int_0^t X_s \, dM_s$ is an $\mathcal{F}_t$-martingale.*
3. *if X_t is $\mathcal{F}_t$-predictable process such that $\int_0^t |X_s| \, \lambda_s \, ds < \infty$ a.s. then $\int_0^t X_s \, dM_s$ is an $\mathcal{F}_t$-local martingale.*

3 Optimal Intensity Control

In this section we study the problem of controlling a point process. In particular, we focus on the case where the controller can affect the intensity of the point process. This type of control differs from impulsive control where the controller has the ability to add or erase some of the point in the sequence.

We consider a point process N_t that we wish to control. The control u belongs to a set $\mathcal{U}$ of *admissible* controls. We will assume that $\mathcal{U}$ consists on the set of real-valued processes defined on $(\Omega, \mathcal{F})$ adapted to $\mathcal{F}_t^N$ in addition for each $t \in [0, T]$ we assume that $u_t \in U_t$. In addition, for each $u \in \mathcal{U}$ the point process N_t admits a $(\mathcal{P}_u, \mathcal{F}_t)$-intensity $\lambda_t(u)$. Here, $\mathcal{F}_t$ is some filtration associated to N_t.

The performance measure that we will consider is given by

$$J(u) = \mathrm{E}_u \left[\int_0^T C_s(u) \, \mathrm{d}s + \phi_T(u) \right]. \tag{1}$$

The expectation in $J(u)$ above is taken with respect to $\mathcal{P}_u$. The function $C_t(u)$ is an $\mathcal{F}_t$-progressive process and $\phi_T(u)$ is a $\mathcal{F}_T$-measurable random variable.

We will consider a problem with *complete information* so that $\mathcal{F}_t \equiv \mathcal{F}_t^N$. In addition, we assume *local dynamics*

$$\begin{aligned} u_t &= u(t, N_t) \quad \text{is } \mathcal{F}_t^N - \text{predictable} \\ \lambda_t(u) &= \lambda(t, N_t, u_t) \\ C_t(u) &= C(t, N_t, u_t) \\ \phi_T &= \phi_T(T, N_T). \end{aligned}$$

Exercise 5:
Consider the cost function

$$J(u) = \mathsf{E}\left[\sum_{0<T_n\leq T} k_{T_n}(u)\right].$$

Where $k_t(u)$ is a nonnegative $\mathcal{F}_t$-measurable process. Show that this cost function can be written in the from given by equation (1).

3.1 Dynamic Programming for Intensity Control

Theorem 1 (Hamilton-Jacobi Sufficient Conditions)

Suppose there exists for each $n \in N_+$ a differentiable bounded $\mathcal{F}_t$-progressive mapping $V(t,\omega,n)$ such that all $\omega \in Omega$ and all $n \in N_+$

$$\frac{\partial}{\partial t}V(t,\omega,n) \quad + \quad \inf_{v\in U_t}\{\lambda(t,\omega,n,v)\left[V(t,\omega,n+1)-V(t,\omega,n)\right]+C(t,\omega,n,v)\}=0 \tag{2}$$

$$V(T,\omega,n)=\phi(T,\omega,n). \tag{3}$$

and suppose there exists for each $n \in N_+$ an $\mathcal{F}_t^N$-predictable process $u^*(t,\omega,n)$ such that $u^*(t,\omega,n)$ achieves the minimum in equation (2). Then, u^* is the optimal control.

Exercise 6: Proof the theorem.

This Theorem lacks of practical applicability because of Value Function is in general path dependent. The analysis can be greatly simplified if we assume that the problem is Markovian.

Corollary 1 (Markovian Control)

Suppose that $\lambda(t,\omega,n,v)$, $C(t,\omega,n,v)$, and $\phi(t,\omega,n)$ do not dependent on ω and that there is a function $V(t,n)$ such that

$$\frac{\partial}{\partial t}V(t,n) \quad + \quad \inf_{v\in U_t}\{\lambda(t,n,v)\left[V(t,n+1)-V(t,n)\right]+C(t,n,v)\}=0 \tag{4}$$

$$V(T,n)=\phi(T,n). \tag{5}$$

suppose that the minimum is achieved by a measurable function $U^*(t,n)$. Then, u^* is the optimal control.

4 Applications to Revenue Management

In this section we present an application of point process optimal control based on the work by Gallego and van Ryzin (1994)*.

*Optimal Dynamic Pricing of Inventories with Stochastic Demand over Finite Horizons, *Mgmnt. Sci.* **40**, 999-1020.

4.1 Model Description and HJB Equation

Consider a seller that owns I units of a product that wants to sell over a fixed time period T. Demand for the product is characterized by a point process N_t. Given a price policy p, N_t admits the intensity $\lambda(p_t)$. The seller's problem is to select a price strategy $\{p_t : t \in [0,T]\}$ (a predictable process) that maximizes the expected revenue over the selling horizon. That is,

$$\max J_p(t,I) := \mathrm{E}_p\left[\int_0^T p_s \,\mathrm{d}N_s\right]$$

$$\text{subject to} \qquad \int_0^T \mathrm{d}N_s \leq I \qquad \mathcal{P}_p \text{ a.s.}$$

In order to ensure that the problem is feasible we will assume that there exist a price p_∞ such that $\lambda(p_\infty) = 0$ a.s. In the case, we define the set of admissible pricing policies $\mathcal{A}$, as the set of predictable $p(t, I - N_t)$ policies such that $p(t,0) = p_\infty$. The seller's problem becomes to maximize over $p \in \mathcal{A}$ the expected revenue $J_p(t,I)$.

Using corollary 1, we can write the optimality condition for this problem as follows:

$$\begin{aligned} \frac{\partial}{\partial t}V(t,n) \quad + \quad & \sup_p \{\lambda(p)\,[V(t,n) - V(t,n-1)] - p\,\lambda(p)\} = 0 \\ & V(T,n) = 0. \end{aligned}$$

Let us make the following transformation of time $t \leftarrow T - t$, that is, t measures the remaining selling time. Also, instead of looking at the price p as the decision variable we use λ as the control and $p(\lambda)$ as the inverse demand function. The revenue rate $r(\lambda) := \lambda\, p(\lambda)$ is assumed to be *regular*, *i.e.*, continuous, bounded, concave, has a bounded maximizer $\lambda^* = \min\{\widetilde{\lambda} = \operatorname{argmax}\{r(\lambda)\}\}$ and such that $\lim_{\lambda\to 0} r(\lambda) = 0$. Under this new definitions the HJB equation becomes

$$\begin{aligned} \frac{\partial}{\partial t}V(t,n) \quad = \quad & \sup_\lambda \{r(\lambda) - \lambda\,[V(t,n) - V(t,n-1)]\} = 0 \\ & V(0,n) = 0. \end{aligned}$$

Proposition 5 *If $\lambda(p)$ is a regular demand function then there exists a unique solution to the HJB equation. Further, the optimal intensities satisfies $\lambda^*(t,n) \leq \lambda^*$ for all n for all $0 \leq t \leq T$.*

Closed-form solution to the HJB equation are generally intractable however, the optimality condition can be exploited to get some qualitative results about the optimal solution.

Theorem 2 *The optimal value function $V^*(t,n)$ is strictly increasing and strictly concave in both n and t. Furthermore, there exists an optimal intensity $\lambda^*(n,t)$ that is strictly increasing in n and strictly decreasing in t.*

The following figure plots a sample path of price and inventory under an optimal policy.

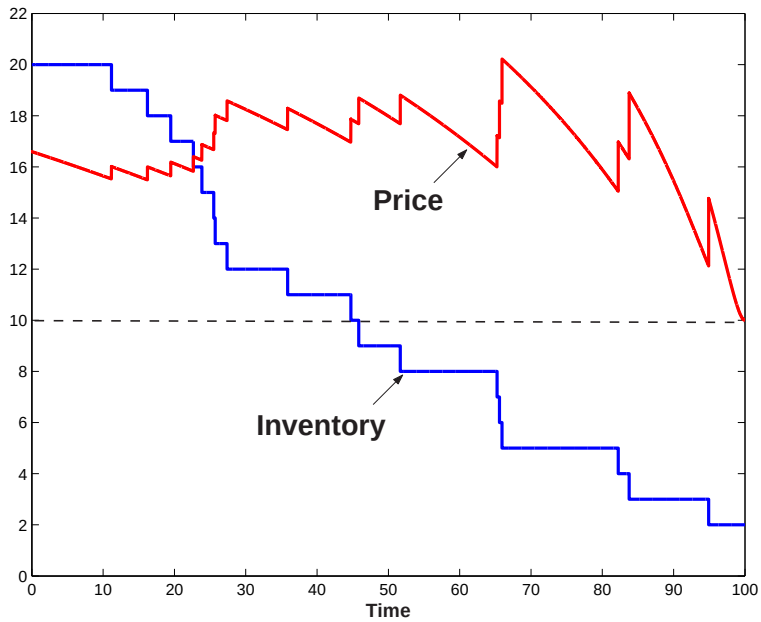


Figure 1: *Path of an optimal price policy and its inventory level. Demand is a time homogeneous Poisson process with intensity* $\lambda(p) = \exp(-0.1p)$*, the initial inventory is* $C_0 = 20$*, and the selling horizon is* $H = 100$*. The dashed line corresponds to the minimum price* $p^{min} = 10$*.*

4.2 Bounds and Heuristics

The fact that the HJB is intractable in most cases creates the need for alternative solution methods. One possibility, that we consider here, is the use of the *certainty equivalent* version of the problem. That is, the deterministic control problem resulting from changing all uncertainty by its expected value. In this case, the deterministic version of the problem is given by

$$V^D(T,n) = \max_{\lambda} \int_0^T r(\lambda_s)\,\mathrm{d}s$$

$$\text{subject to} \qquad \int_0^T \lambda_s \mathrm{d}s \leq x.$$

The solution to this time-homogeneous problem can be found easily. Let $\lambda^0(T,n) = \frac{n}{T}$, that is, the *, run-out rate*. Then, it is straightforward to show that the optimal deterministic rate is $\lambda^D = \min\{\lambda^*, \lambda^0(T,n)\}$ and the optimal expect revenue is $V^D(T,n) = T\ \min\{r(\lambda^*), r(\lambda^0(T,n))\}$.

At least two things make the deterministic solution interesting.

Theorem 3 *If* $\lambda(p)$ *is a regular demand function then for all* $n \geq 0$ *and* $t \geq 0$

$$V^*(t,n) \leq V^D(t,n).$$

Thus, the deterministic value function provides an upper bound on the optimal expected revenue.

In addition, if we fixed the price at p^D and we denote by $V^{FP}(t,n)$ the expected revenue collected from this fixed price strategy then we have the following important result.

Theorem 4

$$\frac{V^{FP}(t,n)}{V^*(t,n)} \geq 1 - \frac{1}{2\sqrt{\min\{n, \lambda^* t\}}}.$$

Therefore, the fixed price policy is asymptotically optimal as the number of product n or the selling horizon t become large.