Incremental Risk Vulnerability[1]

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Abstract

This paper analyses conditions for an increase in an additive independent background risk to increase an agent's risk aversion (incremental risk vulnerability). We, first, present a necessary and sufficient condition on an agent's utility function for a simple mean preserving spread in background risk to increase the agent's risk aversion. Gollier and Pratt (1996) have shown that declining and convex risk aversion as well as standard risk aversion are sufficient for risk vulnerability. We show that these conditions are also sufficient for incremental risk vulnerability, given a mean preserving spread. Second, we present sufficient conditions for a restricted set of stochastic increases in an independent background risk to increase risk aversion. These conditions are differentiated with respect to the type of stochastic increase.

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1 Introduction

Many economic decisions are made in a context where some of the risks are tradable, while others are not. These non-tradable or background risks are not controllable by the decision-maker and yet influence the agent's risk-taking behavior with respect to the tradable claims. Eeckhoudt and Kimball (1992) and Meyer and Meyer (1998) demonstrate this for the demand for insurance, Franke, Stapleton and Subrahmanyam (1998) for portfolio choice. A central question, in this context, is whether an additive background risk makes the agent more risk averse.

Gollier and Pratt (1996) answer this question by considering an agent who starts without background risk and then faces an independent background risk. They introduce the concept of risk vulnerability and show that risk vulnerability is equivalent to the notion that an undesirable risk can never be made desirable by the presence of an independent, unfair risk. Furthermore, the background risk makes the agent more risk averse. Hence, such a background risk reduces the agent's demand for a risky asset, given a choice between a risky and a risk-free asset. Gollier and Pratt derive a necessary and sufficient condition for risk vulnerability. They show that a sufficient condition for risk vulnerability is either that the absolute risk aversion of the agent is declining and convex or that the agent is standard risk averse in the sense of Kimball (1993). In a recent paper Keenan and Snow (2003) relate Gollier and Pratt's condition of local risk vulnerability to compensated increases in risk, introduced by Diamond and Stiglitz (1974). They show that the introduction of a small fair background risk increases risk aversion of agents more, the higher is their index of local risk vulnerability.

Usually, agents have to bear some background risk, but the level of this risk may change. Therefore the relevant question is not so much whether the *presence* of background risk makes the agent more risk averse, but whether an *increase* in this background risk makes the agent more risk averse. An employee, for example, faces labor income risk. When the economy in which he works faces a downturn, his labor income risk may increase because the probability of being fired increases. This may reduce his willingness to take tradable risks like stock market risk. The purpose of this paper to derive conditions on *increases* in background risk and utility functions so that the increase in background risk raises the agent's risk aversion. Under these conditions the agent is said to exhibit *incremental risk vulnerability.*

There are only a few papers dealing with similar issues. Eeckhoudt, Gollier and Schlesinger (1996) consider the issue of this paper in the context of increases in an independent background risk that exhibit second order stochastic dominance. Let x be a random variable denoting the independent background risk. Then they consider an increase in this background risk by adding a random variable v such that the distribution of x second order stochastically dominates the distribution of $(x+v)$. Given this broad set of increases in background risk they derive necessary and sufficient conditions, which leave room only for a small set of utility functions. Kimball (1993) analyzes patent increases in background risk. He (1993, p.603) defines a patent increase as follows. X is patently more risky than x iff for any monotonic, concave utility function u_1 that has decreasing absolute risk aversion and any monotonic, concave utility function u_2, u_2 being globally more risk averse than u_1, the differential risk premium is higher for u_2 than for u_1, given any initial wealth. The differential risk premium, by definition, renders the expected utility of (initial wealth + x - differential risk premium) equal to the expected utility of (initial wealth + X). Kimball shows that a patent increase raises the risk aversion of an agent if it raises the expected marginal utility conditional on his initial wealth and if the agent is standard risk averse. Kimball argues that the background risk X is patently more risky than the background risk x if X can be obtained from x by adding a random variable v such that the distribution of v conditional on x improves for increasing realizations of x according to third order-stochastic dominance. Kimball's claim is rather surprising in view of the very restrictive results of Eeckhoudt, Gollier and Schlesinger (1996). Kimball refers to the large class of standard risk averse utility functions while Eeckhoudt, Gollier and Schlesinger (1996) derive a much smaller class.[1] This is, perhaps, explained by the differences in the assumptions on the nature of the stochastic increase in background risk.

Intuitively, there must be an inverse relation between the set of admissible increases in background risk considered and the set of utility functions that exhibit the characteristic of increased risk aversion. Therefore, in comparison to Eeckhoudt, Gollier and Schlesinger (1996), in this article we consider smaller, but plausible sets of increases in background risk. The benefit is that we obtain a broader set of utility functions that have the desired attribute. In this paper we first consider non-stochastic and then stochastic increases in background risk. In section 2 we define a restricted set of non-stochastic increases which are called *simple mean preserving spreads.* We derive a necessary and sufficient condition for incremental risk vulnerability. It turns out that the sufficient conditions for risk vulnerability given by Gollier and Pratt are also sufficient for incremental risk vulnerability. However, declining risk aversion is not required. All utility functions with a negative third and a negative fourth derivative are also incremental risk vulnerable.

In section 3, we further consider a restricted set of stochastic increases in background

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[1] Eichner and Wagener (2003) discuss the conditions on two-parameter, mean-variance preferences such that the agent is variance vulnerable, i.e. an increase in the variance of an independent background risk induces the agent to take less tradable risk.

risk and derive sufficient conditions for risk aversion to increase. This increase may be independent of the realization of the background risk or the increase may improve with the realization according to nth order-stochastic dominance. General conditions for incremental risk vulnerability are, first, risk vulnerability, and, second, declining absolute risk aversion. If the increase in background risk improves with the realization of the background risk according to nth order-stochastic dominance, the the first $(n+1)$ derivatives of the utility function need to alternate in sign. In addition the set of increases in background risk is restricted more, the slower absolute risk aversion declines in wealth.

2 Characterization of Incremental Risk Vulnerability for Non-Stochastic Increases in Background Risk

In this section we present a necessary and sufficient condition for the utility function to exhibit incremental risk vulnerability, given a subset of non-stochastic increases in background risk, called simple mean preserving spreads. The agent's income, W, is composed of the tradable income w and the non-tradable income y, i.e. $W = w + y$. The non-tradable income represents an additive background risk. y is assumed to be distributed independently of w and to have a zero mean. Moreover, y is assumed to be bounded from below and above, i.e. $y \in (\underline{y}, \bar{y})$. Finally, $W = w + y \in (\underline{W}, \bar{W})$ is assumed. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the probability space on which the random variables are defined.

Rothschild and Stiglitz (1970) define a mean preserving spread of an existing risk as a shift in the probability mass from the center to the tails of the distribution. As pointed out by Eeckhoudt, Gollier and Schlesinger (1996), this is equivalent to a second order degree stochastic dominance shift, provided the mean is fixed. To this definition we add the restriction that the increase in background risk is deterministic and raises the non-tradable income in some states above a threshold level and lowers it in some states below the threshold. We call this increase a *simple mean preserving spread.*

Let y be the independent background risk with $E(y) = 0$, then a simple mean preserving spread is a deterministic change in y, $\Delta(y)$, such that $\Delta(y) \leq [=] [\geq]\ 0$ for $y < [=] [>]\ y_0$ for a given a threshold level y_0, and $E[\Delta(y)] = 0$. In this case, note that the rank order of outcomes both above and below y_0 may change.

Definition 1 *(A Simple Mean Preserving Spread in Background Risk)*
Let y be a background risk with $E(y) = 0$. Then a simple mean preserving spread in the background risk changes y to $y + s\Delta(y)$, with $E(\Delta(y)) = 0$, where $\Delta(y) \leq [=] [\geq]\ 0$ for

$y <$ [=] [$>$] y_0 ,and $s \geq 0$ denotes the scale of the increase.

The agent's utility function is $u(W)$. We assume that the utility function is state-independent, strictly increasing, strictly concave, and four times differentiable on $W \in (\underline{W}, \bar{W})$. We assume that there exist integrable functions on $\omega \in \Omega$, u_0 and u_1 such that

$$u_0(\omega) \leq u(W) \leq u_1(\omega)$$

We also assume that similar conditions hold for the derivatives $u'(W), u''(W)$ and $u'''(W)$. The agent's expected utility, conditional on w, is given by the derived utility function, as defined by Kihlstrom et al. (1981) and Nachman (1982):

$$\nu(w) = E_y[u(W)] \equiv E[u(w+y)|w] \tag{1}$$

where E_y indicates an expectation taken over different outcomes of y. Thus, the agent with background risk and a von Neumann-Morgenstern concave utility function $u(W)$ acts like an individual without background risk and a concave utility function $\nu(w)$. The coefficient of absolute risk aversion is defined as $r(W) = -u''(W)/u'(W)$ and the coefficient of absolute prudence as $p(W) = -u'''(W)/u''(W)$. The absolute risk aversion of the agent‘s derived utility function is defined as the negative of the ratio of the second derivative to the first derivative of the derived utility function with respect to w, i.e.,

$$\hat{r}(w) = -\frac{\nu''(w)}{\nu'(w)} = -\frac{E_y[u''(W)]}{E_y[u'(W)]} \tag{2}$$

It is worth noting that, in the absence of background risk, $\hat{r}(w)$ is equal to $r(w)$, the coefficient of absolute risk aversion of the original utility function.
We are now in a position to define incremental risk vulnerability for non-stochastic increases in background risk.

Definition 2 *(Incremental Risk Vulnerability for non-stochastic increases in background risk)*
An agent is incremental risk vulnerable to a non-stochastic increase in background risk if a simple mean preserving spread in background risk increases the agent's derived risk aversion for all w.

This definition also includes the case in which the agent initially has no background risk. This case is analyzed by Gollier and Pratt (1996). Hence incremental risk vulnerability implies risk vulnerability subject to $E[\Delta(y)] = E[y] = 0$. Gollier and Pratt allow also for a non-random negative y which then necessitates declining risk aversion. Since we consider fair background risks, declining risk aversion is not implied by incremental risk vulnerability.

The first main result of this paper is the following proposition which presents a necessary and sufficient condition for a simple mean preserving spread in background risk to raise derived risk aversion.

Proposition 1 *(Incremental Risk Vulnerability for Simple Mean Preserving Spreads in Background Risk)*

If $u'(W) > 0$ and $u''(W) < 0$, then for any marginal simple mean preserving spread in background risk,

$$\partial\hat{r}(w)/\partial s > [=][<]\ 0,\ \forall(w, y, s) \Longleftrightarrow$$

$$u'''(W_2) - u'''(W_1) < [=][>] - r(W)[u''(W_2) - u''(W_1)],$$

$$\forall\ (W, W_1, W_2),\ \underline{W} < W_1 \leq W \leq W_2 < \bar{W}, W_2 - W_1 < \bar{y} - \underline{y}\ .$$

Since a finite increase in background risk is the sum of marginal increases, this condition is also sufficient for finite increases in background risk to raise risk aversion.

Proof: See Appendix 1.

Proposition 1 allows us to analyze the effect of any simple mean preserving spread in an independent background risk. In order to interpret the necessary and sufficient condition under which a simple mean preserving spread in a background risk will raise the risk aversion of the derived utility function, first consider the special case in which background risk changes from zero to a small positive level. This is the case analyzed previously by Gollier and Pratt (1996) and by Keenan and Snow (2003). In this case, we have

Corollary 1 *Starting with no background risk, for any marginal increase in background risk,*

$$\hat{r}(w) > [=][<]\ r(w) \quad \textit{if and only if} \quad \frac{\partial\theta}{\partial W} < [=][>]\ 0,\ \forall\ W$$

where $\theta(W) \equiv u'''(W)/u'(W)$.

Proof: Let $W_2 - W_1 \to dW$. In this case, $u'''(W_2) - u'''(W_1) \to u''''(W)dW$. Similarly $u''(W_2) - u''(W_1) \to u'''(W)dW$.
Hence, the condition in the Proposition yields, in this case, $u''''(W) < [=][>] -r(W)u'''(W)$. This is equivalent to $\partial\theta/\partial W < [=][>] 0, \ \forall \ W \ \Box$

In Corollary 1, $\theta(W) = u'''(W)/u'(W)$ is a *combined* prudence/risk aversion measure. This measure is defined by the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion. The corollary says that for a small background risk derived risk aversion exceeds [is equal to] [is smaller than] risk aversion if and only if $\theta(W)$ decreases [stays constant] [increases] with W. Hence, it is significant that *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for derived risk aversion to exceed risk aversion. However, the combination of these conditions is sufficient for the result to hold, since the requirement is that the product of the two must be decreasing. The condition in corollary 1 is thus weaker than standard risk aversion, which is characterized by *both* absolute risk aversion and absolute prudence being positive and decreasing. Note that the condition in this case is the same as the 'local risk vulnerability' condition derived by Gollier and Pratt (1996). Local risk vulnerability is $r'' > 2rr'$, which is equivalent to $\theta' < 0$. Keenan and Snow (2003) define $-\theta'$ as the local risk vulnerability index. They show for a small background risk that the difference between derived risk aversion and risk aversion increases in this index.

Since an interior maximum of $r(w)$ implies $r'(w) = 0$ and $r''(w) < 0$, it rules out local risk vulnerability. Therefore, we have

Corollary 2 *Risk vulnerability and incremental risk vulnerability rule out all utility functions with an interior maximum of absolute risk aversion.*

An alternative way to interpret Corollary 1 and Proposition 1 is to assume $u''' > 0$. In this case, Corollary 1 states that a marginal increase in background risk, starting with no background risk, makes the agent more risk averse if and only if temperance $t(W) = -u''''(W)/u'''(W)$ exceeds risk aversion $r(W)$ everywhere. Proposition 1 states that a simple mean preserving spread in background risk makes an agent more risk averse if and only if $-[u'''(W_2) - u'''(W_1)]/[u''(W_2) - u''(W_1)] > r(W)$, for $W_1 \leq W \leq W_2$. The left hand side of this inequality can be interpreted as an average temperance over the range $[W_1, W_2]$. In their analysis of second order stochastic dominance shifts in background risk, Eeckoudt, Gollier and Schlesinger (1996) find the much stronger condition $t(W) \geq r(W'), \forall (W, W')$.

We now apply Proposition 1 to show that standard risk aversion is sufficient for incremental risk vulnerability.

Corollary 3 *Standard risk aversion is a sufficient condition for derived risk aversion to increase with a simple mean preserving spread in background risk.*

Proof: Standard risk aversion requires both positive, decreasing absolute risk aversion and positive, decreasing absolute prudence. Further, $r'(W) < 0 \Rightarrow p(W) > r(W)$ and hence $u'''(W) > 0$. It follows that the condition in the Proposition for an *increase* in the derived risk aversion can be written as [2]

$$\frac{u'''(W_2) - u'''(W_1)}{u''(W_2) - u''(W_1)} < -r(W_1)$$

or, alternatively,

$$p(W_1)\left[1 - \frac{u'''(W_2)}{u'''(W_1)}\right] / \left[1 - \frac{u''(W_2)}{u''(W_1)}\right] > r(W_1)$$

Since $p(W_1) > r(W_1)$, a sufficient condition is that the ratio of the square brackets exceeds 1. This, in turn, follows from decreasing absolute prudence, $p'(W) < 0$. Hence, standard risk aversion is a sufficient condition □

Gollier and Pratt (1996) showed not only that standard risk aversion is sufficient for risk vulnerability, but so also is declining and convex absolute risk aversion $r(w)$. The next corollary shows that the latter condition is also sufficient for incremental risk vulnerability.

Corollary 4 *Declining and convex absolute risk aversion is a sufficient condition for derived risk aversion to increase with a simple mean preserving spread in background risk.*

Proof: From

$$\hat{r}(w) = E_y\left[\frac{u'(W)}{E_y[u'(W)]} r(W)\right],$$

$$\partial\hat{r}(w)/\partial s = E_y\left[\frac{u'(W)}{E_y[u'(W)]} r'(W)\Delta(y)\right] + E_y\left[r(W)\frac{\partial}{\partial y}\left[\frac{u'(W)}{E_y[u'(W)]}\right]\Delta(y)\right] \quad (3)$$

As shown in the appendix, it suffices to consider a three-point distribution of background risk (y_1, y_0, y_2) with $y_1 < 0, y_2 > 0, y_1 < y_0 < y_2$ and $\Delta(y_0) = 0, \Delta(y_1) < 0, \Delta(y_2) > 0$. The first term in equation (3) is positive whenever r is declining and convex. This follows since $E(\Delta(y)) = 0$ and $\Delta(y_2) > \Delta(y_1)$ implies that $E[r'(W)\Delta(y)] \geq 0$. Since $u'(W)$ is

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[2] Note that whenever $r'(W)$ has the same sign for all W, the three-state condition in the Proposition (i.e. the condition on W, W_1, and W_2) can be replaced by a two-state condition (a condition on W_1 and W_2).

declining, it follows that the first term in (3) is positive. Now consider the second term: $\partial[u'(W)/E_y[u'(W)]]/\partial y\ \Delta(y)$ is positive for y_1 and negative for y_2 and has zero expectation. Therefore a declining r implies that the second term is positive. Hence, a sufficient condition for $\partial\hat{r}(w)/\partial s >\ 0$ is a declining and convex r □

Although corollaries 3 and 4 use the property of declining risk aversion, this property is clearly not required for incremental risk vulnerability, as already noted by Gollier and Pratt.

Corollary 5 *: For every utility function with $u'''(W) < 0$ and $u''''(W) \leq 0$ a simple mean preserving spread in background risk raises derived risk aversion.*

Proof: $u''''(W) \leq 0$ implies that the left hand side of the condition in Proposition 1 is non-positive. $u'''(W) < 0$ implies that the right hand side is positive □

A utility function with $u'''(W) < 0$ exhibits negative prudence and increasing risk aversion. Yet this utility function has the property of incremental risk vulnerability if the fourth derivative is also negative. In terms of equation (3), the second term is now negative, but it is overcompensated by a strongly positive first term due to strong convexity of r.
An example of a utility function with the properties stated in corollary 5 is the HARA-function

$$u(W) = \frac{1-\gamma}{\gamma}\left[A + \frac{W}{1-\gamma}\right]^{\gamma}, \text{where } \gamma \in (1,2), W < A(\gamma-1)\ .$$

Finally, it should be noted that corollaries 3 and 4 not only hold for simple increases in background risk requiring $E[\Delta(y)] = 0$, but also allowing for $E[\Delta(y)] < 0$ since these corollaries assume declining absolute risk aversion. This is true since $\Delta(y)$ can be decomposed into a simple mean preserving spread and a negative constant equal to the mean.

3 Characterization of Incremental Risk Vulnerability for Stochastic Increases in Background Risk

A simple mean preserving spread in background risk is a deterministic change relating $\Delta(y)$ to y. A natural generalization is to consider a stochastic change e such that y is replaced by $(y + e)$ with e being distributed independently of w, but perhaps dependently on y. In the case of dependence, the distribution of e is assumed to improve with increasing y according to nth order-stochastic dominance, i.e. the distribution of e conditional on y nth order-stochastically dominates the distribution conditional on a smaller realization of y. It will be assumed throughout that this improvement can be captured by the differential $\partial e/\partial y$. This differential is zero in the case of independence. We, first, derive sufficient conditions on e and on absolute risk aversion to ensure an increase in derived risk aversion and, second, illustrate these conditions.

Definition 3 *(Incremental Risk Vulnerability for stochastic increases in background risk) An agent is incremental risk vulnerable to a stochastic increase in background risk if this increase raises the agent's derived risk aversion for all w.*

We analyse the agent's derived risk aversion $\hat{r}(w)$ in the presence of only the y-risk and the derived risk aversion $\hat{\hat{r}}(w)$ in the presence of the $(y+e)$-risk. For this purpose we define $r_e(w+y)$ as the derived risk aversion over the e-risk, given the income $(w+y)$.

$$r_e(w+y) \equiv \frac{E_e[-u''(w+y+e)]}{E_e[u'(w+y+e)]}; \qquad \forall (w+y).$$

In the following we assume that the assumptions on the utility function in section 2 hold and the utility function is $(n+1)$-times differentiable where n may vary. Proposition 2 provides sufficient conditions for the e-risk to raise the agent's risk aversion.

Proposition 2 *Let e be a random variable which is distributed independently of w, but perhaps dependently on y.*

a) If e is distributed independently, then

$$\hat{\hat{r}}(w) \geq \hat{r}(w), \quad \forall\ w,$$

if

$$r_e(w+y) \geq r(w+y), \quad \forall\ (w+y), \tag{4}$$

and the utility function $u(W)$ exhibits decreasing risk aversion.

b) If e improves with y according to nth order-stochastic dominance, then $\hat{\hat{r}}(w) > \hat{r}(w), \forall w$, if, in addition to condition(4), $r_e(w+y)$ is declining and the first $(n+1)$ derivatives of the utility function $u(W)$ are alternating in sign.

This proposition is proved in Appendix 2. Condition (4) requires the risk aversion of an agent with income $w + y$ to be higher in the presence of the background risk, e. Condition (4) is the risk vulnerability condition of Gollier and Pratt (1996), given wealth $(w + y)$ and a background risk e with nonpositive mean. Therefore we obtain an immediate corrollary to Proposition 2a).

Corollary 6 *Let e be a random variable, distributed independently of y, with nonpositive expectation. Then the increase in background risk from y to $(y + e)$ raises the derived risk aversion if the agent is risk vulnerable and declining risk averse.*

As an example of joint risk vulnerability and declining risk aversion, it suffices that the agent is standard risk averse or risk aversion $r(W)$ is declining and convex. It is more difficult to provide examples for Proposition 2b). Critical is the condition that $r_e(w + y)$ is declining. This condition requires that $r(W)$ is declining since otherwise $r_e(w + y)$ would not be declining for a small e-risk. The following lemma relates the properties of the improvement of e, given an increase in y, to the properties of the risk aversion function $r(W)$. For this purpose, we add the mild constraint on the improvement that $1 + de/dy \geq 0$. In other words, if e improves with y according to nth-order- stochastic dominance, then the sum $(y + e)$ must not decline given an increase in y. This mild condition appears quite natural, given an improvement of e with y.

Lemma 1 *Assume that $r(W)$ is declining and the background income $(y + e)$ does not decline when y increases. Then $r_e(w + y)$ is declining in y, if $r(w + y + e)(1 + de/dy)$ is declining in $e, \forall\ (w + y + e)$.*

The lemma is proved in the appendix. The condition in the lemma states that, for a given increase in e, the decline in risk aversion is not neutralized or overcompensated by an increase in $(1 + de/dy)$. Consider, for example, a binomial distribution of e with $e_1 < e_2$ which changes with y so that $(y + e)$ increases by $(1 + de_1/dy)$ resp. $(1 + de_2/dy)$. Then $r(w + y + e_1)(1 + de_1/dy) \geq r(w + y + e_2)(1 + de_2/dy)$ is required. Hence if $r(w + y + e)$ declines by 1 percent moving from e_1 to e_2, then $(1 + de/dy)$ must not increase by more than 1 percent. Therefore, the faster risk aversion declines, the more $(1 + de/dy)$ may increase.

The conditions stated in Lemma 1 clearly constrain the improvement of the e-distribution with respect to y. Therefore this set of improvements is smaller than the set of second order stochastic dominance increases in background risk analysed by Eeckhoudt, Gollier and Schlesinger (1996). Also simple mean preserving spreads violate these conditions since $y_2 > y_1$ does not imply $y_2 + \Delta(y_2) > y_1 + \Delta(y_1)$. It is straightforward now to find corollaries to Proposition 2b). They are based on the rather restrictive assumption that not only the expectation $E[e]$ is non-positive, but on the stronger assumption that $E[e|y]$ is non-positive for every realization of y.

Corollary 7 *Assume that $E[e|y] \leq 0, \forall y$, the agent is risk vulnerable and the conditions of Lemma 1 hold. Then if the distribution of e improves with increasing y according to second order-stochastic dominance, the increase in background risk replacing y by $(y+e)$ raises the derived risk aversion.*

The proof of corollary 7 is immediate from Proposition 2b), because $r' < 0$, as required by Lemma 1, implies positive prudence. Moreover, condition (4) holds because of $E[e|y] \leq 0$ and risk vulnerability. The next corollary relates to Kimball's larger set of order-stochastic dominance increases in background risk. For a standard risk averse agent $u''' > 0$ and $u'''' < 0$. Since this agent is risk vulnerable for $E[e|y] \leq 0$, we obtain

Corollary 8 *Assume that $E[e|y] \leq 0, \forall y$ and the conditions of Lemma 1 hold. Then, if the distribution of e improves with y according to third order-stochastic dominance, the increase in background risk replacing y by $(y+e)$ raises the derived risk aversion of a standard risk averse investor.*

4 Conclusion

All agents are exposed to background risks. Often the size of these risks changes. Therefore, an important question is how the agent's risk aversion reacts to an *increase* in these risks. This paper considers the effect on derived risk aversion of increases in background risk. We first take the case of non-stochastic increases which are simple mean preserving spreads. We present a necessary and sufficient condition for such an increase to raise the derived risk aversion of an agent. Standard risk aversion and declining, convex risk aversion are shown to be sufficient conditions.

We then analyse the effect of stochastic increases in background risk. If such an increase is independent of the existing background risk and has a non-positive expectation, it raises an agent's derived risk aversion if she is risk vulnerable and declining risk averse. If the distribution of the increase improves with increasing realisations of the existing background risk according to nth order-stochastic dominance and the conditional expectation of the increase is non-positive, then risk vulnerability and declining risk aversion are not sufficient to raise the agent's derived risk aversion. In addition, alternating signs of the first $(n+1)$ derivatives of the utility function and a condition relating the speed of the decline in risk aversion and the improvement in the background risk come into play. These conditions together are shown to be sufficient for incremental risk vulnerability.

Further research should investigate to what extent these conditions can be relaxed, preserving incremental risk vulnerability.

Appendix 1

Proof of Proposition 1

From the definition of $\hat{r}(w)$,

$$\hat{r}(w) = \frac{E_y[-u''(W)]}{E_y[u'(W)]} \tag{5}$$

we have the following condition. For any distribution of y and for any $s \geq 0$,

$$\partial\hat{r}(w)/\partial s > [=][<]\ 0 \Longleftrightarrow f(w,y,s) > [=][<]\ 0, \tag{6}$$

where $f(w,y,s)$ is defined as

$$f(w,y,s) \equiv E_y\left[\Delta(y)\left\{-u'''(W) - u''(W)\hat{r}(w)\right\}\right]. \tag{7}$$

Necessity

We now show that

$$\begin{aligned} f(w,y,s) &> [=][<] & 0 \Longrightarrow \\ u'''(W_2) - u'''(W_1) &< [=][>] & -r(W)\left[u''(W_2) - u''(W_1)\right], \forall\ W_1 \leq W \leq W_2 \end{aligned}$$

Consider a background risk with three possible outcomes, y_0, y_1, and y_2, such that $y_1 < y_0 < y_2$ and $\Delta(y_1) < \Delta(y_0) = 0 < \Delta(y_2)$. Define

$$W_i \quad = \quad w + y_i + s\Delta(y_i), \quad i = 0,1,2,$$

and let q_i denote the probability of the outcome y_i. For the special case of such a risk, equation (8) can be written as

$$f(w,y,s) = q_1|\Delta(y_1)|\left\{-u'''(W_2) + u'''(W_1) - [u''(W_2) - u''(W_1)]\hat{r}(w)\right\} \tag{8}$$

since

$$E[\Delta(y)] = \sum_{i=0}^{2} q_i\Delta(y_i) = 0$$

so that

$$q_1|\Delta(y_1)| = q_2\Delta(y_2)$$

Now $\hat{r}(w)$ can be rewritten from (6) as

$$\begin{aligned}\hat{r}(w) &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}\frac{-u''(W)}{u'(W)}\right\} \\ &= E_y\left\{\frac{u'(W)}{E_y[u'(W)]}r(W)\right\} \qquad (9)\end{aligned}$$

Hence, $\hat{r}(w)$ is the expected value of the coefficient of absolute risk aversion, using the risk-neutral probabilities given by the respective probabilities multiplied by the ratio of the marginal utility to the expected marginal utility. Thus, $\hat{r}(w)$ is a convex combination of the coefficients of absolute risk aversion at the different values of y. For the three-point distribution being considered, $\hat{r}(w)$ is a convex combination of $r(W_0)$, $r(W_1)$, and $r(W_2)$. Suppose that $y_0 = 0$. Then $q_0 \to 1$ is feasible. Hence, as $q_0 \to 1$, $\hat{r}(w) \to r(W_0)$. Therefore, in condition (9) we replace $\hat{r}(w)$ by $r(W_0)$. Since W_0 can take any value in the range $[W_1, W_2]$, $f(w, y, s)$ must have the required sign for *every* value of $r(W_0)$, where $W_1 \leq W_0 \leq W_2$. Thus, since $q_1|\Delta(y_1)| > 0$, the condition as stated in Proposition 1 must hold. As $y \in (\underline{y}, \bar{y})$, $W_2 - W_1 < \bar{y} - \underline{y}$.

Sufficiency

To establish sufficiency we use a method similar to that used by Pratt and Zeckhauser (1987) and Gollier and Pratt (1996).

a) We first show

$$\begin{aligned}u'''(W_2) - u'''(W_1) &< -r(W)\left[u''(W_2) - u''(W_1)\right], \ \forall\ W_1 \leq W \leq W_2 \\ \Longrightarrow f(w, y, s) &> 0, \ \ \forall\ (w, y, s)\end{aligned}$$

We need to show that $f(w, y, s) > 0$, for all non-degenerate probability distributions of y. Hence, we need to prove that the minimum value of $f(w, y, s)$ over *all* possible probability distributions $\{q_i\}$, with $E(\Delta(y)) = 0$, must be positive. In a manner similar to Gollier and Pratt (1996), this can be formulated as a mathematical programming problem, where $f(w, y, s)$ is minimized, subject to the constraints that all q_i are non-negative and sum to one, and $E(\Delta(y)) = 0$. Equivalently, this can be reformulated as a parametric linear program where the non-linearity is eliminated by writing $\bar{r}$ as a parameter

$$\min_{\{q_i\}} f(w, y, s) = \sum_i q_i \left[\Delta(y_i)\left\{-u'''(W_i) - u''(W_i)\bar{r}\right\}\right] \tag{10}$$

s.t.

$$\sum_i q_i \Delta(y_i) = 0 \tag{11}$$

$$\sum_i q_i = 1, \tag{12}$$

the definitional constraint for the parameter $\bar{r}$

$$\bar{r}\sum_i q_i u'(W_i) = -\sum_i q_i u''(W_i) \tag{13}$$

and the non-negativity constraints

$$q_i \geq 0, \quad \forall i. \tag{14}$$

Consider the optimal solution. Since this optimization problem has three constraints, there are three variables in the optimal solution. Number these as $i = 1, 2, a$, with $\Delta(y_1) < 0 < \Delta(y_2)$ and $y_1 + s\Delta(y_1) < y_2 + s\Delta(y_2)$. The associated probabilities are q_1, q_2, q_a, such that $q_1\Delta(y_1) + q_a\Delta(y_a) + q_2\Delta(y_2) = 0$. There are two possibilities with respect to the state a.

Either:
$a = 0$. Then $\Delta(y_a) = \Delta(y_0) = 0$. Hence, we immediately obtain equation (16).

or:
$a \neq 0$. In this case we drop the constraint on $q_0 \geq 0$ (with all the other q_is staying non-negative). Hence the probability associated with y_0 can be negative. Dropping this constraint will lead to a condition that is too demanding. However, since we are searching for a sufficient condition, this is fine. In the original optimisation, all the non-basis variables had nonnegative coefficients in the objective function in the final simplex tableau. Allowing $q_0 < 0$ must result therefore in q_0 replacing either q_1, q_2 or q_a in the optimal basis. Also, the new f-value is either lower or the same as before.

Suppose, first, that q_0 replaces q_a in the optimal basis. Then the new variables in the optimal solution are q_1, q_2 and q_0. Since $\Delta(y_0) = 0$, we can write the objective function (11) as

$$f^*(w, y, s) = q_1\Delta(y_1)\left[-u'''(W_1) - u''(W_1)\bar{r}\right] + q_2\Delta(y_2)\left[-u'''(W_2) - u''(W_2)\bar{r}\right] \tag{15}$$

Since $q_1\Delta(y_1) + q_2\Delta(y_2) = 0$, it follows that (14) can be rewritten as

$$f^*(w, y, s) = q_1\Delta(y_1)\left[(-u'''(W_1) - u''(W_1)\bar{r}) - (-u'''(W_2) - u''(W_2)\bar{r})\right] \tag{16}$$

Hence

$$u'''(W_2) - u'''(W_1) < -\bar{r}\left[u''(W_2) - u''(W_1)\right] \tag{17}$$

is a sufficient condition for $f^* > 0$, given $\bar{r}$.

As shown in equation (10), $\bar{r}$ is a convex combination of $r(W_a)$, $r(W_1)$ and $r(W_2)$ with $W_1 < W_a < W_2$, hence $\bar{r} \in \{r(W)|W \in [W_1, W_2]\}$. Hence, a sufficient condition for (18) is that

$$u'''(W_2) - u'''(W_1) < -r(W)\left[u''(W_2) - u''(W_1)\right] \tag{18}$$

for all $\{W_1 \leq W \leq W_2\}$ as given by the condition of Proposition 1.

Alternatively, suppose that q_0 replaces either q_1 or q_2 in the optimal solution. In this case the above argument remains the same with q_a instead of either q_1 or q_2, in equation (15).

b) By an analogous argument, it can be shown that $\partial\hat{r}(w)/\partial s < [=]\ 0$ is equivalent to $u'''(W_2) - u'''(W_1) > [=]\ -r(W)[u''(W_2) - u''(W_1)]\ \forall\ \{W_1 \leq W \leq W_2\}$ □

Appendix 2

Proof of Proposition 2

We need to show that the conditions given in Proposition 2 are sufficient for $\hat{\hat{r}}(w) \geq \hat{r}(w)$. $E_v[\cdot]$ denotes expectations over v. From the definition of the twice derived risk aversion, $\hat{\hat{r}}$,

$$\hat{\hat{r}}(w) \quad = \quad E_{y+e}\left[\frac{u'(w+y+e)}{E_{y+e}u'(w+y+e)}r(w+y+e)\right]$$

$$
\begin{aligned}
&= E_y\left[\frac{E_e u'(w+y+e)}{E_{y+e}u'(w+y+e)}E_e\left\{\frac{u'(w+y+e)}{E_e u'(w+y+e)}r(w+y+e)\right\}\right]\\
&= E_y\left[\frac{E_e u'(w+y+e)}{E_{y+e}u'(w+y+e)}r_e(w+y)\right],
\end{aligned}
$$

where $r_e(w+y)$ is defined at the beginning of section 3. Hence

$$
\begin{aligned}
\hat{\hat{r}}(w)-\hat{r}(w) &= E_y\left[\left(\frac{E_e u'(w+y+e)}{E_{y+e}u'(w+y+e)}-\frac{u'(w+y)}{E_y u'(w+y)}\right)r_e(w+y)\right]\\
&+ E_y\left[\frac{u'(w+y)}{E_y u'(w+y)}(r_e(w+y)-r(w+y))\right]
\end{aligned}
$$

Condition (4) implies that the second term is positive or zero. The first term is similar to a covariance term since the term in ( ) has zero expectation. Hence the first term is nonnegative if the term in ( ) is single crossing downwards and $r_e(w+y)$ is declining in y.

a) Suppose that e is distributed independently of y. Then declining risk aversion is preserved under the e-risk. Hence $r_e(w+y)$ is declining in y. Therefore, in order to complete the proof, we have to establish the single crossing downwards property of

$$
Z(w+y)\equiv\frac{E_e u'(w+y+e)}{a}-\frac{u'(w+y)}{b},
$$

with a and b being appropriately defined constants.

Differentiating with respect to y yields

$$
\begin{aligned}
Z'(w+y) &= \frac{E_e u''(w+y+e)(1+\frac{\partial e}{\partial y})}{a}-\frac{u''(w+y)}{b}\\
&= -\frac{E_e u'(w+y+e)}{a}r_e(w+y)+\frac{u'(w+y)}{b}r(w+y)+\frac{E_e u''(w+y+e)\frac{\partial e}{\partial y}}{a}. \quad (5)
\end{aligned}
$$

For $Z=0$ it follows that $sgnZ'(w+y)=sgn[r(w+y)-r_e(w+y)+[E_e u'(w+y+e)]^{-1}E_e u''(w+y+e)(\partial e/\partial y)]$. Hence condition (4) implies $Z'(w+y)\leq 0$ at a crossing point if e is distributed independently of y, i.e. $\partial e/\partial y\equiv 0$. Then only one crossing point exists, therefore $Z(w+y)$ is downward sloping.

b) Now suppose that e improves with y according to to nth order-stochastic dominance. By assumption, $r_e(w+y)$ is declining in y. Hence it suffices that $Z(w+y)$ retains the single crossing downwards property. Given condition (4), this is true if the last term on the right hand side of equation (5) is nonpositive. This term denotes the change in $E_e[u'(w+y+e)]$ due to the nth order-stochastic dominance shift in e. For a first order-stochastic dominance shift this term is negative because $u'' < 0$. For a second order-stochastic dominance shift this term is negative if $u'' < 0$ and $u''' > 0$. This generalizes to a nth order-stochastic dominance shift: the sign of the derivatives u^i , $i = 1, \ldots, n+1$, needs to alternate $\square$

Proof of Lemma 1

From the definition

$$r_e(w+y) = \frac{E_e[-u''(w+y+e)]}{E_e[u'(w+y+e)]} \, ,$$

it follows that $r_e(w+y)$ declines in y if

$$\frac{E_e\left[-u'''(w+y+e)\left(1+\frac{de}{dy}\right)\right]}{E_e[-u''(w+y+e)]} \leq \frac{E_e\left[u''(w+y+e)\left(1+\frac{de}{dy}\right)\right]}{E_e\left[u'(w+y+e)\right]} \, .$$

Let $p(.) \equiv u'''(.)/-u''(.)$ denote prudence. Then the last inequality yields

$$E_e\left[\frac{-u''(.)}{E_e[-u''(.)]}\, p(.)\left(1+\frac{de}{dy}\right)\right] \geq E_e\left[\frac{u'(.)}{E_e[u'(.)]}\, r(.)\left(1+\frac{de}{dy}\right)\right] \, .$$

Since, by assumption, $r' < 0$, $p > r$ follows. Moreover, by assumption, $1 + \frac{de}{dy} \geq 0$. Therefore the last inequality holds if

$$E_e\left[\frac{-u''(.)}{E_e[-u''(.)]}\, r(.)\left(1+\frac{de}{dy}\right)\right] \geq E_e\left[\frac{u'(.)}{E_e[u'(.)]}\, r(.)\left(1+\frac{de}{dy}\right)\right] \, .$$

The term on the left hand side can be rewritten as

$$\begin{aligned} & E_e\left[\frac{u'(.)}{E_e[u'(.)]}\,\frac{-u''(.)}{u'(.)}\,\frac{E_e[u'(.)]}{E_e[-u''(.)]}\, r(.)\left(1+\frac{de}{dy}\right)\right] \\ = \;\; & E_e\left[\frac{u'(.)}{E_e[u'(.)]}\, r(.)\left(1+\frac{de}{dy}\right)\right] + cov_e^Q\left[\frac{-u''(.)}{u'(.)}\,\frac{E_e[u'(.)]}{E_e[-u''(.)]}\, ;\, r(.)\left(1+\frac{de}{dy}\right)\right] \end{aligned}$$

with $cov_e^Q(.)$ denoting the covariance under the probability measure Q, derived from the product of the state probability (density) and $u'(.)/E_e[u'(.)]$. If this covariance term is nonnegative, then the last inequality holds. The covariance term can be rewritten as

$$\frac{E_e[u'(.)]}{E_e[-u''(.)]}\, cov_e^Q\left[r(.)\, ;\, r(.)\left(1+\frac{de}{dy}\right)\right] \, .$$

Since $r' < 0$, this term is positive if $r(.)\left(1+\frac{de}{dy}\right)$ declines in e $\square$

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