B60.4308: Applications of Stochastic Control to Revenue Management

Homework #4: Due Thursday, April 24, 2003

Problem 1:

Consider the stochastic control problem

$$V(s,x) = \max_u \mathrm{E}[(X_T)^r], \quad \text{where } 0 < r < 1$$
$$\text{subject to} \quad \mathrm{d}X_t = a\,u\,\mathrm{d}t + u\,\mathrm{d}B_t, \quad X_s = x > 0,$$

where $B_t \in \mathbb{R}$ is a Brownian motion, $a \in \mathbb{R}$ is a constant, and

$$T = \inf\{t > s \,:\, X_t = 0\} \wedge t_1,$$

where $t_1 > s$ is a constant future time.

Solve this control problem and show that the optimal control is

$$u^*(t,x) = \frac{a\,x}{1-r},$$

with corresponding optimal performance

$$V(s,x) = x^r \, \exp\left(\frac{a^2(T-s)r}{2(1-r)}\right).$$

Problem 2:

Consider the stochastic control problem

$$V(s,x) = \sup_u \mathrm{E}\left[\int_s^\infty \exp(-\rho\,t)\,f(X_t)\,\mathrm{d}t\right],$$
$$\text{subject to} \quad \mathrm{d}X_t = r\,u_t\,X_t\,\mathrm{d}t + \alpha\,u_t\,X_t\,\mathrm{d}B_t,$$

where r, α, and $\rho > 0$ are constants, $f(\cdot)$ is a bounded continuous function.

Assume that $V \in C^2$ and the the optimal Markov control u^* exists.

a) Write down the Bellman equation and show that

$$\frac{\partial^2 V(t,x)}{\partial x^2} \le 0.$$

b) Assume that $\frac{\partial^2}{\partial x^2}V(t,x) < 0$. Prove that

$$u^*(t,x) = -\frac{r\frac{\partial V(t,x)}{\partial x}}{\alpha^2 x \frac{\partial^2 V(t,x)}{\partial x^2}}$$

and that

$$2\alpha^2\Big(\exp(-\rho t)f(x) + \frac{\partial V(t,x)}{\partial t}\Big)\frac{\partial^2 V(t,x)}{\partial x^2} - r^2\left(\frac{\partial V(t,x)}{\partial x}\right)^2 = 0.$$

c) Assume that $\frac{\partial^2 V(t,x)}{\partial x^2} = 0$. Prove that $\frac{\partial V(t,x)}{\partial x} = 0$ and

$$\exp(-\rho t)\, f(x) + \frac{\partial V(t,x)}{\partial t} = 0.$$

d) Assume that $u_t^* = u^*(X_t)$ and that b) holds. Prove that $V(t,x) = \exp(-\rho t)\, g(x)$ where $g(x)$ satisfies

$$2\alpha^2(f(x) - \rho g(x))\, g''(x) - r^2\,(g'(x))^2 = 0.$$

Problem 3:

Let X_t denote your wealth at time t. Suppose that at any time t you have a choice between two investments:

1) A risky investment where the unit price P_1 satisfies the equation

$$\mathrm{d}P_1(t) = a_1 P_1(t)\mathrm{d}t + \sigma_1 P_1(t)\mathrm{d}B_t.$$

2) A safer (less risky) investment where the unit price P_2 satisfies

$$\mathrm{d}P_2(t) = a_2 P_2(t)\mathrm{d}t + \sigma_2 P_2(t)\mathrm{d}\tilde{B}_t.$$

The parameters a_i and σ_i are constants such that

$$a_1 > a_2 \qquad \text{and} \qquad \sigma_1 > \sigma_2$$

and B_t and $\tilde{B}_t$ are independent one-dimensional Brownian motions.

a) Let u_t denote the fraction of the fortune X_t which is placed in the risky investment at time t. Show that

$$\mathrm{d}X_t = X_t(a_1 u_t + a_2(1-u_t))\,\mathrm{d}t + X_t(\sigma_1 u_t \mathrm{d}B_t + \sigma_2(1-u_t)\,\mathrm{d}\tilde{B}_t).$$

b) Assuming that u is a Markov control, *i.e*, $u_t = u(t, X_t)$, find the generator of (t, X_t).

c) Write down the HJB equation for the stochastic control problem

$$V(s,x) = \sup_{u} \mathrm{E}\left[\sqrt{X_T}\right],$$

where $T = \min\{t > s \,:\, X_t = 0\} \wedge t_1$.

d) Find the optimal control u^* for the problem in c).

Problem 4:

Consider the following control problem (*Bounded Velocity Follower*):

$$\begin{aligned} & V(x) = \min_{\xi} \mathrm{E}\left[\int_0^\infty \exp(-\alpha t)\,(x + B_t - \xi_t)^2\,\mathrm{d}t\right] \\ \text{subject to} \quad & -\infty < \theta_0 \le \dot{\xi}_t \le \theta_1 < \infty, \end{aligned}$$

where B_t is a standard one-dimensional Brownian motion.

a) Write down the HJB equation and show that a bang-bang solution solves the problem.

b) Using the the smoothness fit assumption $V(x) \in C^2$ solve the control problem.

Problem 5:

Consider the following singular control problem.

$$\begin{aligned} & V(x) = \min_{\theta_t} \lim_{T\to\infty} \frac{1}{T}\mathrm{E}\left[\int_0^T \theta_t^2 \mathrm{d}t + pU_T + rL_T\right] \\ \text{subject to} \quad & \mathrm{d}X_t = -\theta_t\,\mathrm{d}t + \mathrm{d}B_t + \mathrm{d}L_t - \mathrm{d}U_t, \end{aligned}$$

where the pair (U_t, L_t) is the two-sided regulator for the interval $[0,b]$. p and r are the cost of "hitting" the upper and lower boundaries $x = b$ and $x = 0$, respectively.

Write down the HJB equation and find the optimal control policy θ^*.