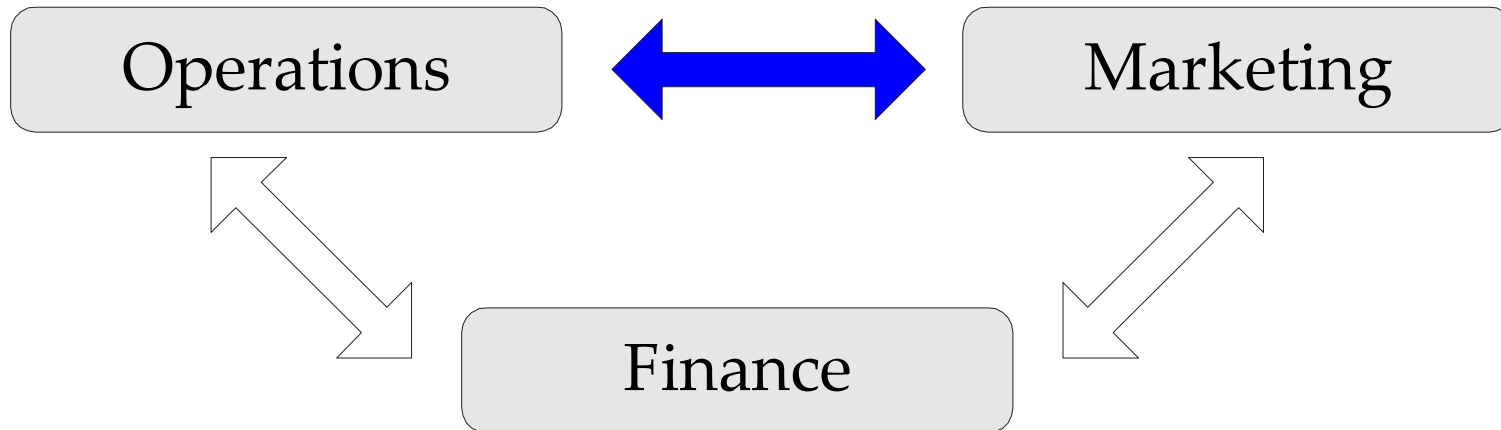


B60.4308: Applications of Stochastic Control to Revenue Management

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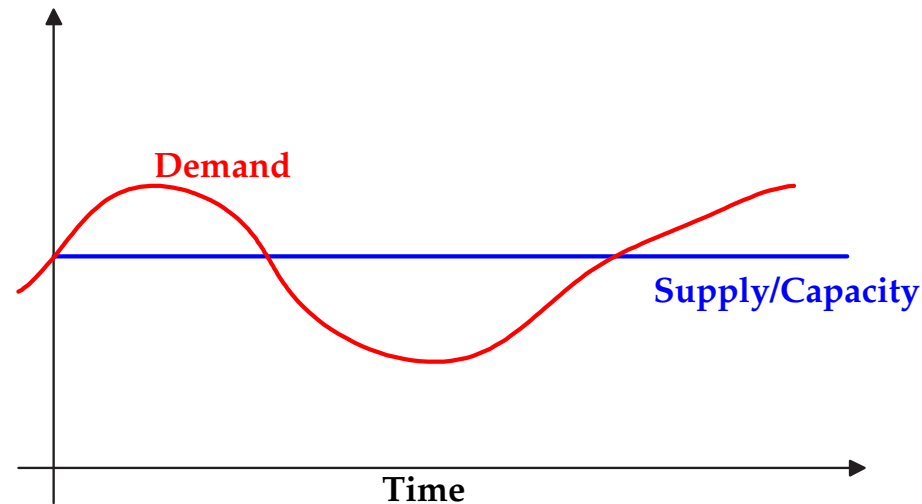
Motivación



- There is lack of coordination among the three areas \implies inefficiency... but also implies an opportunity!
- There is a need to integrate traditional OM/OR models (e.g. Inventory and Capacity Management, Scheduling and Logistics, Queueing Theory, etc.) with Marketing and Finance.
- In this class, we will be discussing one specific connection between Operations and Marketing using the Revenue Management framework.

Revenue Management: Introduction

Supply Chain Management and its tight connection to Inventory Theory studies the optimal strategy to match supply and demand.



For many industries capacity is only adjustable in the long run and the corresponding costs can be considered sunk in the short run.

Airlines, Hotels, Car Rentals, Movie Theaters, Opera Halls, Cruise Lines, Bandwidth and Internet Providers, Passenger Railways, The Yankee Stadium, etc.

In this context, the **Price** is a very effective variable that managers can use to encourage or discourage demand in the short run.

Revenue Management: Introduction

Problem Characteristics:

1. Capacity is perishable with a selling horizon that is fixed and predefined.
2. Capacity is irreversible.
3. Demand is stochastic.
4. Price can be used to segment demand.

Objective:

“Sell the Right product to the Right customer at the Right price.”

Decisions/Controls:

- At any specific time, how many units of a product should be offered and at what price?
- How these quantities should change over time?

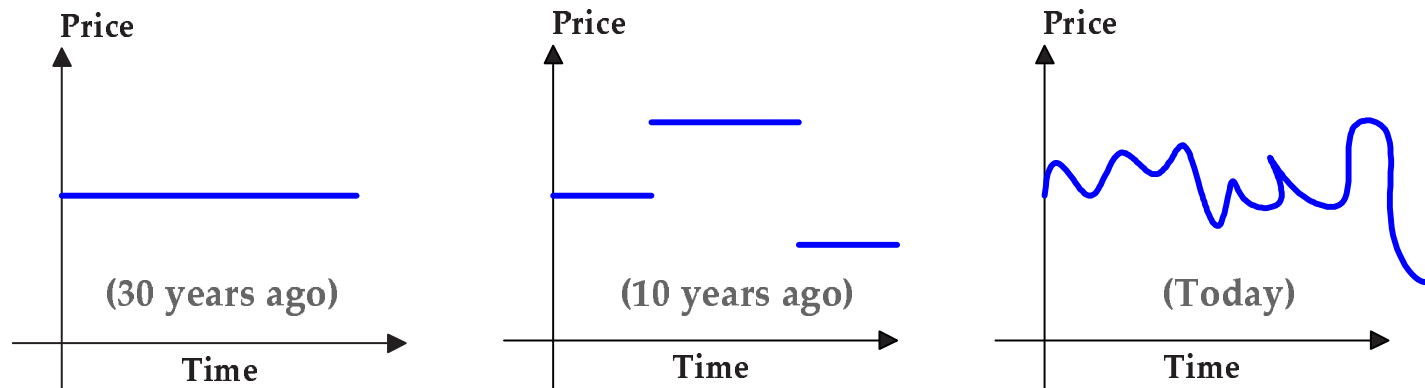
Revenue Management: Introduction

Requirements:

An information system capable to (i) collect real-time information about demand, sales, inventories and (ii) implement dynamic pricing policies and capacity control efficiently.

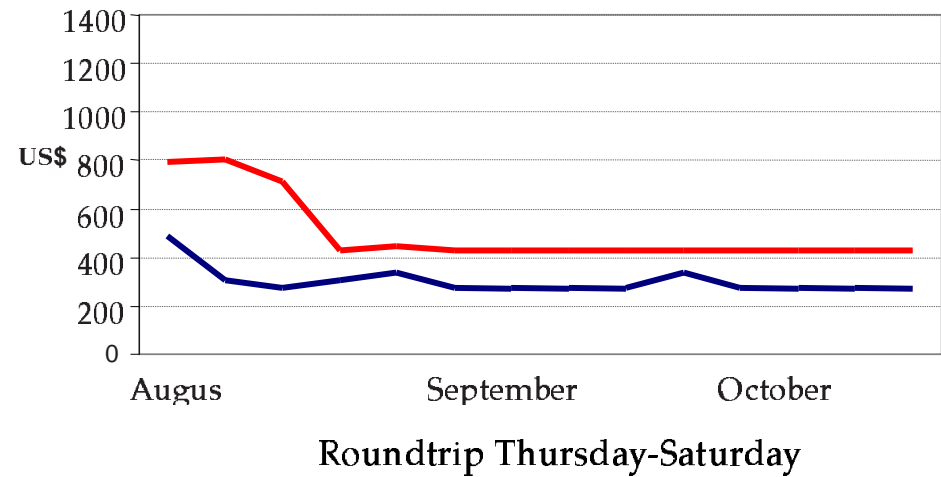
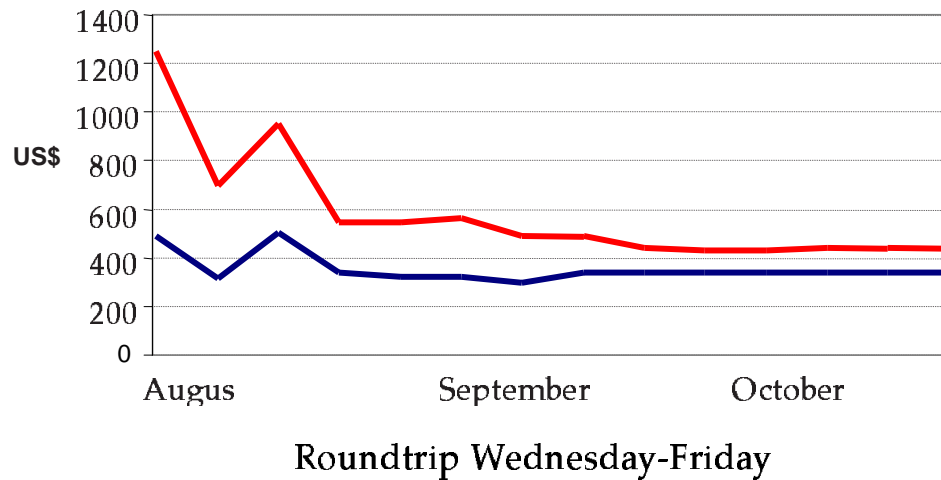
Implications:

- Better use of available resources
- Increase in revenues (usually between 5% to 10%)
- Increase in price volatility.



Revenue Management: Introduction

New York - Chicago



A Wednesday-Friday ticket is 15%-20% more expensive than a Thursday-Saturday ticket!!

Revenue Management: History

- Kincaid & Darling (1963)
- Rothstein (1971-74) and Littlewood (1971)
- Belobaba (1987) – AA (Sabre System)
- Smith *et al.* (1992) – Yield Management at AA – F. Edelman Prize
- Gallego & van Ryzin (1994) – Bitran & Mondschein (1997)
- Feng & Xiao (1999)-(2000), Bertsimas, Popescu, de Boer (2001)
- van Ryzin & Vulcano (2002).

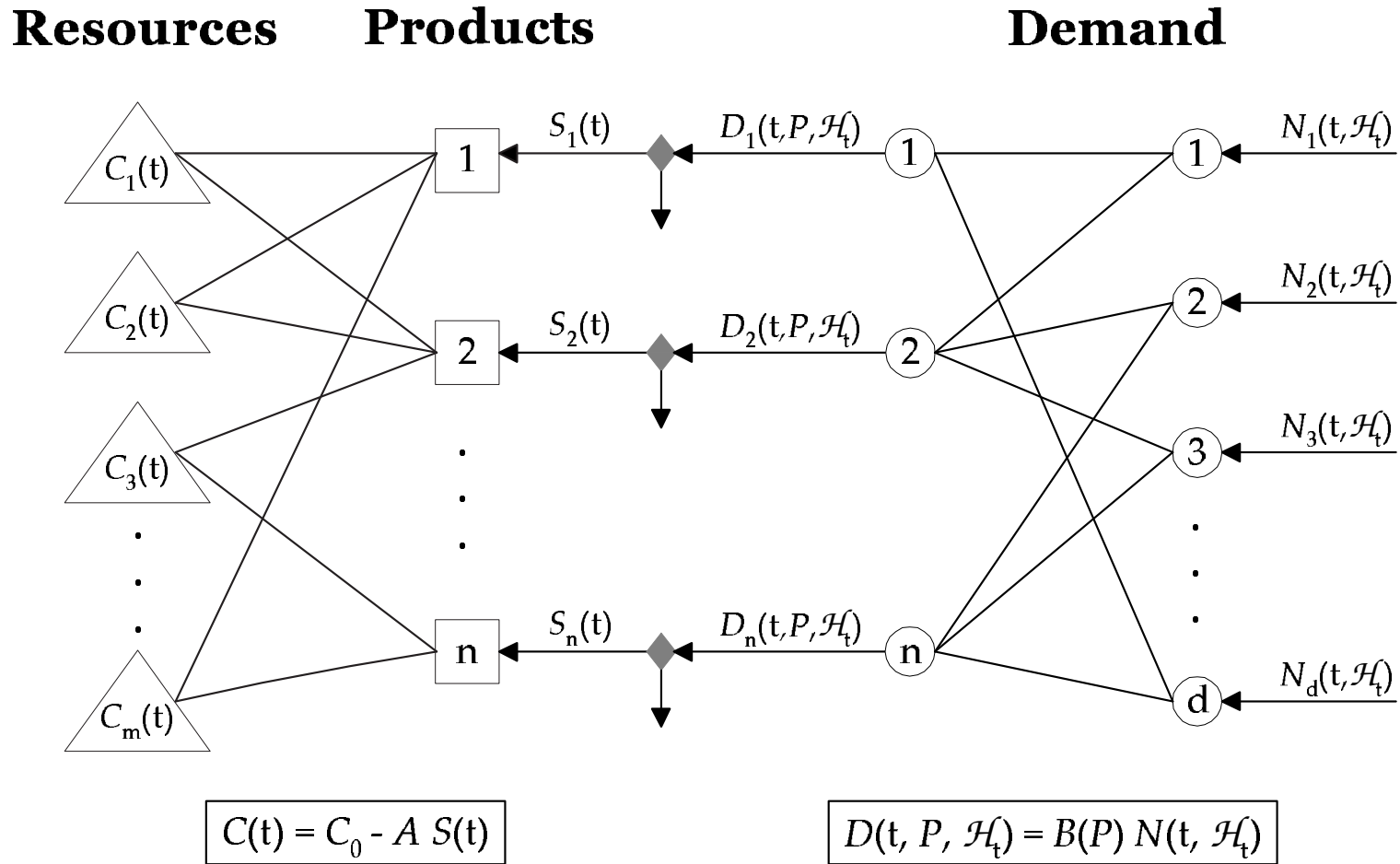
Revenue Management: Formulation

$$\sup_{P,S} \mathbb{E} \left[\int_0^T p_t dS(t) \right]$$

subject to: $C_t = C_0 - A S(t) \geq 0$ for all $t \in [0, T]$,
 $0 \leq S(t) \leq D(t, P, \mathcal{H}_t)$ for all $t \in [0, T]$,
 $P \in \mathcal{P}$, and $S(t) \in \mathcal{H}_t$.

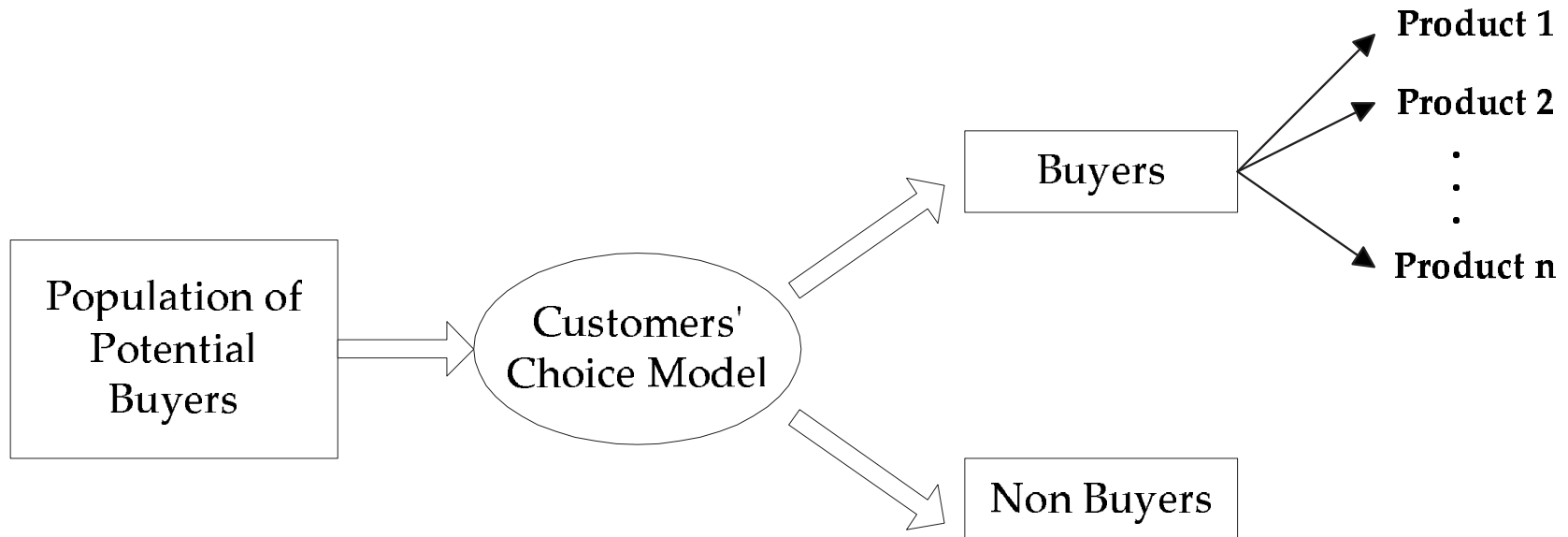
- $T > 0$: Selling horizon.
- $C_t \in \mathbb{R}_+^m$: vector of available resources at time t .
- $A \in \mathbb{R}_+^{m \times n}$: “Product-Resource” matrix.
- $(\mathcal{H}_t)_{0 \leq t \leq T}$: “observed history”.
- $P = \{p_t \in \mathbb{R}_+ : t \in [0, T]\} \in \mathcal{P}$: Price policy. \mathcal{P} is the set of admissible policies.
- $D(t, P, \mathcal{H}_t) \in \mathbb{R}_+^n$: Cumulative demand process up to time t .
- $S(t) \in \mathbb{R}_+^n$: Cumulative sales up to time t .

Revenue Management: Formulation



Revenue Management: Formulation

Modelling Demand



Choice Models

- Reservation Price
- Multinomial Logit/Probit
- Hierarchical Models

Diffusion Models

- Life Cycle / Bass Model
- Innovators / Imitators
- Inventory Effects

Revenue Management: Formulation

Modelling Pricing Policies

- Static or Dynamic: $p_t = \bar{p}$ for all $t \in [0, T]$ vs. $p_t \in C^k[0, T]$.
- Discrete or Continuous: $p_t \in C^k[0, T]$ vs. $p_t \in \{p_1, p_2, \dots, p_\eta\}$.
- Fixed number of price changes. $\Delta p_t := p_t - p_{t-} = 0$ for all $t \notin \{t_1, t_2, \dots, t_\tau\}$.
- Joint Price Constraints: $p_t - p_s \leq \xi$ for all $t, s \in [0, T]$.
- Cost-Based Pricing: $p_t \geq c_t$ for all $t \in [0, T]$.

Revenue Management: Taxonomy

	Elements	Descriptor
A	Resource	Discrete/Continuous
B	Capacity	Fixed/Nonfixed
C	Prices	Predetermined/Set Optimally/Set Jointly
D	Willingness to Pay	Buildup/Drawdown
E	Discount Price Classes	1/2/3/.../I
F	Reservation Demand	Deterministic/Mixed/Random-independent/ Random-correlated
G	Show-Up of Discount Reservation	Certain/Uncertain without Cancellation/ Uncertain with cancellation
H	Show-Up of Full-Price Reservation	Certain/Uncertain without Cancellation/ Uncertain with cancellation
I	Group Reservations	No/Yes
J	Diversion	No/Yes
K	Displacement	No/Yes
L	Bumping Procedure	None/Full-price/Discount/FCFS/Auction
M	Asset Control Mechanism	Distinct/Nested
N	Decision Rule	Simple Static /Advanced Static / Dynamic

Source: Weatherford & Bodily (1992), *Ops. Res.* **40**, 831-844.

Example: A2-B1-C1-E2-N3

Revenue Management: Example

Seat Inventory Control (P. Belobaba (1989), *Ops. Res.* **37**, 183-197):

$$V(C, T) = \max_S E [p_1 S_1(T) + p_2 S_2(T)] \quad (\text{Suppose that } p_1 \geq p_2)$$

subject to

$$S_1(t) + S_2(t) \leq C \quad \text{for all } t \in [0, T]$$

$$S_1(t) \leq D_1(t) \quad \text{for all } t \in [0, T]$$

$$S_2(t) \leq D_2(t) \quad \text{for all } t \in [0, T]$$

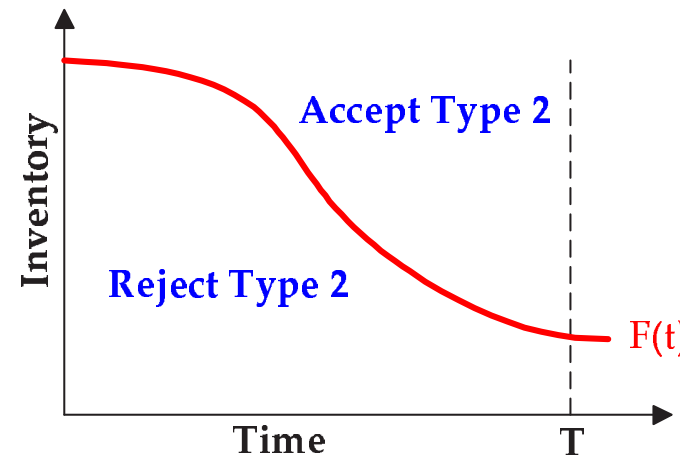
$S(t)$ nondecreasing and non-anticipative with respect to $D(t)$.

Intuitively, the solution to this problem should look like...

1. Always accept type 1 if there is available capacity.

2. Accept type 2 at time t if available capacity ($C(t)$) is "sufficiently large":

$$C(t) \geq F(t)$$



Revenue Management: Example

Solution Method: Suppose we know $V(C, t)$ where

$V(C, t)$: Optimal revenue from if the available capacity at time $T - t$ is C .

Then, if at time $T - t$ when available capacity is C there is a Type 2 request then we should accept it if:

$$p_2 \geq V(C, t) - V(C - 1, t) := \Delta_C V(C, t).$$

Approximations:

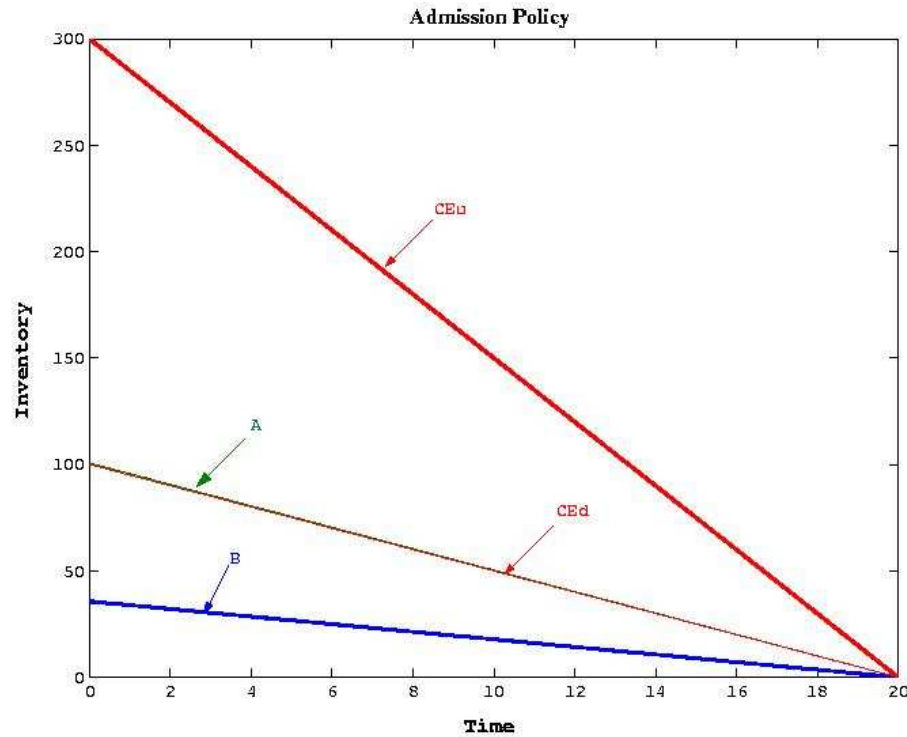
★ Belobaba: $V^B(C, t) = E [p_1 \min\{D_1(T - t), C\}]$.

★ Certainty Equivalent:

$$V^{CE}(C, t) = p_1 \min\{E[D_1(T - t)], C\} + p_2 \min\{E[D_2(T - t)], (C - E[D_1(T - t)])^+\}.$$

★ Anticipative: $V^A = E [p_1 \min\{D_1(T - t), C\} + p_2 \min\{D_2(T - t), (C - D_1(t))^+\}]$.

Revenue Management: Example



$D_i(t)$ exponentially distributed with mean $\mu_i t$.
 Data: $p_1 = \$100$, $p_2 = \$70$, $\mu_1 = 5$, $\mu_2 = 10$, $C = 300$, $T = 20$.

	V^B	V^{CEu}	V^A
Expected Revenue	22357	10009	19333
Std. Dev.	700.37	1000.31	781.13
Type 2 rejections	11.57%	100%	33.2%

Revenue Management: Solution Methods

Certainty Equivalent:

Idea: Replace every source of uncertainty by its corresponding expected value.

Implications: Deterministic Control Problem.

$$V^{\text{det}}(C_0, T) = \max_{P, S} \int_0^T p_t dS(t)$$

$$\begin{aligned} \text{subject to: } C_t = C_0 - A S(t) &\geq 0 \quad \text{for all } t \in [0, T], \\ 0 \leq S(t) &\leq E[D(t, P, \mathcal{H}_t)] \quad \text{for all } t \in [0, T], \end{aligned}$$

Under mild assumptions on the demand, we can show that

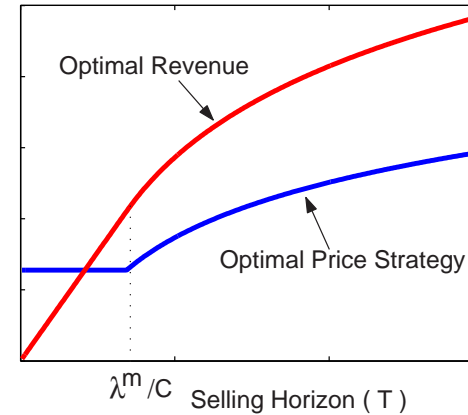
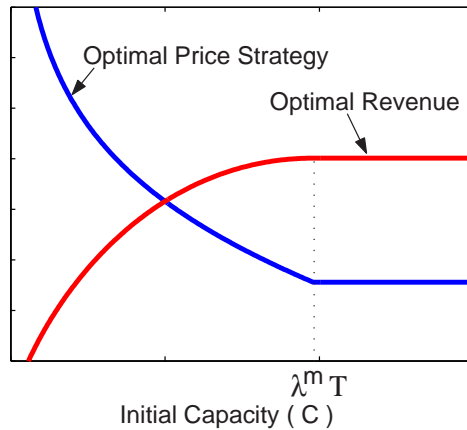
* A fixed price policy is optimal.

$$* V_{\text{opt}}^{\text{stoc}} \leq V_{\text{opt}}^{\text{det}}$$

* $1 \geq \frac{V_{\text{opt}}^{\text{stoc}}(p^{\text{det}})}{V_{\text{opt}}^{\text{stoc}}} \geq 1 - \frac{\nu}{2}$ where ν is the coefficient of variation of the cumulative demand at a price p^{det} .

Revenue Management: Solution Methods

Properties of the Deterministic Solution:



$$\text{Deterministic demand: } \lambda(T, p) = \lambda (1 + \exp(p))^{-1}$$

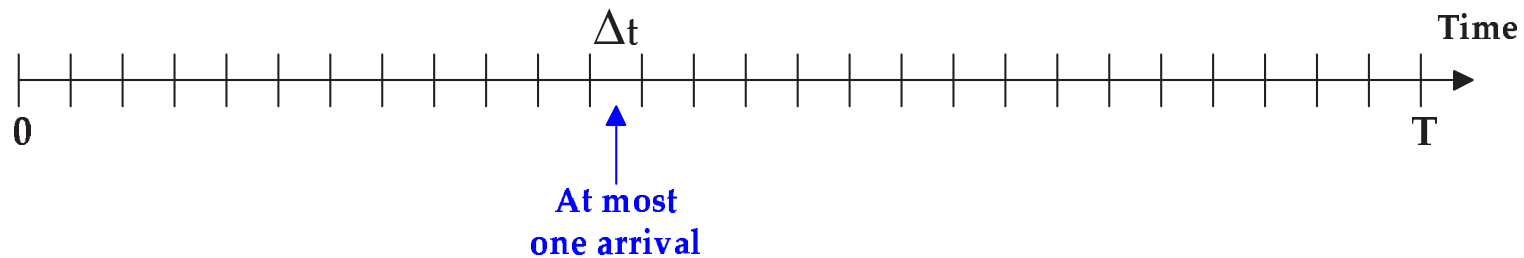
The optimal price is:

- ★ Decreasing with the level of inventory C .
- ★ Increasing on the selling horizon T .
- ★ There is an “efficient” inventory target $C^* = \lambda^* T$.

Revenue Management: Solution Methods

Discrete Time Dynamic Programming:

We divide the $[0, T]$ interval into N disjoint sub-intervals (periods) of length Δt ($T = N \Delta t$) so that at most one arrival during each sub-interval.



Suppose that with fixed probability q_n a potential buyer arrives during the sub-interval $n = 1, \dots, N$.

What type of demand process satisfies this assumption?

Suppose that at a price p an arriving potential buyer buys a unit of product with fixed probability $\beta(p) \in [0, 1]$ for all p .

We define $V_n(C)$ as the optimal revenue from period n to the end if the available inventory at the beginning of the period is C .

Revenue Management: Solution Methods

Bellman Equation:

$$V_n(C) = \max_p \{q_n \beta(p)(p + V_{n+1}(C - 1)), 1 - q_n \beta(p)V_{n+1}(C)\},$$

with boundary conditions $V_n(0) = 0$ for all n and $V_{N+1}(C) = 0$ for all C .

Let define the demand rate $\lambda_n(p)$ as follows $q_n \beta(p) = \lambda_n(p) \Delta t$. Then, under some smoothness conditions on $V_n(C)$, taking limit as $\Delta t \downarrow 0$ on the discrete time DP we can get

$$\boxed{-\frac{\partial V_t(C)}{\partial t} = \max_p \{\lambda_t(p)[p - (V_t(C) - V_t(C - 1))]\} .}$$

Bellman Equation

Optimal price policies require knowing (or approximating) the value of the “opportunity cost” $V_t(C) - V_t(C - 1)$ (Bid Price). Under mild assumption (Gallego & van Ryzin (1994)), the optimal price path $p_t^*(C)$ satisfies:

* $p_t^*(C)$ is decreasing in C * $p_t^*(C)$ is increasing in $T - t$.

Problem: How we solve the Bellman equation? The discrete time DP for one product, 200 units of initial inventory and 100 sub-intervals requires the solution of 20.000 optimization problems. With three products about 800 millions opt. problems!

Revenue Management: Solution Methods

DP in Discrete Time:

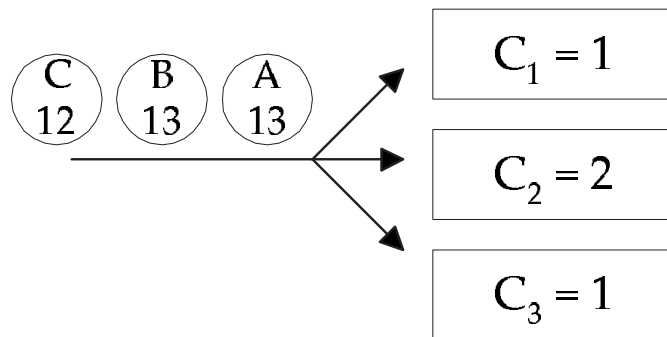
Suppose the price can be modify only at fixed a predefined instants.

In this situation, the characterization of the dynamics of C_n , the inventory position, can be computationally difficult.

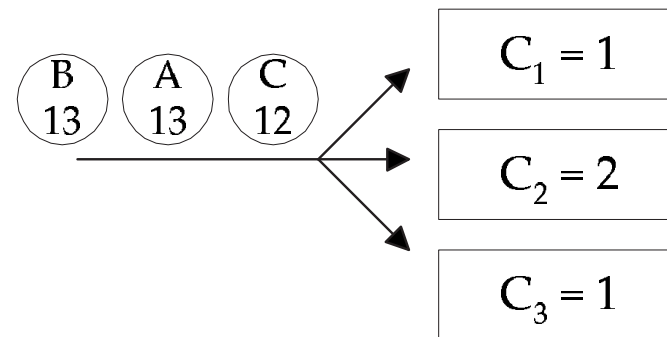
For example, consider the following example with three products:

Inventory at the beginning of period n is $C_n = (1, 2, 1)$. During period n customers A, B, and C arrive. Customers A and B want to buy product 1 but if 1 is not available then they will try to get 3. Customer C wants product 1 first but if not available then s/he will try to get 2.

Case 1



Case 2



What is the inventory C_{n+1} at the beginning of next period for Case 1 and Case 2?

Revenue Management: State of the Art

Solution Methods:

- ★ Heuristics: Certainty Equivalent, Rolling Horizon, etc.
- ★ Bid Price method, bounds on $V_t(C) - V_t(C - A.i)$.
- ★ MonteCarlo Simulation.
- ★ Approximated DP.

Research Opportunities:

- ★ Multi Product
- ★ Strategic Buyer Behavior
- ★ Market Competition
- ★ Product Design and Bundling