Midterm Review

(B60.4308: Appl. Stoc. Contr. to Rev. Mgmnt.)

Definition 1 *Let T be a transformation defined on an open domain D in a normed space X and having range in a normed space Y. If for fixed $x \in D$ and each $h \in X$ there exists $\delta T(x;h) \in Y$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\| \to 0} \frac{\|T(x+h) - T(x) - \delta T(x;h)\|}{\|h\|} = 0$$

then T is said to be Fréchet differentiable *at x and $\delta T(x;h)$ is said to be the* Fréchet differential *of T at x with increment h,*

Problem 1:

a) If the Fréchet differential of T exists at x, then the Gâteaux differential exits at x and they are equal.

b) Let $X = C[0,1]$ and let $f(x) = \int_0^1 g(x(t), t)\, \mathrm{d}t$, where it is assumed that the function g_x exits an is continuous with respect to x and t. Show using the definition that

$$\delta f(x;h) = \int_0^1 g_x(x,t)\, h(t)\, \mathrm{d}t$$

is a Fréchet differential.

Problem 2:

Let f be areal-valued functional defined on a vector space X. Suppose that x_0 minimizes f on the convex set $\Omega \subseteq X$ and that f is Gâteaux differentiable at x_0. Then,

$$\delta f(x_0; x - x_0) \geq 0 \quad \text{for all } x \in \Omega.$$

Problem 3:

a) Consider the following problem

$$\min_{x(t), t_2} J(x, t_2) = \int_{t_1}^{t_2} f(t, x(t), \dot{x}(t))\, \mathrm{d}t$$
$$\text{subject to} \qquad x(t_1) = x_1 \quad \text{and} \quad (t_2, x(t_2)) \in (t, g(t)).$$

That is the right end point is allowed to vary along a curve S described by the function $x = g(t)$. Show that in this case, the set of necessary conditions is

$$f_x(t, x, \dot{x}) = \frac{\mathrm{d}}{\mathrm{d}t} f_{\dot{x}}(t, x, \dot{x}) \qquad \text{Euler Condition}$$
$$\{f(t, x, \dot{x}) + [\dot{g}(t) - \dot{x}(t)] f_{\dot{x}}(t, x, \dot{x})\}\Big|_{t=t_2} = 0. \qquad \text{Transversality Condition}$$

b) Use the result in part (a) to show that the minimum arc length $x(t)$ from a point (say the origin (0,0)) to the curve $g(t)$ is orthogonal to the tangent of g at the intersection point.

Problem 4:

The standard calculus of variation problem that assumes that the extremal are continuously differentiable functionals. In some problems, it is necessary to consider a broader class of functionals. Suppose that we seek to minimize (or maximize)

$$J[x] = \int_a^b F(t, x(t), \dot{x}(t))\, \mathrm{d}t$$

among all functions x for which $x(a) = A$, $x(b) = B$ and which have continuous derivatives on $[a, b]$ except possibly at a single point in $[a, b]$. If the extremum has a continuous derivative except at $c \in [a, b]$ show that $F_x = \mathrm{d}F_{\dot{x}}/\mathrm{d}t$ on the interval $[a, c)$ and $(c, b]$ and that the functions $F_{\dot{x}}$ and $F - \dot{x}F_{\dot{x}}$ are continuous at c. These are called the Weierstrass-Erdman corner conditions. Apply these considerations to the functional

$$J[x] = \int_{-1}^{1} x^2(1-\dot{x})^2\, \mathrm{d}t, \qquad x(-1) = 0,\ x(1) = 1.$$

Problem 5:

Let

$$J(x) = \int_{t_0}^{t_1} K(s,t)x(s)x(t)\, \mathrm{d}s\, \mathrm{d}t + \int_{t_0}^{t_1} (x(t))^2 - \int_{t_0}^{t_1} x(t)f(t)\, \mathrm{d}t$$

where f and K are given and $K(s, t) = K(t, s)$.

a) Find $\delta J(x; y)$. Show that a necessary condition for $J(x)$ to be minimum at x^*, among all continuous functions on $[t_0, t_1]$, is that x^* be a solution of the integral equation

$$2\int_{t_0}^{t_1} K(s,t)x(s)\, \mathrm{d}s + 2x(t) = f(t).$$

Problem 6:

Let

$$J(x) = \int_{t_0}^{t_1} \left(x^4(t) + x(t)\,\dot{x}(t) + \frac{1}{2} \right) \mathrm{d}t.$$

a) Verify that the only extremal is $x^*(t) = 0$.

b) For $(t_0, x_0) = (0, 1)$ and $(t_1, x_1) = (1, 0)$, show that the infimum of $J(x)$ on $\mathcal{D}$ –the vector space of all real-valued piecewise C^1 functions on $[t_0, t_1]$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$– is 0 but that $J(x)$ has no minimum on $\mathcal{D}$. *Hint.* For $r = 1, 2, \ldots$ consider $x(t) = 1 - rt$ if $0 \le t \le r^{-1}$ and $x(t) = 0$ for $r^{-1} \le t \le 1$.

Problem 7:

A collection of $N \ge 2$ objects is observed randomly and sequentially one at a time. The observer may either select the current object observed, in which case the selection process terminates, or reject the object and proceed to observe the next object. The observer can rank each object relative to those already observed, and the objective is to maximize the probability of selecting the "best" object according to some criterion. It is assumed that no two objects can be judged to be equal. Let r^* be the smallest positive integer r such that

$$\frac{1}{N-1} + \frac{1}{N-2} + \cdots + \frac{1}{r} \leq 1.$$

Show that an optimal policy requires that the first r^* objects be observed. If the r^*th object has rank 1 relative to the other already observed, it should be selected; otherwise, the observation process should be continued until an object of rank 1 relative to those already observed is found. *Hint*: Assume that, if the rth object has rank 1 relative to the previous $(r-1)$ objects, then the probability that it is best is r/N. For $k \geq r^*$, let $J_k(0)$ be the maximal probability of finding the best object assuming k objects have been selected and the kth object is not the best relative to the previous $(k-1)$ objects. Show that

$$J_k(0) = \frac{k}{N}\left(\frac{1}{N-1} + \frac{1}{N-2} + \cdots + \frac{1}{k}\right).$$

Problem 8:

Consider the following production scheduling problem: It is known that, at time t_0, the production rate $p(t)$ is $p(t_0) = p_0$; it is required that, at time $t_1 > t_0$, the production rate be $p(t_1) = p_1$. The objective is to minimize total cost which is the integral over the interval $[t_0, t_1]$ of the sum of two partial cost rates C_1 and C_2.

It is assumed that the cost of changing the rate of production $p(t)$ varies as the square of the rate of change $\dot{p}(t)$ of the production rate; this is teh cost associated with the mechanical changes accompanying a change in the production rate. Hence

$$C_1 = k_1(\dot{p}(t))^2$$

where k_1 is a constant.

In addition, there are certain personnel costs associated with changing the production rate. On the one hand, if the production rate $p(t)$ is increased, there are hiring costs for new workers; these are proportional to the rate of change of the production rate, and they increase linearly with time. On the other hand, if the production rate decreases, certain personnel costs are eliminated; these saving are assumed to be proportional to the rate of change of the production rate (by the same proportionality factor as hiring costs), and they increase linearly with time. Hence the personnel costs may be expressed as

$$C_2 = k_2\, t\, \dot{p}(t)$$

where k_2 is a constant.

Find the optimal production scheduling that minimizes costs.

Problem 9:

Consider the optimization problem

$$\min J(x) = \int_{t_0}^{t_1} L(t, x, \dot{x}, \ddot{x})\, \mathsf{d}t$$

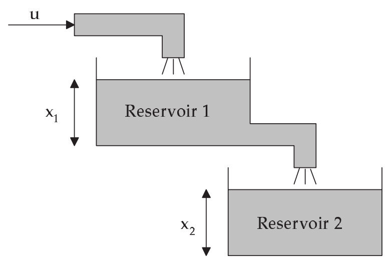
subject to $\quad x(t_0) = x_0,\ x(t_1) = x_1,\ \dot{x}(t_0) = \dot{x}_0,\ \dot{x}(t_1) = \dot{x}_1$

where x_0, x_1, $\dot{x}_0$, and $\dot{x}_1$ are constants. Show that the necessary condition for optimality is in this case

$$L_x - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} L_{\ddot{x}} = 0.$$

Problem 10:

Consider the system of reservoir shown in the figure.



The system of equations is

$$\dot{x}_1(t) = -x_1(t) + u(t),$$
$$\dot{x}_2(t) = x_1(t),$$

and the control constraint is $0 \le u(t) \le 1$ for all t, Initially

$$x_1(0) = x_2(0) = 0.$$

We want to maximize $x_2(1)$ subject to $x_1(1) = 0.5$. Solve the problem.

Problem 11:

Consider a model for one sector of the economy. A production function $f(k)$ gives the rate of increase of capital per worker due to production. If the population has growth rate constant β, there is a rate of decrease $-\beta k$ of capital per worker due to population growth. A fraction u of the new production is retained in the economy and the remaining fraction $1-u$ is consumed. Hence the equation

$$\dot{k} = u\, f(k) - \beta\, k$$

governs the capital per worker in the economy. Let $h(c)$ denote de utility to the economy of consuming at rate c. Consider the problem of choosing a saving plan $u(t)$ to increase the capital per worker from k_1 to k_2 in the fixed time T, while maximizing

$$\int_0^T h([1-u(t)]f(k(t)))\,\mathrm{d}t.$$

For $f(k) = \alpha k$ and $h(c) = c$ compute the extremal control law. Notice that $0 \le u(t) \le 1$ must be satisfied.

Problem 12:

For the simplest problem in calculus of variation consider the minimum as a function of the initial endpoint $s = t_0$, $y = x_0$ with fixed final end point (t_1, x_1). Thus let χ_{sy} be the class of piecewise C^1 functions $x(t)$ on $[s, t_1]$ such that $x(s) = y$, $x(t_1) = x_1$ and let

$$V(s, y) = \inf_{\chi_{sy}} \int_s^{t_1} L(x(t), \dot{x}(t))\,\mathrm{d}t.$$

Show that partial differential equation od dynamic programming is

$$\text{(a)} \qquad V_s(s,y) + \min_u \{L(y,u) + u\, V_y(s,y)\} = 0$$

and if $x(t)$ is optimal in χ_{sy} that

$$\text{(b)} \qquad V_s(t,x(t)) + L(x(t),\dot{x}(t)) + \dot{x}(t)\, V_y(t,x(t)) = 0$$

Show that if V is C^2 then (a) (b) imply that the Euler equation

$$\frac{\mathsf{d}}{\mathsf{d}t} L_{\dot{x}}(x(t),\dot{x}(t)) = L_x(x(t),\dot{x}(t))$$

holds.

Problem 13:

Consider a simple queueing process Q_t adapted to $\mathcal{F}_t$ and with input point processes A_t and D_t. Suppose that D_t has $\mathcal{F}_t$-intensity μ_t. Then, the departure process also admits the $\mathcal{F}_t$ intensity $\mu_t\, \mathbb{1}(Q_t > 0)$.

Problem 14:

Let Q_t be a simple queue with $\mathcal{F}_t$-parameters λ_t and μ_t where λ_t and μ_t are bounded. Define $p_n(t)$, $\lambda_n(t)$, and $\mu_n(t)$ by

$$\begin{aligned} p_n(t) &= \mathsf{E}[Q_t = n] \\ \lambda_n(t) &= \mathsf{E}[\lambda_t | Q_t = n] \\ \mu_n(t) &= \mathsf{E}[\mu_t | Q_t = n]. \end{aligned}$$

Show that for all $n \geq 0$

$$p_n(t) = p_n(0) + \int_0^t (p_{n-1}(s)\lambda_{n-1}(s)\mathbb{1}(n>0) - p_n(s)(\lambda_n(s) + \mu_n(s)\mathbb{1}(n>0)) + p_{n+1}(s) + \mu_{n+1}(s))\, \mathsf{d}s.$$