ELSEVIER Journal of Banking & Finance 21 (1997) 1487–1513

Journal of BANKING & FINANCE

The valuation of American options on bonds [1]

T.S. Ho [a], Richard C. Stapleton [a,2], Marti G. Subrahmanyam [b,*]

[a] *Department of Accounting and Finance, The Management School, Lancaster University, Lancaster LA1 4YX, UK*

[b] *Leonard N.Stern School of Business, New York University, Management Education Center, 44 West 4th Street, Suite 9–190, New York, NY 10012–1126, USA*

Abstract

We value American options on bonds using a generalization of the Geske–Johnson (Geske, R., Johnson, H., 1984. Journal of Finance 39, 1151–1542) (GJ) technique. The method requires the valuation of European options, and options with multiple exercise dates. It is shown that a risk-neutral valuation relationship (RNVR) along the lines of Black–Scholes (Black, F., Scholes, M., 1973. Journal of Political Economy 81, 637–659) model holds for options exercisable on multiple dates, even under stochastic interest rates, when the price of the underlying asset is lognormally distributed. The proposed computational procedure uses the maximized value of these options, where the maximization is over all possible exercise dates. The value of the American option is then computed by Richardson extrapolation. The volatility of the underlying default-free bond is modeled using a two-factor model, with a short-term and a long-term interest rate factor. We report the results of simulations of American option values using

* Corresponding author. Tel.: +1 212 998 0348; fax: +1 212 995 4233; e-mail: msubrahm@stern.nyu.edu.

[1] Earlier versions of this paper have been presented at the European Finance Association meeting in Rotterdam, at the Financial Options Research Centre, Warwick University, and at the Hong Kong University of Science and Technology. We thank Anurag Gupta for able research assistance. We would also like to thank an anonymous referee for detailed comments on an earlier draft of the paper.

[2] Tel.: +44 1524 59 36 37; fax: +44 1524 84 73 21; e-mail: dj@staplet.demon.co.uk.

PII S0378-4266(97)00034-4

our method and show how they vary with the key parameter inputs, such as the maturity of the bond, its volatility, and the option strike price.

JEL classification: G11

Keywords: American bond options; Stochastic interest rates

1. Introduction

The valuation of American-style bond options involves two important aspects that need to be modeled carefully. First, stochastic interest rates influence the volatility of the price of the bond, the underlying asset, in a complex fashion as the bond approaches maturity. The behavior of the volatility over time influences the value of the option if held to maturity, as well as the incremental value of the early exercise (American) feature. Second, the early exercise decision for such options is affected by the term structure of interest rates on future dates, since the live value of the claim on each future date depends on the discount rates on that date.

In this paper, we model the volatility of the default-free bond price using a two-factor model. Hence, the bond's volatility is determined by the volatilities of the two interest rate factors and by the sensitivity of the bond price to changes in the two factor rates. This model allows us to capture the effect of non-parallel shifts in the term structure of interest rates, that may have a significant effect on the volatility of the bond price over time, and hence, on the value of the contingent claims. In order to analyse the early exercise decision, we derive a model for the value of an option exercisable on one of many dates, which permits speedy computation of option values and hedge parameters. The model assumes that (continuously compounded) interest rates are normally distributed, i.e., the prices of zero-coupon bonds are lognormally distributed. [3] In this case, it has been shown that the Black–Scholes risk-neutral valuation relationship (RNVR) holds for the valuation of European options, even under stochastic interest rates. [4]

Also, since an American option can be thought of as an option with multiple exercise dates, where the number of dates becomes very large, it is necessary to

[3] The well-known drawback of this assumption is that interest rates could be negative with positive probability. However, this disadvantage may be less important than the possible contradictions in assuming that both the coupon bond prices and zero-coupon bond prices are lognormally distributed.

[4] This has been established in the case of a single-factor interest rate model by Jamshidian (1989) and, for the general case, by Satchell et al. (1997).

establish a similar RNVR for options exercisable on one of many dates. Once such a RNVR is established, American-style options can be valued using an extension of the Geske and Johnson (1984) (GJ) approach, i.e., by Richardson extrapolation, using a series of options that can be exercised on one of a number of discrete dates. The series consists of a European option, an option exercisable on one of two dates, and so on.

GJ apply their methodology to the case of American put options on stocks. However, the GJ approach can be applied to any American-style option whose value depends upon the underlying asset price as a state variable. [5] In particular, it is applied here to American options where early exercise is generated by the changing volatility of the underlying asset, or by the nature of the exercise schedule. The approach also extends to the valuation of American options on assets, including bonds, when interest rates are stochastic. [6]

Since the formulae derived here for approximating the value of American options involve multivariate cumulative-normal density functions, the implementation can be simplified by approximating the normal distribution by discrete binomial distributions along the lines of Cox et al. (1979). However, the binomial methodology needs to be generalized to handle the changing volatilities of the asset (both conditional and unconditional) over time. Also, the method has to take into account the possibility that the term structure of interest rates and, in turn, bond prices are driven by a multifactor model. The method we use to capture changing volatilities is similar in spirit to that suggested in Nelson and Ramaswamy (1990), generalized to multiple state variables by Ho et al. (1995).

In the single-variable approach with a constant volatility of the price of the underlying asset, the Cox et al. (1979) method involves building a binomial tree centered around the forward price of the asset, rather than around its expected spot price. For a European option, the payoff is computed at each node of the tree on the expiration date and the expected value of this payoff is discounted to determine the option value. For American options with multiple possible exercise dates, the procedure is somewhat more complex. First, the method used here entails building binomial trees of the asset price and the discount factor, where the *conditional* expectation of each variable is its forward price for delivery on each of the possible future exercise dates of the option. Hence, the fundamental no-arbitrage condition on the evolution of the asset price and the discount factor is satisfied. Next, the contingent exercise decisions on each future state and date are determined and the value of the option on each future

[5] Huang et al. (1996) develop an alternative method where the early exercise boundary is first estimated and then the value of American options is determined by extrapolation.

[6] See Ho et al. (1997) for an application of the GJ approach to the general problem of valuation of options on assets when interest rates are stochastic.

date is computed. Finally, these values are discounted using the appropriate zero-coupon bond price to determine the current value of the option.

In the case of American-style options on finite-life, coupon bonds the GJ method has to be adapted somewhat. Since the volatility of a finite-life bond tends to decline over time, with the approach of the bond maturity, an American-style option on the bond is a *wasting asset*. Even in the case of European options, a long-maturity option on such a bond may have less value than a shorter-maturity option. For this reason we use a GJ type approximation where the European option and the option with two possible exercise dates are chosen so as to maximize the value of the options. Thus our benchmark, or minimum possible, value for the American-style option is the value of the European option with the maximum value, where the maximum is taken with respect to the feasible lives of the option. In the case of a typical 10-year coupon bond, this maximized European option value may be that of a two-year or three-year maturity option. Similarly, in the case of the option with two possible exercise dates, we take the maximum option value taken over all possible pairs of exercise dates. [7] GJ-type extrapolation is then performed, using an exponential rather than a linear approximation to generate estimates of the American-option price. [8] We also demonstrate that only a relatively small increase in accuracy is obtained when options exercisable on one of three dates are added to the extrapolation. This small increase can be obtained only with a relatively large amount of computational effort. Simulations show that it is far more important to obtain accurate estimates of the volatility and the forward price inputs, than to consider options exercisable on more than two dates.

Our solution provides a rather simple prescription for the answer to a problem of considerable complexity. The method presented in this paper may be applied to the valuation of any American option under stochastic interest rates, given that the distributional assumptions are satisfied. It is consistent with approaches using a multifactor model of the term structure of interest rates, but is simpler and more efficient than other approaches, because it involves the evaluation of options with only a small number of exercise dates. It is also more general than alternative approaches using a particular factor model for the evolution of the term structure of interest rates, although it uses a two-factor mod-

[7] This method was proposed and tested in a somewhat different form by Bunch and Johnson (1992). Since Bunch and Johnson do not value options on finite-lived assets, they take the first term in the extrapolation as the value of the European option whose maturity equals that of the American option. They then find the maximum option value with two exercise dates, given that the second exercise date is the final maturity date of the American option. In the case of finite-lived assets such as bonds, the Bunch and Johnson approach has to be modified along the lines proposed here, since the volatility of the bond declines as it approaches maturity.

[8] Ho et al. (1994) modify the linear Richardson approximation technique used by both GJ and Bunch and Johnson, adapting it for long-maturity options using exponential approximation.

el for generation of the volatility inputs. Furthermore, the model is arbitrage free, while avoiding the complex problems involved in modeling the full evolution of the term structure. However, the important restriction, as in the case of the Black and Scholes (1973) model, is that asset prices must follow a multivariate lognormal distribution.

In Section 2, we discuss the modification of the GJ approach to the case of American options in the context of other approaches in the literature. In Section 3, we present a valuation model for American-style options on bonds and establish the requisite RNVRs.

In Section 4, we proceed to illustrate the method by applying it to American options on a variety of bonds. We show, using simulations, that for reasonable exercise schedules, the GJ method can be applied in modified form using options exercisable on one date (European options), and on one of two possible dates, only.

2. Bond options and the use of the GJ methodology

Much of the work in recent years on the valuation of contingent claims on bonds and interest rates uses a factor model to characterize the evolution of the term structure of interest rates. For example, Ho and Lee (1986), Black et al. (1990), and Jamshidian (1989) all build a process for the evolution of the term structure that is based on a single-factor model. Although Heath et al. (1990a, b, 1992) provide a framework for the pricing of claims using a general multifactor approach to characterize the term structure, the implementation of this methodology using a binomial lattice becomes difficult when the number of factors increases, due to the computational problems associated with building a multidimensional lattice of bond prices or interest rates. [9] In addition to the cumbersome procedure for building a multidimensional lattice, the problem of the valuation of American-style options requires an examination of the optimality of early exercise at each node of the lattice, which is even more complex. The computational limits of the multifactor lattice approaches are illustrated by Amin and Bodurtha (1995) who find even a 10-stage lattice very costly to implement when two or more factors are involved. In contrast, the GJ methodology can be implemented without any restrictive assumptions involving the factor model underlying term structure movements.

In view of the limitations of the lattice-based approaches, it is worthwhile to explore the possibility of using the GJ methodology to value American bond options. GJ originally suggested the use of the Richardson approximation to extrapolate the value of an American option from the values of a series of

[9] See Hull and White (1994) for details of implementation of a two-factor model.

options: a European option, an option with two possible exercise dates, an option with three possible exercise dates, and so on. A number of subsequent papers have extended and modified the basic GJ approach. For example, Omberg (1987) and Breen (1991) approximate the distribution of the price of the underlying asset with a binomial process. However, Omberg (1987) shows that there could be problems of non-uniform convergence in some cases. Essentially, in these cases, the computed value of the American option is not monotonic in the number of options considered for the Richardson extrapolation. Bunch and Johnson (1992) modify the GJ method by showing that it may be more efficient to compute the prices of all options with two exercise dates and select the one with the maximum value. In this manner, one can obtain the best approximation with the extrapolation. Ho et al. (1994) point out that the accuracy of the GJ technique can be improved, particularly in the case of long-term options, such as warrants and bond options, by using an exponential rather than a linear approximation in the extrapolation. In addition, Ho et al. (1997) show that the GJ technique can be extended successfully to the multidimensional case where interest rates as well as the price of the underlying asset are stochastic.

In the present paper, we use all these extensions and modifications of the GJ technique, and apply them to the problem of valuation of bond options. First, we use the binomial methodology of Omberg (1987) and Breen (1991), but avoid the non-convergence problem by using a two-point extrapolation on the lines of Bunch and Johnson (1992). We also use the exponential approximation proposed by Ho et al. (1994) to improve the results for long-term options. Also, since we necessarily have to address the issue of stochastic rates when valuing bond options, we use the results in Ho et al. (1997) where it is shown that a RNVR exists for the pricing of claims even in this case.

3. The valuation model

We are interested in valuing American-style options on bonds, given the exercise schedule, i.e., the relationship between the exercise price of the option and the exercise date. [10] The options could, in principle, be standard call or put options or more complex exotic options whose characteristics are defined by the respective payoff functions. The exercise schedule is defined by the function

$$K_{t_i} = K(t_i), \quad i = 1, 2, \ldots, J, \tag{1}$$

[10] The exercise schedule, which represents the changing exercise price of the option over its time to maturity, is specified as part of the bond option contract. It is a feature of many bond option contracts, particularly those that are embedded as part of the bond.

where t_i are the exercise dates, t_1 the earliest date on which the option can be exercised, $i = 1, 2 \ldots J$, $t_J = T$ the maturity date of the option, and J the number of dates between the current date, 0, and the maturity date t_J on which the option can be exercised.

The value of the underlying bond at time t_i is denoted as S_{t_i}. Thus, the live value of the option, i.e., its market value if it is not exercised at or before time t_i, is C_{t_i}, and its value, just prior to the exercise decision at time t_i is

$$\max[g(S_{t_i}), C_{t_i}], \quad i = 1, 2, \ldots, J, \tag{2}$$

where $g(S_{t_i})$ is the payoff function of the option. Since we are concerned here with the possible early exercise of such options, the price of the option on intermediate dates between 0 and t_J is relevant. We denote the price of the option at time t_i, with J possible exercise dates over its life, as

$$C_{t_i,J}(t_1, t_2, \ldots, t_J; K_{t_1}, K_{t_2}, \ldots, K_{t_J}), \quad i = 1, 2, \ldots, J. \tag{3}$$

In general, the GJ approach to the valuation of American options estimates the American option price by Richardson extrapolation from the values of a series of options, with $1, 2, \ldots, J$ exercise dates. We denote the estimated, time 0, American option prices as $\hat{C}_2$, $\hat{C}_3$, using a series of two and three option prices respectively. For example, $\hat{C}_2$ is the estimated price of the American option using the values of two options: a European option and an option with two exercise dates.

We first establish conditions under which options can be valued using formulae analogous to those of Black and Scholes (1973). The central idea here is the concept of a RNVR, which can be defined as follows for European options:

Definition 1. A RNVR exists for a European option if it can be valued by taking the expected value of its payoff, using a distribution for the asset price which is identical to the true distribution but with the mean shifted to equal the forward price of the asset.

The model of Black and Scholes (1973) can be thought of as a RNVR, under the assumption of continuous trading (or a lognormal pricing kernel) and a lognormal distribution for the asset price on the expiration date of the option. As shown by Merton (1973) and extended by several others including Heath et al. (1990a), Turnbull and Milne (1991), and Satchell et al. (1997), this result can be extended to the case of stochastic interest rates. In the case of American-style options under stochastic interest rates, the definition of a RNVR has to be broadened somewhat along the following lines:

Definition 2. A RNVR exists for the valuation of an option that has multiple exercise dates, if the option can be valued by taking the expected values of its payoff using distributions of the asset price at the various exercise dates, and discounting them using the relevant zero-coupon bond prices. The distributions are identical to those of the true distributions except for a mean shift which makes the conditional expected value of each of the prices equal to their respective (conditional) forward prices.

The concept of a RNVR for European options can thus be generalized to American options. The key aspect of the RNVR for American options is that it yields a valuation model based only on the (conditional) forward price of the asset for delivery at various future dates before the expiration date of the option and the corresponding volatilities. We now define the implications of a RNVR more precisely and then establish conditions under which the price of an option with two possible exercise dates, $C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2})$, can be found, if we know: (a) the forward price at time 0 of the asset for delivery at t_1, (b) the (conditional) forward price of the asset at time t_1 for delivery at t_2, (c) the forward price at time 0 of the zero-coupon bond for delivery at t_1 which pays one unit of currency at t_2, plus all the relevant volatilities. More formally,

Proposition 1. *If a RNVR exists for the valuation of an option with two possible exercise dates, then*

$$C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2}) = B_{0,t_1} E_0[Y_{t_1}], \tag{4}$$

where

$$Y_{t_1} = \max[g(S_{t_1}), C_{t_1,1}(t_2; K_{t_2})], \tag{5}$$

and where

$$C_{t_1,1}(t_2; K_{t_2}) = B_{t_1,t_2} E_{t_1}[Y_{t_2}], \tag{6}$$

$$Y_{t_2} = g(S_{t_2}), \tag{7}$$

and all the relevant conditional distributions of the three random variables, S_{t_1}, S_{t_2}, and B_{t_1,t_2}, have means equal to their respective forward prices.

Proof. Y_{t_1} and Y_{t_2} are the option values (or cash flows accruing to the holders of the option) at times t_1 and t_2. A positive cash payoff occurs at t_1 if the value of the option at t_1 if not exercised, $C_{t_1,1}(t_2; K_{t_2})$, is less than the payoff from early exercise. The positive payoff Y_{t_2} occurs only if the early exercise condition at t_1 is not fulfilled and the option ends up in-the-money at t_2.

From the definition of a RNVR, we know that the option value is the value, discounted at B_{0,t_1} and B_{t_1,t_2}, of the expected payoffs on the option. Hence Eq. (4) is correct, if expectations are taken with respect to the shifted distributions of S_{t_1}, S_{t_2} and B_{t_1,t_2}. Also, if the RNVR holds, the exercise decision at t_1 can be taken by valuing the option at t_1, using Eqs. (6) and (7). Note that there are two random variables at t_1 that affect this decision, the price of the underlying bond, S_{t_1}, and the zero-coupon bond price, B_{t_1,t_2}. The latter affects the spot price of the option (if unexercised). Eq. (6) values the option at t_1 with a RNVR. The expectation of S_{t_2}, as of time t_1, is the (conditional) forward price of S_{t_2} at time t_1. □

Corollary 1. *If a RNVR exists for the valuation of an option with exercise dates, t_1 and t_2, a RNVR exists for the valuation of European options with exercise dates t_1 and t_2, respectively.*

Proof. As an illustration we prove the statement for call options. If we make $K_{t_2} = \infty$, Eq. (4) becomes

$$C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2}) = B_{0,t_1} E_0[Y_{t_1}],$$

i.e.

$$C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2}) = B_{0,t_1} E_0[g(S_{t_1})], \tag{8}$$

since

$$C_{t_1,1}(t_2; K_{t_2}) = 0.$$

This confirms the RNVR, for a European call option of maturity t_1. The same approach extends for any type of European contingent claim. Also, if we make $K_{t_1} = \infty$, Eq. (4) becomes

$$C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2}) = B_{0,t_1} E_0[B_{t_1,t_2} g(S_{t_2})],$$

i.e.

$$C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2}) = B_{0,t_2} E_0[g(S_{t_2})]. \tag{9}$$

This confirms the RNVR, for a European option of maturity t_2. □

The implications of Proposition 1 for the computation of the C_2 price are illustrated, for the case of a call option, in Fig. 1, where, for the sake of compactness, we adopt the shortened notation C_2 for $C_{0,2}(t_1, t_2; K_{t_1}, K_{t_2})$. The figure illustrates the computation of the value of a call option exercisable at times t_1 or t_2. The value of the option at time t_1 is Y_{t_1} and at t_2 is Y_{t_2}. In states 0–h_1 the option is exercised at time t_1. The exercise decision is indicated by E and the payoff is $S_{t_1} - K_{t_1}$, where S_{t_1} is the asset price at time t_1 and K the exercise price. In states h_1–n_1 exercise does not occur at t_1. This is indicated by NE and the option value in these states is the discounted value of the expected time t_2 payoff, where the discount factor is B_{t_1,t_2}. States at t_2 are indicated by 0–n_2. States at t_2 in which exercise did not occur at time t_1 and in which exercise may occur at t_2 are indicated by h_2–n_2. The payoff at time t_2 in these states is the larger of $S_{t_2} - K_{t_2}$ and 0. The arrows indicate period by period discounting of the option values. The option is valued by discounting the payoffs period-by-period, taking the optimal exercise decision into account, and using the discount factors in each state. The value on each date is the expectation of the discounted payoffs

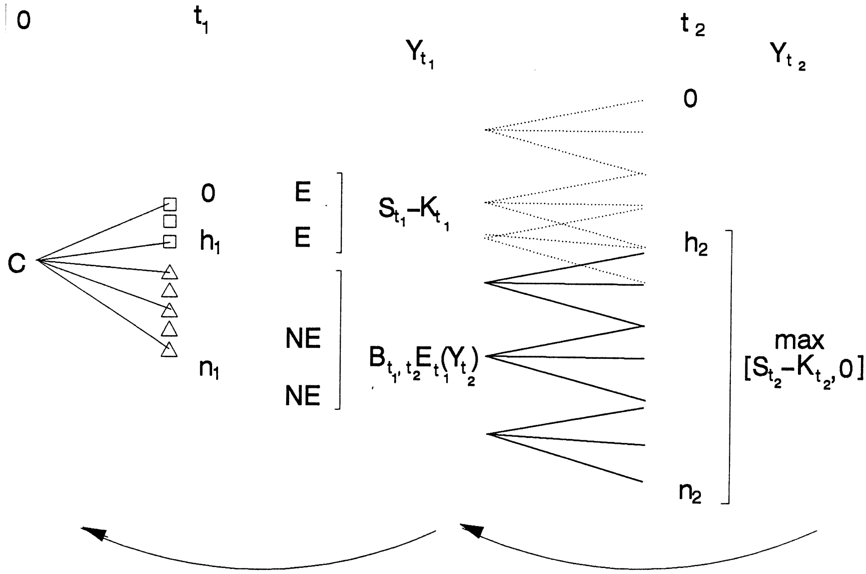

Fig. 1. Computation of C_2, the early exercise decision and option payoffs.

under the risk-neutral distribution. There are n_1 states at time t_1, where a state is defined as a pair of values of the asset price (S_{t_1}) and the zero-coupon bond price (B_{t_1,t_2}). The expected value of each variable is its respective forward price. In each state, a call price is computed using Eq. (6). This is compared with the early exercise payoff, $S_{t_1} - K_{t_1}$. In Fig. 1, states 0–h_1 indicate states in which early exercise occurs. In all other states the option is not exercised at t_1. In the states where exercise occurs, Y_{t_1} is equal to $S_{t_1} - K_{t_1}$. In all other states $Y_{t_1} = C_{t_1}$. If the option is not exercised at t_1 it may pay off at t_2. This occurs in states h_2–n_2 at t_2. Note that the probability of the Y_{t_1} values occurring are joint probabilities over the pair of variables (S_{t_1}, B_{t_1,t_2}). The probability of the payoff $Y_{t_2} = \max[0, S_{t_2} - K_{t_2}]$ values occurring are joint probabilities over the triplet of variables $(S_{t_1}, B_{t_1,t_2}, S_{t_2})$. Proposition 1 implies that the expected values of Y_{t_1} and Y_{t_2} can be computed using distributions of the three random variables each with a conditional mean equal to its forward price. The call price can then be computed by discounting the time t_1 payoff or option value at the zero-coupon bond prices B_{0,t_1}.

Ho et al. (1997) establish sufficient conditions for the existence of a RNVR relationship for the valuation of an option exercisable on one of many dates when the asset prices on a future date are joint-lognormally distributed. Specifically, this involves the derivation of conditions that are strong enough to guarantee that the risk-neutral distributions of the underlying asset price, $\{S_{t_i}, i = 1, 2, \ldots, J\}$ are joint lognormal with their conditional means being equal to the respective forward prices. These conditions are that the price process for the underlying asset and the (conditional) pricing kernels, at time 0 for cash flows at time t_1, and at time t_1 for cash flows at time t_2, ψ_{t_1} and ψ_{t_1,t_2}, respectively, are joint-lognormally distributed. [11]

The result holds for the general case with J exercise dates. However, to avoid cumbersome notation, we state Proposition 2 for the case of options that are exercisable on one of two dates:

Proposition 2. *Suppose that the prices of an asset at t_1 and t_2, S_{t_1} and S_{t_2}, and the price at t_1 of the zero-coupon bond which matures at t_2, B_{t_1,t_2}, are joint-lognormally distributed. Then, if there exist joint-lognormally distributed pricing variables ψ_{t_1}, ψ_{t_1,t_2}, which satisfy*

$$F_{0,t_1} = E_0(S_{t_1}\psi_{t_1}), \quad E_0(\psi_{t_1}) = 1, \tag{10}$$

$$F_{t_1,t_2} = E_{t_1}(S_{t_2}\psi_{t_1,t_2}), \quad E_{t_1}(\psi_{t_1,t_2}) = 1, \tag{11}$$

then a RNVR exists for the valuation of an option with two possible exercise points.

Proof. See Ho et al. (1997). □

Since, by Proposition 2, a RNVR exists for the option with two exercise dates, it follows from Proposition 1 that the option can be valued given appropriate forward price and volatility inputs. The same argument applies to the case of an option exercisable on one of J dates.

4. The application of the GJ technique to bond options

Ho et al. (1997) extend the GJ methodology to the case of American options with stochastic interest rates. In its simplest form, the GJ technique estimates the value of an American option by Richardson extrapolation as

$$\hat{C}_2 = C_2 + [C_2 - C_1], \tag{12}$$

where $\hat{C}_2$ is the estimated option price using options with just one and two exercise dates, and as

$$\hat{C}_3 = C_3 + \frac{7}{2}[C_3 - C_2] - \frac{1}{2}[C_2 - C_1], \tag{13}$$

[11] The pricing kernel can be thought of as a (state-dependent) random variable that adjusts for the risk aversion in the economy.

where $\hat{C}_3$ is the estimated option price using options with one, two, and three exercise dates. For simplicity of notation, we have used here the compact notation C_1, C_2, and C_3 for the values, at time 0, of the options with one, two, and three exercise dates, respectively.

In applying this technique to the case of options on bonds, this procedure needs to be modified because of the changing volatility of the underlying asset. To see this, consider the case of stock options to which the GJ technique was first applied. The reason why the simple GJ technique works quite well for stock options is that the (non-annualized) volatility of the underlying asset increases with time in this case. Hence, in this case, the European option C_1 with an expiration date T has the highest value of any of the European options with maturities in the range $[0,T]$. Similarly, the C_2 option with the highest value is, at least approximately, the one with exercise dates at $T/2$ and T, and the C_3 option with the highest value is close to the one with exercise dates $T/3$, $2T/3$ and T.

The pattern of volatility of a bond price over the life of the bond is quite different from that for stock prices because of the finite life of the bond. A default-free bond with a finite maturity of N years tends to have declining (annualized) volatility over its life, with the volatility declining to zero at maturity. This means that the (non-annualized) variance of the bond price, as a function of time, rises and then eventually falls to zero, at maturity. The changing volatility of the bond price creates a problem in applying the GJ technique, since it is no longer clear which values of options exercisable on a finite number of dates (C_1, C_2, and C_3) should be used in the extrapolation in Eqs. (12) and (13). For example, suppose the American option that we wish to value has an expiration date of $T \leqslant N$, where N is the maturity date of the underlying bond. Now, consider a European option on the bond with the same expiration date, T. The value of the European option C_1 depends on the expiration date, since the volatility of the underlying bond price changes over time, depending on the value of T. In the extreme case, where $T = N$, the volatility is zero, since the price of the underlying default-free bond is known with certainty. The option C_1, therefore, has zero insurance value. Similarly, when T is very small in relation to N, the time to expiration of the option is too low for the option to have much value. However, if T is somewhere in between, say at $N/2$, then the value is likely to be much higher. [12]

A practical solution to this problem is to use the "maximizing" modification, of Bunch and Johnson (1992), to the basic GJ technique. Under this modification, the C_1, C_2, and C_3 values that are used are the *maxima* over all

[12] This highlights an important difference between the option on a finite-life bond and an option on an infinite-life asset, such as a stock. In the case of stock options, the call option with the longest life is the one with the highest value.

possible exercise dates. Thus, C_1^* is the value of the European option with the highest value, where C_1 is maximized over all possible exercise dates in the range $[0,T]$. [13] Similarly, C_2^* is maximized over all possible pairs of exercise dates, and C_3^* over all possible triplets of exercise dates. [14] The Bunch and Johnson (1992) technique, which provides only a marginal improvement in accuracy in the case of stock options is, therefore, essential in applying the GJ methodology to bond options. [15]

In Ho et al. (1994), a further modification of the GJ methodology is suggested. It is shown that, for long-dated options, the accuracy of the GJ approach can be improved by assuming an exponential relationship between the prices of options with different numbers of exercise dates. Combining the idea of the "exponential" technique and the Bunch and Johnson (1992) "maximization" technique, we use the following predictor of the value of an American option. Using just C_1^* and C_2^* values, for example, the approximation for the value of the American option is given by

$$\hat{C}_2 = [C_2^*/C_1^*]C_2^*. \tag{14}$$

The value of the American option is the asymptotic value of the series of maximized option values. The methodology is illustrated in Fig. 2, which shows a plot of option values as a function of the number of exercise points. The range A–A′ shows the European option values for different feasible maturities. C_1^* is the maximum European option value. The range D–D′ shows the option values for options with two possible exercise dates. C_2^* is the maximum of these option values. Similarly, C_3^* is the maximum value of options with three possible exercise dates. The asymptotic point B is the estimated value of the American option. When there is only one exercise point, the option

---

[13] The European option with the highest time 0 value is
$C_1^* = C_1(t^*) = \max_t[C_1(t)], \quad t \in (0,T],$
where T is the final maturity date of the American option.

[14] The "mid-Atlantic" option with two exercise dates which has the highest value at time 0 is worth
$C_2^* = C_2(t_1^*, t_2^*) = \max_{t_1,t_2}[C_2(t_1,t_2)], \quad t_1 \leqslant t_2, \ t_1, t_2 \in (0,T],$
where T is the final maturity date of the American-style option. C_3^* is defined analogously by
$C_3^* = C_3(t_1^*, t_2^*, t_3^*) = \max_{t_1,t_2,t_3}[C_3(t_1,t_2,t_3)], \quad t_1 \leqslant t_2 \leqslant t_3, \ t_1,t_2,t_3 \in (0,T].$

[15] Bunch and Johnson (1992) found that the increased accuracy produced by their maximization technique meant that inclusion of options with more than two exercise dates was unnecessary (except for deep-in-the-money options). We conducted a similar test using options with three exercise points and the estimate $\hat{C}_3$. The maximization procedure is more complex with three exercise points, so the time taken to compute the prices is considerably increased. We found that the prices were very similar from the two models, showing that penny accuracy (i.e., to within 1%) was produced by the model with just two possible exercise dates.

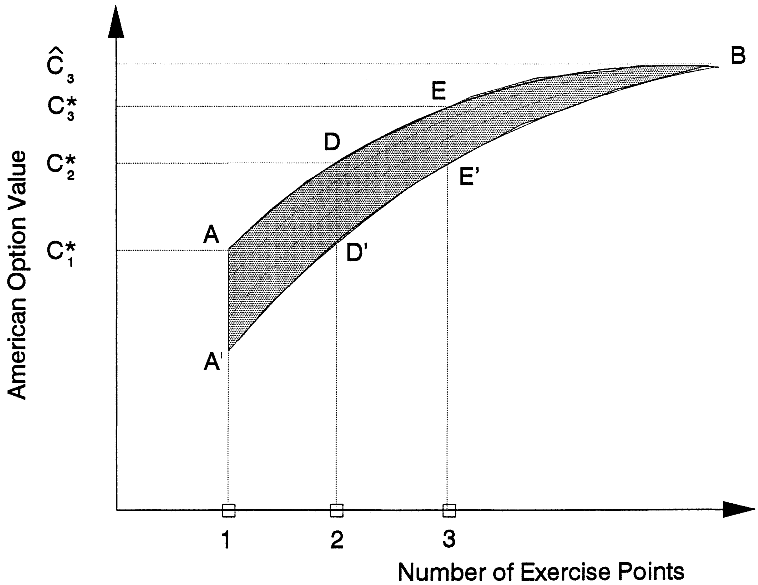

Fig. 2. Approximating American call option values using "maximized" values of European options and options exercisable on one, two and three dates.

values lie in the range A–A′, the highest value being at A. Similarly, for two and three exercise points, the maximum values are at D and E, respectively. Using the values at A, D and E, the asymptotic value at B is obtained by extrapolation.

4.1. Inputs required for the calculation of the option prices C_1 and C_2

Ho et al. (1995) describe a method which can be used to construct a multivariate-binomial approximation to a joint-lognormal distribution. This approximation can be used to value an option with two possible exercise dates. The key step in this methodology is the construction of a binomial tree with the required mean, variance and covariance characteristics. In this section, we describe the required inputs for the model.

The important inputs required for the calculation of option prices are the forward prices of the asset for each exercise date, and volatility of the asset price over the relevant time periods. For example, since we need the maximum European option price, we need the forward price and volatilities for all possible future exercise dates. In the examples that follow, we maximize the option prices by calculating the prices of options with maturities that increase by six-monthly intervals. Similarly, when calculating C_2 values, we consider a set of possible exercise dates on a grid of six-monthly spaced points. We also consider bonds with semi-annual coupon payments. Therefore, in the examples, we simply take the forward price of the bond, F_{0,t_i}, to be a constant. In general, however, the forward prices need to be computed in the usual way by compounding the spot prices of the bond up to the exercise date and adjusting for the value of any intermediate coupon interest payments. [16]

The model first requires volatility inputs for computing the European option prices, for all maturities $t_i \in (0, T]$, where T is the final maturity date of the American option. As discussed earlier, the price of the underlying bond has a time-dependent volatility due to its fixed final maturity date. For the valuation of the options with two possible exercise dates, we require both unconditional and conditional volatilities on the relevant dates. For example, if we wish to value an option with two exercise dates, t_1 and t_2, we need the unconditional volatilities σ_{0,t_1} and σ_{0,t_2} and the conditional volatility σ_{t_1,t_2}. [17]

A number of approaches to estimating these volatilities are possible. First, the volatilities could simply be assumed to be given exogenously. Second, we could generate the volatilities using a factor model. Third, we could build a model of the evolution of the term structure of interest rates, value bonds given these interest rates, and then price the options using these prices.

The first approach has been used in many practical applications of the Black and Scholes (1973) model to the pricing of European options on bonds. The second approach was employed by Brennan and Schwartz (1979) and Schaefer and Schwartz (1987), for pricing bond options. The former paper uses a two-factor model, with the long rate and the spread between the short and long rate as factors. The latter paper uses a one-factor duration model to generate bond volatilities. The third approach builds a no-arbitrage term structure and was first used by Ho and Lee (1986) and then by Heath et al. (1990a, b, 1992). In this paper, we use a variation of the second of the approaches outlined

[16] In the case of a bond, the forward price of the underlying asset for delivery at time t_i, F_{0,t_i}, depends upon the coupon-interest payments on the bond. If the bond pays no interest then by spot-forward parity the forward price would be

$$F_{0,t_i} = \frac{S_0}{B_{0,t_i}}, \quad i = 1, 2, \ldots, J.$$

However, given semi-annual coupon payments of $c/2$ paid at $\tau = \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, N$, this simple relationship has to be modified as follows using spot-forward parity:

$$F_{0,t_i} = \left[S_0 - \sum_{\tau=\frac{1}{2}}^{N} \frac{c}{2} B_{0,\tau} \right] \Big/ B_{0,t_i},$$

where $c/2$ is the semi-annual coupon and N the maturity date of the bond. Note that coupons paid before time t_i are deducted from the bond price.

[17] Since conditional and unconditional volatilities are required for any combination of exercise dates, we need to ensure consistency between the volatility estimates. The bond volatility, for example, should be a declining function of time, as the maturity of the bond approaches. This is roughly analogous to ensuring consistency between spot and forward interest rates.

above, but with the important additional feature of being arbitrage-free, in line with the spirit of the third approach. We do so for the following reasons.

First, we need so many volatility inputs that the first approach is somewhat impractical when a large number of simulations are to be performed. The third approach on the other hand, which was used by Jamshidian (1989) to value bonds options, is extremely complicated to apply, except in the case of one-factor models. Thus, there is a tradeoff between the number of factors used to describe the movements in the term structure and the level of detail in defining the evolution over time. We, therefore, use the second approach and assume that an exogenously given two-factor model of interest rates generates the yields on bonds. In such a model, we run the risk of not satisfying the requirements of a complete term structure model. However, at a practical level, this risk is perhaps worth taking, given the computational effort that would be required to build a full, arbitrage-free two-factor model of the term structure. The volatility of a bond over a specified period depends on the volatility of the term structure of interest rates. Here, we assume that the term structure is generated by two factors, a short-term rate factor x_t and an orthogonal second factor y_t. The second factor can be thought of as a spread between the short-term interest rate and the long-term interest rate. The τth interest rate at time t is given by the linear relationship

$$r_\tau = a_\tau x_t + b_\tau y_t, \quad \tau = 1, 2, \ldots, I, \tag{15}$$

where I is the longest maturity date. When $a_1 = 1$, $b_1 = 0$, it follows that $r_\tau = x_t$. We further assume that the short-term interest rate factor follows a mean-reverting process of the form

$$x_t = \mu + (x_{t-1} - \mu)(1 - \alpha_x) + \epsilon_t, \tag{16}$$

where μ is the long-run mean of the process, α_x is the periodic mean reversion and ϵ_t is a white noise error term. In this discrete version of the Vasicek-type model, the (non-annualized) variance of x_t over any period (0,t) is

$$\mathrm{Var}_{0,t}(x) = \mathrm{Var}_{t-1,t}(x)[1 - (1 - \alpha_x)^{2t}]/[1 - (1 - \alpha_x)^2]. \tag{17}$$

Eq. (17) shows the relationship between the degree of mean reversion of the short-term interest rate factor and its volatility over a finite time-period. If the short rate mean reverts strongly, the volatility will be a steeply declining function of time. Thus, on an annualized basis, the volatility of the short-term interest rate over a long period will be significantly less than its volatility looked at over a short period. On the other hand, we assume here that the long-rate spread factor, y_t, follows a random walk. This implies that the long-rate factor has a constant volatility, looked at over different time intervals, (0,t).

The price of a default-free bond, with principal amount of \$1, coupon rate c, and final maturity date N, at time t is modeled as the linear sum of the discounted cash flows. We denote the discount factor for the bond cash flows that occur at time $t+\tau, \tau = (\frac{1}{2}, 1, 1\frac{1}{2}, \ldots, N-t)$ as $B_{t,t+\tau}$. Time τ is counted in half-years, since we model the price of a bond paying semi-annual coupons. Assuming that time t is a coupon-payment date, the ex-coupon price of the coupon bond at time t, denoted by $B^c_{t,N}$ is

$$B^c_{t,N} = \sum_{\tau=\frac{1}{2}}^{N-t} \frac{c}{2} B_{t,t+\tau} + B_{t,N}, \tag{18}$$

where

$$B_{t,t+\tau} = \mathrm{e}^{-r_\tau \tau}, \tag{19}$$

and r_τ is given by the two-factor model in Eq. (15). We can now model the volatility of the coupon bond price as a function of the volatilities of the two interest rate factors x_t and y_t. First, we invoke the following approximation: [18]

$$\begin{aligned}\mathrm{Var}[f(x_t, y_t)] \approx {} & \left(E\left[\frac{\partial f(x_t, y_t)}{\partial x_t}\right]\right)^2 \mathrm{Var}(x_t) \\ & + \left(E\left[\frac{\partial f(x_t, y_t)}{\partial y_t}\right]\right)^2 \mathrm{Var}(y_t)\end{aligned} \tag{20}$$

given that x_t and y_t are independent. To apply this relationship in the case of our two-factor model, we first define

$$f(x_t, y_t) = \ln B^c_{t,N}, \tag{21}$$

and then derive

$$\frac{\partial f(x_t, y_t)}{\partial x_t} = \frac{\partial \ln B^c_{t,N}}{\partial x_t} = -\frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} a_{t+\tau} B_{t,t+\tau} + (N-t) a_N B_{t,N}}{B^c_{t,N}}, \tag{22}$$

$$\frac{\partial f(x_t, y_t)}{\partial y_t} = \frac{\partial \ln B^c_{t,N}}{\partial y_t} = -\frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} b_{t+\tau} B_{t,t+\tau} + (N-t) b_N B_{t,N}}{B^c_{t,N}}. \tag{23}$$

Note that the expectation in Eq. (20) in our case is the expectation under the risk-neutral measure where the mean is the forward price of the asset. It follows, therefore, that we can use the following approximation for the mean of the partial derivatives:

$$E\left[\frac{\partial \ln B^c_{t,N}}{\partial x_t}\right] \simeq -\frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} a_{t+\tau} F_{0,t,t+\tau} + (N-t) a_N F_{0,t,N}}{F^c_{0,t,N}}, \tag{24}$$

[18] See Stuart and Ord (1987), p. 324.

$$E\left[\frac{\partial \ln B^c_{t,N}}{\partial y_t}\right] \simeq -\frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} b_{t+\tau} F_{0,t,t+\tau} + (N-t) b_N F_{0,t,N}}{F^c_{0,t,N}}, \tag{25}$$

where $F^c_{0,t,N}$ is the forward price of the coupon bond and $F_{0,t,t+\tau}$ the forward price for delivery at t of a zero-coupon bond with final maturity $t+\tau$. [19]

For convenience, we now, define the "duration"-type terms as follows:

$$D_x = \frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} a_{t+\tau} F_{0,t,t+\tau} + (N-t) a_N F_{0,t,N}}{F^c_{0,t,N}}, \tag{26}$$

$$D_y = \frac{\sum_{\tau=\frac{1}{2}}^{N-t} \tau \frac{c}{2} b_{t+\tau} F_{0,t,t+\tau} + (N-t) b_N F_{0,t,N}}{F^c_{0,t,N}}. \tag{27}$$

It follows, after substituting in Eq. (20), that the variance of the logarithm of the coupon-bond price is:

$$\mathrm{Var}_{0,t}[\ln B^c_{t,N}] \simeq D_x^2 \mathrm{Var}_{0,t}(x) + D_y^2 \mathrm{Var}_{0,t}(y), \tag{28}$$

where the variances are given by Eq. (17). Finally, we have the expression for the coupon-bond volatility in terms of the annualized volatilities of x_t and y_t:

$$\sigma_{0,t} = \sqrt{D_x^2 \sigma^2_{0,t,x} + D_y^2 \sigma^2_{0,t,y}}. \tag{29}$$

In order to price options with two possible exercise dates, t_1 and t_2, we require unconditional volatilities from (29) and also the conditional volatilities. The conditional volatilities are computed from the same model, simply recognizing the maturity of the underlying bond at time t_1. Hence, the "duration" terms become

$$D'_x = \frac{\sum_{\tau=\frac{1}{2}}^{N-t_2} \tau \frac{c}{2} a_{t+\tau} F_{0,t_2,t_2+\tau} + (N-t_2) a_N F_{0,t_2,N}}{F^c_{0,t_2,N}}, \tag{30}$$

$$D'_y = \frac{\sum_{\tau=\frac{1}{2}}^{N-t_2} \tau \frac{c}{2} b_{t+\tau} F_{0,t_2,t_2+\tau} + (N-t_2) b_N F_{0,t_2,N}}{F^c_{0,t_2,N}}, \tag{31}$$

and the conditional volatility is

$$\sigma_{t_1,t_2} = \sqrt{(D'_x)^2 \sigma^2_{t_1,t_2,x} + (D'_y)^2 \sigma^2_{t_1,t_2,y}}. \tag{32}$$

---

[19] The approximation in Eqs. (24) and (25) ignores the effect of non-linearity due to Jensen's inequality. In particular, the effect of the covariances of $F_{0,t,t+t\tau}$ an $F_{0,t}$ are ignored. This has the effect of slightly understating the volatilities by ignoring second-order (convexity) and higher-order effects.

The use of these duration measures allows us to model the effect of declining maturity on the conditional volatility of the coupon bond. However, in order to capture the no-arbitrage condition at the intermediate dates, we also need to adjust the conditional probabilities of up movements in the bond process. The no-arbitrage condition is that the conditional forward price must equal the conditional expected value of the bond price under the risk-adjusted measure. In the paper Ho et al. (1995), a multivariate binomial distribution with varying conditional probabilities is used to approximate a multivariate lognormal distribution with given volatility characteristics. In the following simulations we ensure that the no-arbitrage condition is met using such a change in probability. The conditional probability at a node reflects the zero-bond price, and the forward price at the node.

4.2. Estimation of American option values

The computational efficiency of the method is achieved by predicting the value of an American option using a European option and an option with two possible exercise dates. [20] However, as illustrated in Fig. 2, it is only the maximized option prices denoted by

$$C_1^* = \max_t \ C_1, \quad t \in (0, T],$$

$$C_2^* = \max_{t_1,t_2} \ C_2, \quad t_1 \leqslant t_2, t_1, t_2 \in (0, T]$$

for simplicity, that are relevant. In Fig. 2, the options with one exercise point are the European options. Point A denotes the option with price C_1^*, point D denotes the option with price C_2^*, and point E denotes the option with price C_3^*. Ho et al. (1994) argued that an exponential relationship could be assumed to exist between the American option value and the number of possible exercise points. This is illustrated by the line ADE in the figure. The resulting American value is represented by the point B. In Section 5, we examine the comparative statics of the predicted value of the American option.

5. Comparative statics of the model

In this section, we examine the characteristics of the American bond option prices generated by our model in some detail. We demonstrate that the model values American bond options to "penny accuracy" using only the prices of European options and options with two exercise dates. We consider two types of simulations of our model:

[20] Breen (1991) shows the efficiency of the GJ approximation in the binomial case.

(a) *Sensitivity analysis of the computational method.*
Here, we examine the effect the size of the binomial lattice (i.e., the number of binomial stages, n).

(b) *Comparative statics and analysis of key input parameters.*
The parameters we consider are the exercise price, volatility, and time to expiration.

In the simulations reported below, the parameters used in the base case are:
Maturity of the underlying bond, $N = 10$ years.
Annual coupon rate of bond, $c = 10.8\%$.
Time-grid size for the underlying bond = 0.5 years.
Short-term interest rate volatility, $\sigma_{0,t,x} = 0.0055$. [21]
Long-rate spread volatility, $\sigma_{0,t,y} = 0.0040$.
Mean reversion coefficient, $\alpha_x = 0.05$.
Exercise price, $K = 100$. [22]

5.1. Sensitivity analysis of the computational method: The effect of changing the density of the binomial lattice

Table 1 shows the estimated American call option value for different sizes of the binomial lattice, n. The grid size used in the maximization process is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatilities of the short- and long-term interest rate factors are, respectively, $\sigma_{0,t,x} = 0.0055$ and $\sigma_{0,t,y} = 0.0040$, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for C_1^* the maximum valued European option value, where the maximum is taken over all possible option maturities. C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2$ is the exponential estimate of the American call option value.

Table 1 shows the estimated values of the option, $\hat{C}_2$, with a maturity equal to that of the underlying bond of 10 years, based on the extrapolation of two option prices, as a function of the number of binomial stages, n. For example, for $n = 60$, the maximum European option price is estimated with $t^* = 3.0$ years, resulting in a value of $C_1^* = 0.7987$. The combination of (t_1^*, t_2^*) which gives the maximum value of $C_2^* = 0.9466$, is $t_1^* = 1.5$ years and $t_2^* = 4.0$ years.

[21] The interest rate volatility numbers, $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$ are chosen so that they provide reasonable estimates for bond price volatility when multiplied by the "duration"-type terms in Eq. (29).

[22] Although it is possible to make the strike price a function of t we simply choose
$K(t_i) = K, \quad \forall i,$
a constant, in the following simulations.

Table 1
American call option values as function of the size of the binomial lattice

Size of binomial lattice n	Maturity t^*	Maximum European option value C_1^*	Maturity t_1^*	t_2^*	Maximum two-exercise point option C_2^*	Exponential American option value $\hat{C}_2$
5	3.0	0.7583	1.5	4.0	0.9246	1.1273
6	3.0	0.7640	2.0	4.5	0.9557	1.1954
7	3.0	0.7680	1.5	4.0	0.9411	1.1533
8	3.0	0.7725	2.0	4.5	0.9515	1.1720
9	3.0	0.7764	1.5	4.0	0.9477	1.1568
10	3.0	0.7794	2.0	4.5	0.9477	1.1523
12	3.0	0.7838	2.0	4.5	0.9444	1.1379
14	3.0	0.7869	2.0	4.5	0.9416	1.1267
16	3.0	0.7891	1.5	4.0	0.9418	1.1240
18	3.0	0.7907	1.5	4.0	0.9447	1.1287
20	3.0	0.7920	1.5	4.0	0.9467	1.1316
25	3.0	0.7943	1.5	4.0	0.9558	1.1262
30	3.0	0.7957	1.5	4.0	0.9496	1.1332
35	3.0	0.7967	1.5	4.0	0.9458	1.1229
40	3.0	0.7973	1.5	4.0	0.9482	1.1276
45	3.0	0.7979	1.5	4.0	0.9484	1.1274
50	3.0	0.7982	1.5	4.0	0.9459	1.1208
55	3.0	0.7985	1.5	4.0	0.9487	1.1272
60	3.0	0.7987	1.5	4.0	0.9466	1.1217

The estimated $\hat{C}_2$ in this case is 1.1217. The model values exhibit the normal fluctuations associated with the binomial lattice method as a function of n, which get dampened as n gets larger. These values and other simulations not shown here with different exercise prices show that the values in the range of $n = 11$–15 provide a reasonable approximation to the asymptotic $\hat{C}_2$ value. The advantage of using a relative small n is the obvious computational efficiency in relation to competing methods that use numerical (polynomial) approximations for bivariate and trivariate normal distribution. [23]

[23] We also investigated the increased accuracy resulting from using a model with three exercise dates. Again, the option price used was the maximum of the values across exercise dates, where the three exercise dates are chosen with $t_1 \leqslant t_2 \leqslant t_3$. The principal finding was that only a marginal increase in accuracy is obtainable by considering options exercisable on three dates. The $\hat{C}_3$ model requires a far more complex calculation and optimization procedure than the $\hat{C}_2$ model, since the value of the option must be maximized over combinations of three different exercise dates. The marginal increase in accuracy obtained may not be justified by the increase in computational time.

5.2. Sensitivity analysis of key input parameters

We now consider the effect of changing three key input parameters, the exercise price, the maturity of the underlying bond and the volatility inputs.

5.2.1. Sensitivity of option prices to changes in exercise price

We next investigate the impact of the change in the exercise price on value of the American-style option, $\hat{C}_2$. In each case, the maturity of the option is the same as that of the underlying bond, 10 years. This has the effect of investigating the valuation characteristic of the model for options which are deep-in-the-money to options which are deep-out-of-the-money. Because of the convergence of the option prices when the option is very deep-in-the-money and deep-out-of-the-money, the results reported are tabulated in Table 2 for exercise prices of $K = 95$–109 only.

The table shows the estimated American call option value for different values of the exercise price, K. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatility of the short-term interest rate factor, $\sigma_{0,t,x}$, is 0.0055, volatility of the long-term interest rate factor, $\sigma_{0,t,y}$, is 0.0040, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%. In the table, t^* is the maturity at which the maximum is obtained for the European option value, C_1^* is the maximum-valued European option value where the maximum is taken over all possible option maturi-

Table 2
American call option values for different values of the exercise price

Exercise price K	Maturity t^*	Maximum European option value C_1^*	Maturity t_1^*	t_2^*	Maximum two-exercise point value, option C_2^*	Exponential American option $\hat{C}_2$
95	1.0	4.4036	0.5	1.0	4.6693	5.0150
96	1.0	3.5084	0.5	1.0	3.7733	4.0582
97	1.0	2.6341	0.5	1.0	2.8686	3.1239
98	1.5	1.8152	1.5	2.0	2.0488	2.3124
99	2.5	1.2228	1.0	3.0	1.4310	1.6748
100	3.0	0.7838	2.0	4.5	0.9444	1.1379
101	3.5	0.4957	2.0	5.0	0.6031	0.7336
102	4.0	0.3002	2.5	5.5	0.3582	0.4274
103	4.5	0.1630	2.0	5.0	0.2012	0.2484
104	4.5	0.0893	2.5	5.5	0.0994	0.1106
105	5.0	0.0446	2.0	5.5	0.0479	0.0516
106	5.0	0.0194	3.0	6.0	0.0187	0.0180
107	4.5	0.0089	1.0	4.5	0.0089	0.0089
108	5.0	0.0035	1.5	5.0	0.0035	0.0035
109	5.0	0.0012	1.0	5.0	0.0012	0.0012

ties, C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The combinations of dates for the maximum are (t_1^*, t_2^*). $\hat{C}_2$ is the exponential estimates of the American call option values.

The simulations show that as the call option is further out-of-the-money, the value of $\hat{C}_2$ approaches zero. Using the case where $\sigma_{0,t,x} = 0.0055$ and $\sigma_{0,t,y} = 0.0040$ as the call option gets deep-in-the-money the value of $\hat{C}_2$ increases from an at-the-money ($K = 100$) price of 1.1379 to a price of 5.0150 for $K = 95$. The well-behaved characteristics of the option prices, which are quite similar to those found in the Black–Scholes model, are clearly depicted in Fig. 3. The graph plots American call option values against exercise prices for fixed volatilities of the short- and long-term interest rate factors, $\sigma_{0,t,x} = \sigma_{0,t,y} = 0.002, 0.0055, 0.008, 0.01$. The other parameters used in the calculations of option values are as follows: The size of the binomial lattice, n, is 12, the grid size 0.5 years, the mean reversion coefficient, α_x, 0.05, the bond maturity, N, 10 years with an annual coupon, c, of 10.8%.

In addition, Fig. 3 shows that as $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$ increase the value of the call option also increases. The call values are therefore shown to be sensitive to the forward prices (as represented by changing the exercise price, K) and the

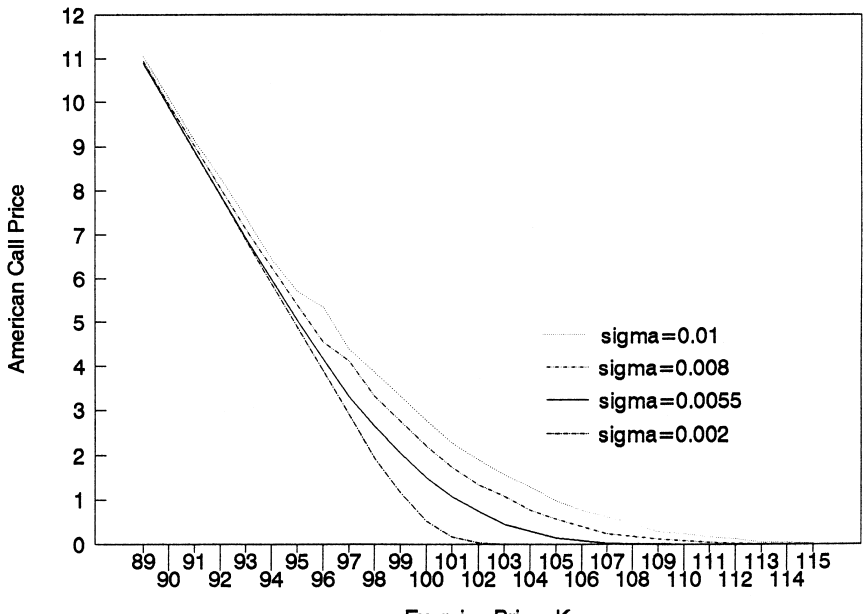

Fig. 3. Sensitivity of American call option values to changes in the exercise price for different volatilities of the short-term and long-term interest rates factors.

estimates of $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$. The sensitivity, however, is more pronounced for at-the-money options.

5.2.2. Sensitivity of option prices to the maturity of the underlying bond

The next comparative statics exercise investigates the pricing characteristics of the $\hat{C}_2$ estimate for the valuation of options on 10.8% coupon bonds with maturities of 5, 10, 15 and 20 years. The option maturity is the same as that of the underlying bond. The other parameters used in the model are listed in Table 3. The table shows the estimated American call option value for different maturities of the underlying bond. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatility of the short-term interest rate factor, $\sigma_{0,t,x}$, is 0.0055, volatility of the long-term interest rate factor, $\sigma_{0,t,y}$, is 0.0040, the bond maturity, N, varies from 5 to 20 years, with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for the European option, C_1^* the maximum European option value, where the maximum is taken over all possible option maturities, C_2^* the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2$ is the exponential estimate of the American call option value. It can readily be seen from the table that the price of $\hat{C}_2$ increases with bond maturities for a given estimate of the volatility of the short-term rate ($\sigma_{0,t,x}$) and long-term ($\sigma_{0,t,y}$) interest rate spread factors.

5.2.3. Sensitivity of $\hat{C}_2$ to volatility inputs

Lastly, we investigate whether the results above, on the accuracy of the $\hat{C}_2$ estimation, are sensitive to the volatility inputs used. The table shows the estimated American-style bond option values for varying volatilities of the short- and long-term interest rate factors, $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$, respectively. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the

Table 3
American call option values for different bond maturities

Bond maturity N	Maturity t^*	Maximum European option value C_1^*	Maturity t_1^*	t_2^*	Maximum two-exercise point option C_2^*	Exponential American option value $\hat{C}_2$
5	2.0	0.3934	1.0	2.5	0.4896	0.6094
10	3.0	0.7838	2.0	4.5	0.9444	1.1379
15	3.5	1.1262	2.5	6.0	1.3706	1.6438
20	4.0	1.3626	3.0	7.5	1.6600	2.0224

Table 4
American call option values for varying short and long interest rate volatilities

Short and long rate factors volatility	Maturity	Maximum European option value	Maturity		Maximum two-exercise point option	Exponential American option value
$\sigma_{0,t,x} = \sigma_{0,t,y}$	t^*	C_1^*	t_1^*	t_2^*	C_2^*	$\hat{C}_2$
0.0020	2.5	0.3564	1.5	3.5	0.4196	0.4940
0.0040	2.5	0.7435	2.0	4.5	0.8788	1.0387
0.0055	2.5	1.0351	1.5	4.0	1.2216	1.4418
0.0080	2.5	1.5235	1.5	4.0	1.7935	2.1114
0.0100	2.5	1.9161	1.5	4.0	2.2498	2.6417

maturity at which the maximum is obtained for the European option, C_1^* is the maximum European option value, where the maximum is taken over all possible option maturities, C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2$ is the exponential estimate of the American call option value. The results tabulated in Table 4 show as expected that $\hat{C}_2$ increases with increases in the volatilities of the short rate and the long-rate spread, i.e., $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$.

6. Conclusions

An American option can be thought of as the limit of a series of options exercisable on one of many exercise dates. However, in the case of an option with a general exercise schedule, on an asset with an arbitrary volatility structure, the limit is one of a series of maximized option prices. We propose a model which uses just a European and an option exercisable on one of two dates. We show in the simulations of the model, that a binomial version of the model, with just 12 stages in the binomial process is sufficient for penny accuracy. Also we show, using simulations of bond option prices, that the model has characteristics which are similar to those of the Black and Scholes (1973) model with respect to changes in strike prices and volatility.

In future research, we hope to extend the results reported here in two directions. First, we could compare the accuracy of our method to that of models that explicitly characterize term structure movements using a one-factor model. If our method proves to be reasonably accurate, it would have the significant advantage of computational efficiency, over competing approaches. Second, we could define term structure movements with a complete two-factor structure, and eliminate the duration-type estimates of volatility that are used here. Such a revised model may be more computationally intensive, but may be worthwhile if specific aspects of the two-factor structure are relevant to the valuation of securities, as in the case of mortgage-backed securities.

References

Amin, K.I., Bodurtha, J.N., 1995. Discrete-time valuation of American options with stochastic interest rates. Review of Financial Studies 8, 193–234.

Black, F., Derman, E., Toy, W., 1990. A One-factor model of interest rates and its application to treasury bond options. Financial Analysts Journal 46, 33–39.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–659.

Breen, R., 1991. The accelerated binomial option pricing model. Journal of Financial and Quantitative Analysis 26, 153–164.

Brennan, M.J., Schwartz, E.S., 1979. A continuous time approach to the pricing of bonds. Journal of Banking and Finance 3, 133–155.

Bunch, D.S., Johnson, H., 1992. A simple numerically efficient valuation method for American puts using a modified Geske–Johnson approach. Journal of Finance 47, 809–816.

Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option pricing: A simplified approach. Journal of Financial Economics 7, 229–263.

Geske, R., Johnson, H., 1984. The American put valued analytically. Journal of Finance 39, 1511–1542.

Heath, D., Jarrow, R.A., Morton, A., 1990a. Bond pricing and the term structure of interest rates: A discrete time approximation. Journal of Financial and Quantitative Analysis 25, 419–440.

Heath, D., Jarrow, R.A., Morton, A., 1990b. Contingent claim valuation with a random evolution of interest rates. Review of Futures Markets 9, 55–75.

Heath, D., Jarrow, R.A., Morton, A., 1992. Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation. Econometrica 60, 77–105.

Ho, T.S.Y., Lee, S.B., 1986. Term structure movements and pricing interest rate contingent claims. Journal of Finance 41, 1011–1021.

Ho, T.S., Stapleton, R.C., Subrahmanyam, M.G., 1991. The valuation of American options in stochastic interest rate economies. Proceedings of the 18th European Finance Association Conference, Rotterdam, August.

Ho, T.S., Stapleton, R.C., Subrahmanyam, M.G., 1994. A simple technique for the valuation and hedging of American options. Journal of Derivatives 2, 55–75.

Ho, T.S., Stapleton, R.C., Subrahmanyam, M.G., 1995. Multivariate binomial approximations for asset prices with non-stationary variance and covariance characteristics. Review of Financial Studies 8, 1125–1152.

Ho, T.S., Stapleton, R.C., Subrahmanyam, M.G., 1997. The valuation of American options with stochastic interest rates: A generalization of the Geske–Johnson technique. Journal of Finance 52, 827–840.

Huang, J., Subrahmanyam, M.G., Yu, G.G., 1996. Pricing and hedging American options: A recursive integration method. Review of Financial Studies 9, 277–300.

Hull, J., White, A., 1994. Numerical procedures for implementing term structure models II: Two-factor models. Journal of Derivatives 2, 7–16.

Jamshidian, F., 1989. An exact bond option formula. Journal of Finance 44, 205–209.

Jamshidian, F., 1990. Bond and Option Evaluation in the Gaussian Interest Rate Model, Merrill Lynch.

Merton, R.C., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141–183.

Nelson, D.B., Ramaswamy, K., 1990. Simple binomial processes as diffusion approximations in financial models. Review of Financial Studies 3, 393–430.

Omberg, E., 1987. A note on the convergence of binomial-pricing and compound-option model. Journal of Finance 42, 463–469.

Satchell, S., Stapleton, R.G., Subrahmanyam, M.G., 1997. The pricing of marked-to-market contingent claims in a no-arbitrage economy. Australian Journal of Management 22, 1–20.

Schaefer, S., Schwartz, E., 1987. Time-dependent variance and the pricing of bond options. Journal of Finance 42, 1113–1128.

Stuart, A., Ord, J.K., 1987. Kendall's advanced theory of statistics, vol. 1, 5th ed. Charles Griffin, London.

Turnbull, S.M., Milne, F., 1991. A simple approach to interest-rate option pricing. Review of Financial Studies 4, 87–120.