Introduction to Deterministic Optimal Control

For a fixed interval $[t_0, t_1]$, we define the control problem:

$$\min_{u \in \mathcal{U}} J(x_0, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \qquad \text{(system dynamics)}$$

$$\phi(x(t_1)) = 0 \qquad \text{(boundary conditions)},$$

where $\mathcal{U}$ is the set of piece-wise continuous functions from $[t_0, t_1]$ to a closed subset $U \subseteq \mathbb{R}^m$, a C^1 function $\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^k$, and a continuous function $f : \mathbb{R} \times \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ having continuous first partial derivatives with respect to x. A control problem in this form is called *Langrange problem*.

Equivalent problem (*Mayer problem*):

$$\min_{u \in \mathcal{U}} \phi_1(x(t_1))$$

$$\text{subject to} \quad \dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \qquad \text{(system dynamics)}$$

$$\phi(x(t_1)) = 0 \quad k = 2, \dots, k \qquad \text{(boundary conditions)}.$$

Weak & Strong Extremals

Let $\mathcal{H}[a,b]$ be a subset of piecewise right-continuous function with left-limit (*càdlàg*). We define on $\mathcal{H}[a,b]$ two norms

$$\text{for } x \in \mathcal{H}[a,b] \qquad \|x\| = \sup_{t\in[a,b]} \{|x(t)|\} \quad \text{and} \quad \|x\|_1 = \|x\| + \|\dot{x}\|.$$

A set $\{x \in \mathcal{H}[a,b] \ : \ \|x - x^*\|_1 < \epsilon\}$ is called a *weak neighborhood* of x^*. A solution x^* is called a *weak solution* if $J(x^*) \leq J(x)$ for all x in a weak neighborhood containing x^*.

A set $\{x \in \mathcal{H}[a,b] \ : \ \|x - x^*\| < \epsilon\}$ is called a *strong neighborhood* of x^*. A solution x^* is called a *strong solution* if $J(x^*) \leq J(x)$ for all x in a strong neighborhood containing x^*.

Example:

$$\min_x J(x) = \int_{-1}^{1} (x(t) - \text{sign}(t))^2 \, \mathrm{d}t + \sum_{t\in[-1,1]} (x(t) - x(t^-))^2,$$

where $x(t^-) = \lim_{\tau\uparrow t} x(\tau)$.

Necessary Conditions

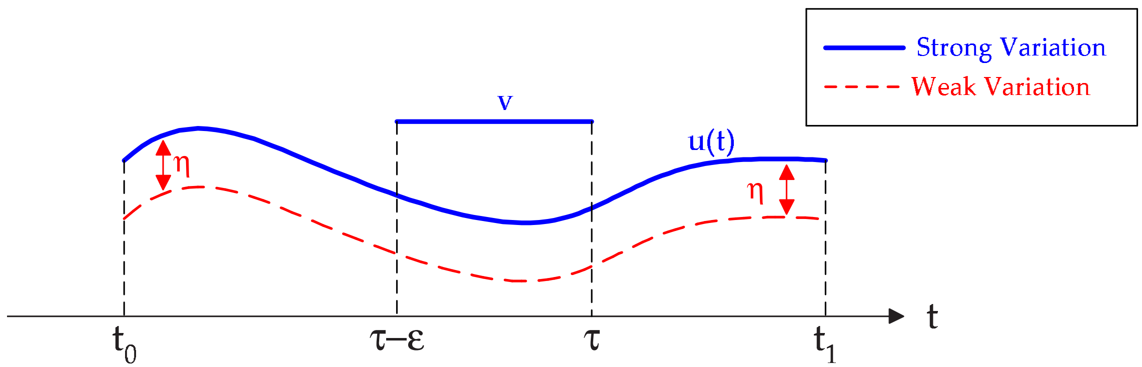
Given a control $u \in \mathcal{U}$ with corresponding *trajectory* $x(t)$, we consider the following family of variations:

For a fixed direction $v \in U$, $\tau \in [t_0, t_1]$, and $\eta > 0$ small, we defined the "strong" variation ξ of $u(t)$ in the direction v by the function

$$\begin{aligned} \xi : 0 \leq \epsilon \leq \eta \quad &\longrightarrow \quad \mathcal{U} \\ \epsilon \quad &\longrightarrow \quad \xi(\epsilon) = u^{\epsilon}, \end{aligned}$$

where

$$u^{\epsilon}(t) = \begin{cases} v & \text{if } t \in (\tau - \epsilon, \tau] \\ u(t) & \text{if } t \in [t_0, t_1] \cap (\tau - \epsilon, \tau]^c. \end{cases}$$



Lemma 1: For a real variable ϵ, let $x^\epsilon(t)$ be the solution of $\dot{x}^\epsilon(t) = f(t, x^\epsilon(t), u(t))$ on $[t_0, t_1]$ with initial condition

$$x^\epsilon(t_0) = x(t_0) + \epsilon y + o(\epsilon).$$

Then,

$$x^\epsilon(t) = x(t) + \epsilon\delta(t) + o(t, \epsilon),$$

where $\delta(t)$ is the solution of

$$\dot{\delta}(t) = f_x(t, x(t), u(t))\,\delta(t), \quad t \in [t_0, t_1] \text{ and } \delta(t_0) = y.$$

Lemma 2: If x^ϵ are solutions to $\dot{x}^\epsilon(t) = f(t, x^\epsilon(t), u^\epsilon(t))$ with the same initial condition $x^\epsilon(t_0) = x_0$ then

$$x^\epsilon(t) = x(t) + \epsilon\delta(t) + o(t, \epsilon),$$

where $\delta(t)$ solves

$$\delta(t) = \begin{cases} 0 & \text{if } t_0 \le t < \tau \\ f(\tau, x(\tau), v) - f(\tau, x(\tau), u(\tau)) + \int_\tau^t f_x(s, x(s), u(s))\,\delta(s)\,\mathrm{d}s & \text{if } \tau \le t \le t_1. \end{cases}$$

Pontryagin Principle For Free Terminal Conditions

Mayer's formulation: Let $P(t)$ be the solution of

$$\dot{P}(t) = -P(t)\, f_x(t, x(t), u(t)), \qquad P(t_1) = -\phi_x(x(t_1)).$$

A necessary condition for optimality of a control u is that

$$P(t)\,[f(t, x(t), v) - f(t, x(t), u(t))] \leq 0$$

for each $v \in U$ and $t \in (t_0, t_1]$.

Lagrange's formulation: We define the *Hamiltonian* H as

$$H(t, x, u) := P(t) f(t, x, u) - L(t, x, u).$$

Where $P(t)$ solves

$$\dot{P}(t) = -\frac{\partial}{\partial x} H(t, x, u)$$

with boundary condition $P(t_1) = 0$. A necessary condition for a control u to be optimal is

$$H(t, x(t), v) - H(t, x(t), u(t)) \leq 0 \quad \text{for all } v \in U,\ t \in [t_0, t_1].$$

Pontryagin Principle with Terminal Conditions

Mayer's formulation: Let $P(t)$ be the solution of

$$\dot{P}'(t) = -P'(t)\, f_x(t, x(t), u(t)), \qquad P(t_1) = -\lambda' \phi_x(t_1, x(t_1)).$$

A necessary condition for optimality of a control $u \in U$ is that there exists λ, a nonzero k-dimensional vector with $\lambda_1 \leq 0$, such that

$$P(t)' \left[f(t, x(t), v) - f(t, x(t), u(t)) \right] \leq 0$$

$$P(t_1)' f(t_1, x(t_1), u(t_1)) = -\lambda' \phi_t(t_1, x(t_1)).$$

Example: Solve

$$\min_u \int_0^T (u(t) - 1) x(t)\, \mathrm{d}t,$$

subject to $\dot{x}(t) = \gamma\, u(t)\, x(t) \quad x_0 > 0,$

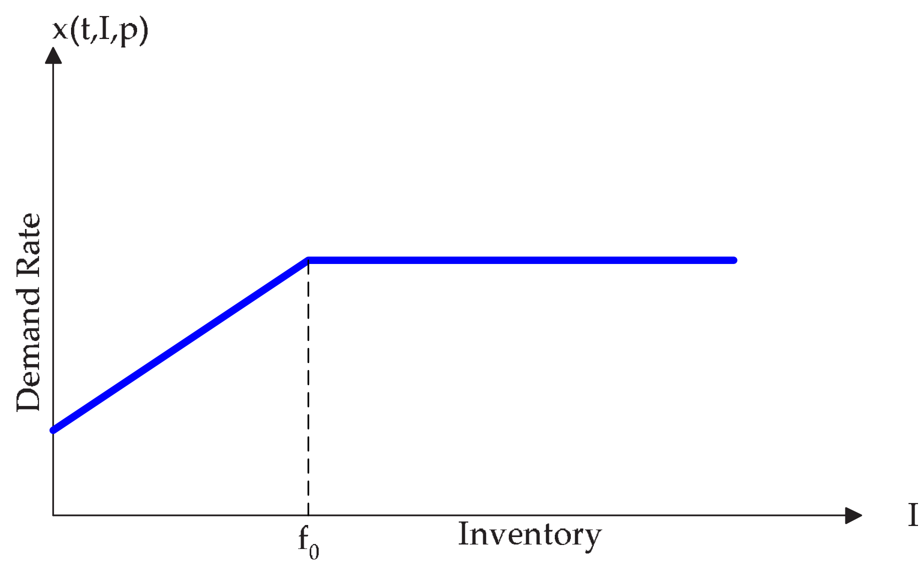
$0 \leq u(t) \leq 1, \quad \text{for all } t \in [0, T].$

Clearance Pricing and Inventory Policies for Retail Chains

S. Smith & D. Achabal (*Mgmnt. Sci.* **44**,*285-300, 1998.*)

Summary: The paper studies pricing and inventory policies for a single perishable product under the assumption that demand rate is deterministic and sensitive to both price and inventory levels. The main objective is to compute optimal *clearance* pricing policies.

Let $x(t, I(t), p(t))$ be the demand rate at time t if the price is $p(t)$ and the available inventory is $I(t)$.



Model Formulation

1. $x(t, I(t), p(t))$: Deterministic demand rate.
2. $p(t)$: Dynamic pricing policy.
3. $s(t)$: Sales trajectory.
4. $I(t) = I_0 - s(t)$: Inventory trajectory.
5. $H(t)$: On-hand inventory trajectory.
6. H_0, I_0: Initial conditions.
7. c_e: Per unit residual value.

<u>System Dynamics:</u>

$$s(t) = I_0 - I(t) = I_0 - \int_{t_0}^{t} x(\tau, I(\tau), p(\tau))\, \mathrm{d}\tau \qquad \Longleftrightarrow$$

$$\dot{I}(t) = -x(t, I(t), p(t)), \quad I(0) = I_0.$$

<u>Objective Function:</u>

$$R(I_0) = \max_p \int_{t_0}^{t_e} p(t)\, x(t, I(t), p(t))\, \mathrm{d}t + c_e\, I(t_e).$$

<u>Boundary Constraints:</u>

$$I(t_e) \geq 0.$$

Model Formulation

$$R(I_0) = \max_p \int_{t_0}^{t_e} p(t)\, x(t, I(t), p(t))\, \mathsf{d}t + c_e\, I(t_e)$$

subject to $\dot{I}(t) = -x(t, I(t), p(t)), \quad I(0) = I_0$

$I(t_e) \geq 0.$

Hamiltonian: $H(t, I(t), p(t), \lambda(t)) = (p - \lambda)\, x.$

Adjoint Condition: $\dot{\lambda}(t) = -(p - \lambda)\, x_I$ and $\lambda(t_e) = c_e + \theta.$

Maximum Principle: $0 = (p - \lambda)x_p + x.$

Solution:

$$\dot{\lambda} = \frac{x\, x_I}{x_p} \qquad \text{and} \qquad p = \lambda - \frac{x}{x_p}.$$

Separable Demand Rate

Suppose that $x(t, I, p) = k(t)\, y(I)\, \exp(-\gamma p)$. In this case the solution satisfies the ODE:

$$\dot{p}(t) = \frac{1}{\gamma}\, k(t)\, \dot{y}(I(t))\, \exp(-\gamma p(t)).$$

Lemma 1: For the multiplicative separable demand case we have

$$\frac{x(t, I(t), p(t))}{k(t)} = y(I(t))\, \exp(-\gamma\, p(t)) = \text{constant}.$$

This result implies that there are a pair of constants p_e and y_e such that the optimal price trajectory as a function of the inventory is given by

$$p(I) = p_e + \frac{1}{\gamma} \ln\left(\frac{y(I)}{y_e}\right)$$