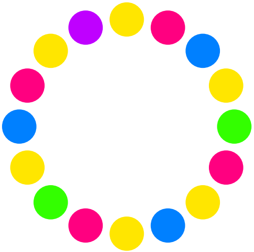
Recharging Bandits

Bobby Kleinberg **Cornell University**

Joint work with Nicole Immorlica.



NYU Machine Learning Seminar **New York, NY** **24 Oct 2017**

Prologue

Can you construct a dinner schedule that:

- never goes **2** days without macaroni and cheese
- never goes **3** days without pizza
- never goes **5** days without fish?

Answer: Impossible. For $N \geq 60$,

$$\lfloor N/2 \rfloor + \lfloor N/3 \rfloor + \lfloor N/5 \rfloor > N.$$

Prologue

Can you construct a dinner schedule that:

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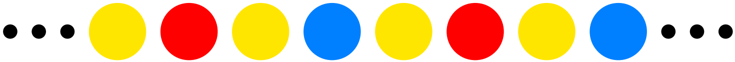
- never goes **4** days without pizza
- never goes **5** days without fish?

Answer: Possible.



Prologue

Can you construct a dinner schedule that:

- never goes **2** days without macaroni and cheese
- never goes **3** days without pizza
- never goes **100** days without fish?

Answer: Impossible.

Prologue

Can you construct a dinner schedule that:

- never goes **2** days without macaroni and cheese
- never goes **5** days without pizza
- never goes **100** days without fish
- never goes **7** days without tacos?

Answer: Impossible.

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Can you construct a dinner schedule that:

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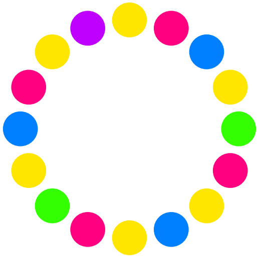
Prologue

The Pinwheel Problem

Given $g_1, \ldots, g_n$, can $\mathbb{Z}$ be partitioned into $S_1, \ldots, S_n$ such that S_i intersects every interval of length g_i?

E.g., $(g_1, \ldots, g_5) = (3, 4, 6, 10, 16)$



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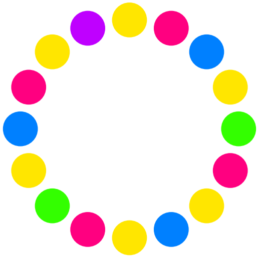
Given $g_1, \ldots, g_n$, can $\mathbb{Z}$ be partitioned into $S_1, \ldots, S_n$ such that S_i intersects every interval of length g_i?

What is the complexity of this decision problem?



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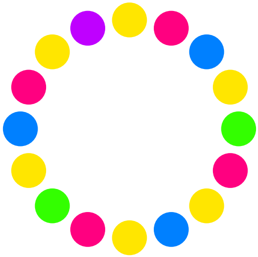
What is the complexity of this decision problem?

It belongs to PSPACE.

No non-trivial lower bounds known.

Later in this talk: PTAS for an optimization version.



The Multi-Armed Bandit Problem

Stochastic Multi-Armed Bandit Problem: A decision-maker ("gambler") chooses one of n actions ("arms") in each time step.

Chosen arm yields random payoff from unknown distrib. on $[0,1]$.

Goal: Maximize expected total payoff.

Recharging Bandits

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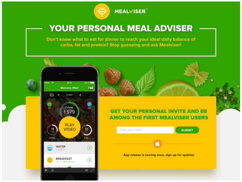

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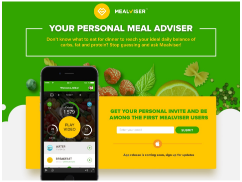

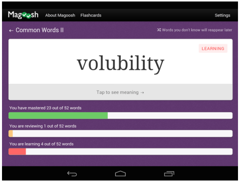

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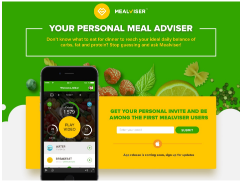

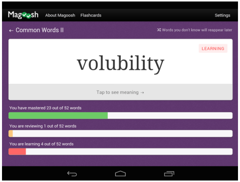

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Recharging Bandits

- Pulling arm i at time t, when it was last pulled at time s, yields random payoff with expectation $H_i(t-s)$.
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Concavity assumption implies free disposal: in step t, pulling i is better than doing nothing because

$$H_i(u-t) + H_i(t-s) \geq H_i(u-s).$$

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With known $\{H_i\}$: a special case of deterministic restless bandits.

General case is PSPACE-hard [Papadimitriou & Tsitsiklis 1987].

Which reinforcement learning problems have a PTAS?

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Plan of attack:

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Greedy $\frac{1}{2}$-Approximation

Greedy algorithm: always maximize payoff in current time step.

Greedy/OPT ratio can be arbitrarily close to 1/2

- $H_1(t) = 1 - \varepsilon, \quad H_2(t) = t.$
- Greedy always pulls arm 2.
- "Almost-OPT" pulls arm 1 for $T \gg 1$ time steps, then arm 2.
- Net payoff $(2 - \varepsilon)T + 1$ over $T + 1$ time steps.

Greedy $\frac{1}{2}$-Approximation

Greedy algorithm: always maximize payoff in current time step.

Greedy/OPT is never less than 1/2

- Imagine allowing the algorithm (but not OPT) to pull two arms per time step.
- At each time, supplement the greedy selection with the arm selected by OPT, if they differ.
- This at most doubles the payoff in each time step.
- Net payoff of supplemented schedule $\geq$ OPT.
  (free disposal property)

Rate of Return Function

For $0 \leq x \leq 1$, let $R_i(x)$ denote maximum long-run average payoff achievable by playing i in at most x fraction of time steps.

$$R_i(x) = \sup \left\{ \frac{1}{T} \sum_{j=1}^{\ell} H_i(t_j - t_{j-1}) \;\middle|\; \begin{array}{l} T < \infty,\ \ell \leq x \cdot T, \\ 0 = t_0 < t_1 < \cdots < t_\ell \leq T \end{array} \right\}.$$

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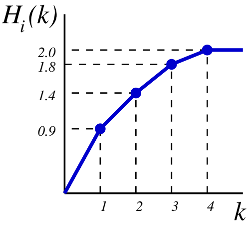

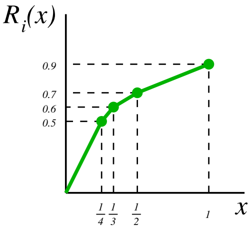

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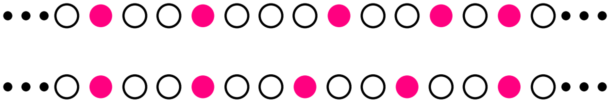
Proof sketch: The optimal sequence $0 = t_0 < \cdots < t_\ell \leq T$ has at most two distinct gap sizes, $\lfloor \frac{1}{x} \rfloor$ and $\lceil \frac{1}{x} \rceil$.

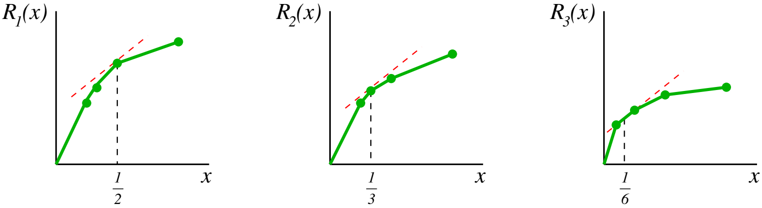


Concave Relaxation

The problem

$$\max \left\{ \sum_{i=1}^{n} R_i(x_i) \;\middle|\; \sum_i x_i \leq 1, \forall i\, x_i \geq 0 \right\}$$

specifies an upper bound on the value of the optimal schedule.



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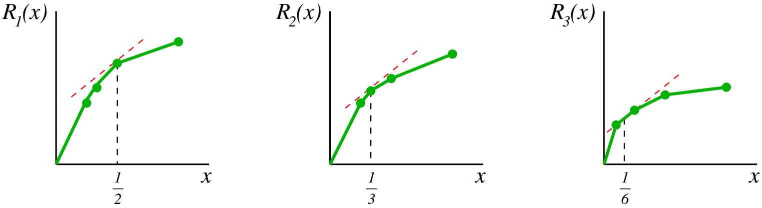

Mapping $(x_1, \ldots, x_n)$ to a schedule: pinwheel problem!

Independent Rounding

First idea: every time step, sample arm i with probability x_i.

Then $\tau_i =$ delay of arm $i = t_j(i) - t_{j-1}(i)$ is geometrically distributed with expectation $1/x_i$.

Rounding scheme gets $x_i \cdot \mathbb{E} H_i(\tau_i)$ whereas relaxation gets $R_i(x_i) = x_i H_i(1/x_i) = x_i \cdot H_i(\mathbb{E}\tau_i)$.

Fact: if H is concave and non-decreasing and Y is geometrically distributed then $\mathbb{E} H(Y) \geq \left(1 - \frac{1}{e}\right) H(\mathbb{E} Y)$.

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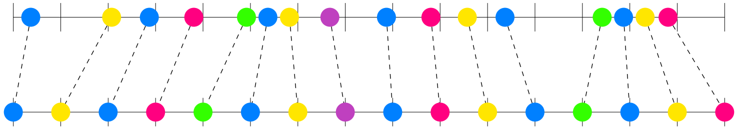
To do better, need rounding scheme that reduces variance of τ_i.

Interleaved Arithmetic Progressions

Second idea: round continuous-time schedule to discrete time.

In continuous time, pull i at $\{\frac{r_i+k}{x_i} \mid k \in \mathbb{N}\}$ where $r_i \sim_{\mathsf{Unif}} [0, 1)$.

Map this schedule to discrete time in an order-preserving manner.



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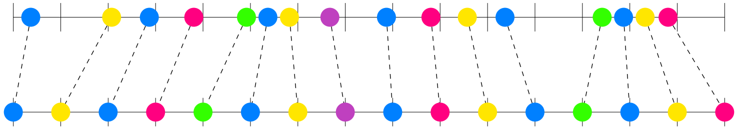

Between two pulls of i, we pull j either $\lfloor x_j/x_i \rfloor$ or $\lceil x_j/x_i \rceil$ times.

$$\tau_i = 1 + \sum_{j \neq i} Z_j$$

$\{Z_j\}$ are independent, each supported on 2 consecutive integers.

Convex Stochastic Ordering

Definition

If X, Y are random variables, the *convex stochastic ordering* defines $X \leq_{\mathrm{cx}} Y$ if and only if $\mathbb{E}\phi(X) \leq \mathbb{E}\phi(Y)$ for every convex function ϕ.

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Lemma

If X is a sum of independent Bernoulli random variables and Y is Poisson with $\mathbb{E}Y = \mathbb{E}X$ then $X \leq_{\mathrm{cx}} Y$.

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$$\begin{aligned} \tau_i &= 1 + \textstyle\sum_{j \neq i} Z_j \leq_{\mathrm{cx}} 1 + \mathrm{Pois}(\tfrac{1}{x_i} - 1) \\ x_i \cdot \mathbb{E}H_i(\tau_i) &\geq x_i \cdot \mathbb{E}H_i(1 + \mathrm{Pois}(\tfrac{1}{x_i} - 1)) \end{aligned}$$

Approximation Ratio for Interleaved AP Rounding

Fact 1: If H is concave and non-decreasing and Y is Poisson, then

$$\mathbb{E}H(1+Y) \;\geq\; (1-\tfrac{1}{2e})H(1+\mathbb{E}Y)$$

Fact 2: If H is concave and non-decreasing and Y is Poisson with $\mathbb{E}Y \geq m$, then

$$\mathbb{E}H(1+Y) \;\geq\; \left(1-\frac{1}{\sqrt{2\pi m}}\right)H(1+\mathbb{E}Y)$$

Conclusion: Interleaved AP rounding is

- a $1-\frac{1}{2e} \approx 0.816$ approximation in general
- a $1-\delta$ approximation for "small arms" to whom the concave relaxation assigns $x_i < \delta^2$

PTAS for Recharging Bandits

Let $\varepsilon > 0$ be a small constant. Two easy cases . . .

1. All arms are big. Every arm that gets pulled in the optimal schedule is pulled with frequency ε^2 or greater.

   Then the optimal schedule uses only $1/\varepsilon^2$ arms. Brute-force search takes polynomial time.

2. All arms are small. If the optimal concave program solution has $x_i < \varepsilon^2$ for all i, then randomly interleaved arithmetic progressions get $1 - \varepsilon$ approximation.

Combine the cases using "partial enumeration". For $p = O_\varepsilon(1) \ldots$

Outer loop: iterate over p-periodic schedules of arms and gaps.

Inner loop: fit small arms into gaps using interleaved AP rounding.

PTAS Difficulties

- Gaps in the p-periodic schedule may not be equally spaced.
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  - Eliminate intermediate arms by finding $k \leq 1/\varepsilon$ such that arms with $x_i \in (\varepsilon^{4(k+1)}, \varepsilon^{4k}]$ contribute less than $\varepsilon \cdot \mathrm{OPT}$.
  - Conclusion: $\#$ of big arms $\leq (1/\varepsilon)^{O(1/\varepsilon)}$.

PTAS Difficulties

- Gaps in the p-periodic schedule may not be equally spaced.
- Why can we assume big arms are scheduled with period $p = O_\varepsilon(1)$?
  - We need existence of a p-periodic schedule that matches two properties of OPT
    1. rate of return from big arms
    2. amount of time left over for small arms
  - Existence proof is surprisingly technical; omitted.
  - Conclusion $p = (\#\text{big})/\varepsilon^2$ suffices.

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- Grand conclusion: PTAS with running time $n^{(1/\varepsilon)^{(24/\varepsilon)}}$.

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- Grand conclusion: PTAS with running time $n^{(1/\varepsilon)^{(24/\varepsilon)}}$.
- Remark: the 0.816-approximation runs in time $O(n^2 \log n)$.

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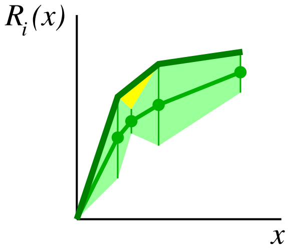
Idea: divide time into "planning epochs" of length $\phi = O(n/\epsilon)$.

In each epoch ...

1. Compute $\bar{H}_i(x)$, an upper confidence bound on $H_i(x)$, $\forall i$.
2. Run approx alg. on $\{\bar{H}_i\}$ to schedule arms within epoch.
3. Update empirical estimates and confidence radii.

Main challenge: Although H_i is concave, $\bar{H}_i$ may not be.



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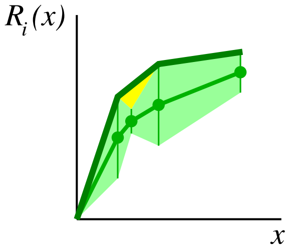
Solution: Work with $\bar{R}_i$ and "iron" the non-concavity, without disrupting the approximation guarantee.



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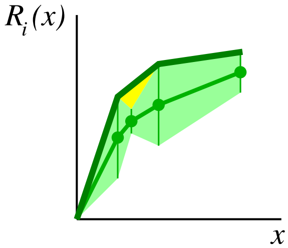
Approx alg. is almost black box.

Can plug in greedy, interleaved AP rounding, or PTAS.

Approx. factor reduced by $1 - \varepsilon$, plus $O\left(n \log(n) \sqrt{T \log(nT)}\right)$ regret.



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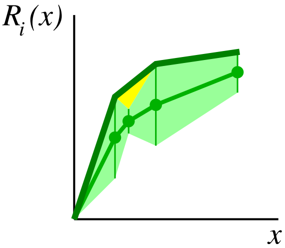

Summary

Recharging bandits: A model for learning to schedule recurring tasks (interventions) whose benefit increases with latency.

Approximation algorithms:

- simple greedy ($\frac{1}{2}$);
- rounding concave relaxation using interleaved arithmetic progressions ($1 - \frac{1}{2e}$);
- partial enumeration and concave rounding ($1 - \varepsilon$).

Nice connections to pinwheel problem in additive combinatorics.

Open Questions

1. Pinwheel problem
    1. Complexity? (Could be in P. Could be PSPACE-complete.)
    2. Is $(g_1, \ldots, g_n)$ always feasible if $\sum_i g_i^{-1} \leq 5/6$?
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2. Reinforcement learning: What other special cases admit PTAS?

Open Questions

3. Applications: extend recharging bandits model to incorporate domain-specific features such as . . .
    1. (fighting poachers) Strategic arms with endogenous payoffs. [Kempe-Schulman-Tamuz '17]
    2. (invasive species removal) Externalities between arms. Movement costs.
    3. (education) Payoffs with more complex history-dependency. [Novikoff-Kleinberg-Strogatz '11]