Zeroing In: Asset Pricing Near the Zero Lower Bound

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Abstract

Over the past decade, many central banks reduced interest rates to near-zero levels. I show that this has important implications for the dynamics of asset prices. In both the US and Japan, one such effect was that the correlation of stock and nominal bond returns decreased sharply as the short rate approached zero. To explain this fact, alongside the changing dynamics of stock and bond risks near the Zero Lower Bound (ZLB), I propose a New Keynesian framework with nominal rigidities. Specifically, I find that the probability that the ZLB binds in the near future represents a new source of macroeconomic risk. As this probability of hitting the ZLB increases, expected dividends drop and equity risk premia increase. This combination causes stock prices to fall. In contrast, long-term bond prices increase as investors expect future short rates and bond risk premia to drop. These opposite exposures to the risk of a binding ZLB constraint sharply lower the stock-bond return correlation and turn it negative. I develop and calibrate a model that endogenously generates these observed changes while respecting unconditional macroeconomic and asset pricing moments.

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1 Introduction

Long treated as a mere theoretical possibility, the Great Recession led many central banks in developed markets to reduce short-term nominal interest rates to near-zero levels for prolonged periods of time. In the United States, the Federal Reserve targeted its policy rate between zero and 25 basis points for seven years starting in December 2008: U.S. nominal short rates were stuck at their Zero Lower Bound (ZLB). The ZLB is of genuine concern to policy makers because if an economy is stuck at its lower bound, the central bank can no longer stimulate it nor absorb adverse shocks by reducing the nominal short rate. A question that naturally arises is how asset prices behave when conventional monetary policy tools are no longer effective.

This paper studies how the behavior of asset prices, risk premia, and asset correlations changes near and at the ZLB. To the best of my knowledge, this is the first paper to look at the effect of the ZLB on both nominal bond and stock prices, bond and equity risk premia, as well as the co-movement of stock and nominal bond returns. In particular, I find that the ZLB can help account quantitatively for the puzzling changes in the correlation between stock and bond returns that are observed in the data.

Although this correlation fluctuates, Table 1 confirms that it tended to be positive in the Pre-ZLB era but has on average been significantly lower and negative at the ZLB.\footnote{Figure 6 in the Appendix also shows that the stock-bond return correlation in Japan also turned negative as Japan hit the Zero Lower Bound in the mid 1990s.} Moreover, Table 1 documents that the shift in the stock-bond correlation coincides with inflation changing from being countercyclical pre-ZLB to being procyclical in the ZLB era. Nominal bonds are risky when inflation is countercyclical but become a hedge when inflation is procyclical.\footnote{There is substantial interest in the time-varying nature, including sign-switches, of the stock-bond return correlation. A number of recent papers including Baele et al. (2010), Burkhardt and Hasseltoft (2012), Campbell et al. (2017), Campbell et al. (2015), David and Veronesi (2013), and Song (2017) have documented that the correlation was particularly negative during the recent recession.}

The first part of this paper starts with a stylized three-period New Keynesian model of asset pricing with imperfect price adjustment and a ZLB on nominal interest rates. This model serves to illustrate the economic mechanism at work. The second part of the paper develops and calibrates an infinite horizon New Keynesian model to match salient asset pricing features in the data. This allows me to quantitatively assess the model and to conduct monetary policy experiments.
Table 1: Average Correlations before and at the ZLB

<table>
<thead>
<tr>
<th>Variables</th>
<th>Pre-ZLB</th>
<th>At ZLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Return - 5Y Bond Return</td>
<td>0.14</td>
<td>-0.35</td>
</tr>
<tr>
<td>Inflation - Real Consumption</td>
<td>-0.15</td>
<td>0.27</td>
</tr>
<tr>
<td>Growth</td>
<td>-0.05</td>
<td>0.53</td>
</tr>
<tr>
<td>Inflation - Real Dividend Growth</td>
<td>-0.14</td>
<td>0.56</td>
</tr>
<tr>
<td>Inflation - Real Stock Return</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

My main finding from the stylized one-shock model is that the presence of the ZLB generates a new source of macroeconomic risk: the risk that the ZLB will bind in the future. This new source of risk is priced in both stocks as well as bonds near and at the ZLB. When the policy rate is far above the ZLB, a positive demand shock induces agents to increase savings and cut spending which would lower output, but is met instead by an appropriate drop in the nominal interest rate to discourage a rise in the quantity saved.

However, when the probability of the ZLB binding in the near future increases, current short rates cannot fall (much), and investors expect future short rates to drop, which causes long-term bond prices to rise. This increases the holding period return of nominal bonds and thus lowers the bond risk premium. Bonds are a good hedge against such a “ZLB shock”. Simultaneously, an increase in the probability of the ZLB binding, lowers expected future output and future dividends. As dividends drop, investors require a higher compensation to hold the risky stock which increases the equity risk premium.

So as the economy approaches the ZLB, bonds become a hedge whereas stocks become riskier. These opposite exposures to the same risk result endogenously in a negative correlation between stock and bond returns as the economy is near or at the ZLB.

Next, I develop a quantitative infinite-horizon New Keynesian model to match key asset pricing features and solve it using a non-linear projection method. Two crucial new features of my model, relative to standard New Keynesian models exposited in for example Woodford (2003) and Galí (2015), are that investors have Epstein-Zin recursive utility preferences and the economy faces a small probability of a disaster occurring. These low probability events, such as economic depressions, wars or natural disasters, have a large impact on the economy and help account for asset pricing puzzles, as in Rietz (1988), Barro (2006), and Gabaix (2012). The quantitative model I develop matches several features of the data. It successfully replicates all the correlation sign switches from Table 1 in a fully endogenous manner without relying on different regimes. Although the quantita-
tive model also features supply shocks, the main intuition from the stylized model carries over. Far from the ZLB, demand shocks have little effect because demand fluctuations are largely neutralized by changes in the monetary policy rate. A positive supply shock increases consumption but decreases inflation (as the marginal cost drops). Dominant supply shocks give rise to a negative covariance between consumption growth and inflation. When the ZLB binds, demand shocks are amplified because they can no longer be counteracted by a change in the interest rate. A positive demand shock decreases both consumption and inflation giving rise to a positive covariance between consumption growth and inflation.

As the cyclicality of inflation changes depending on which shock dominates, bond and equity risks also change. Away from the ZLB, when supply shocks dominate, inflation is countercyclical. Nominal bonds are risky when inflation is countercyclical because this implies procyclical bond returns (high marginal utility states of the world have high inflation, this increases nominal yields, decreases bond prices and lowers bond returns). At the ZLB, when demand shocks dominate, inflation is procyclical and nominal bonds are a hedge (countercyclical bond returns). When marginal utility is high, dividend growth is low which always gives rise to procyclical equity returns. To summarize, away from the ZLB nominal bond returns are procyclical which gives rise to a positive correlation between stock and bond returns. At the ZLB, nominal bond returns are countercyclical and hence the stock-bond return correlation is negative.

In the final part of this paper, I use my quantitative model to conduct a policy experiment. Although on average the stock-bond return correlation is negative at the ZLB, Figure 1 examines the correlation around the “taper tantrum” in the summer of 2013. The taper tantrum ensued after the FOMC meeting in June 2013, when Chairman Bernanke suggested that the Federal Reserve ought to begin conversation of winding down (i.e., tapering) their quantitative easing (QE) bond-buying program and tighten monetary policy. Figure 1 plots the implied 2-year Federal Funds (FF) futures rate, a measure of the expected nominal short rate 2 years out and hence a measure of the expected duration of the ZLB, against the stock-bond return correlation. The figure shows a dramatic revision of financial markets’ expectations to leave the ZLB within 2 years, as reflected by the FF futures rate increasing from about 25 basis points (i.e., the ZLB) to more than double that. Over that period, the stock-bond return correlation turns positive. The figure illustrates that as the expected duration of the ZLB decreased, the stock-bond return correlation increased. My model is not only able to replicate this phenomenon, but also match the magnitudes.
Figure 1: Stock-Bond return correlation and 2-year Fed Funds futures rate around the “Taper Tantrum” in 2013.

Relation to the literature

My research draws on several literatures. Theoretically, I build on the large and growing literature which studies the macroeconomic consequences of a binding zero lower bound (ZLB). Influential papers in this literature, including Krugman (1998) and Eggertsson and Woodford (2003), argue that the ZLB can lead to a significant drop in output. In these models, the ZLB depresses consumption because the natural interest rate – the equilibrium real interest rate in the absence of nominal rigidities – is negative. In the majority of these models, a large enough rise in agents’ discount factor (i.e., an increase in their patience) generates a ZLB episode. Fernández-Villaverde et al. (2015) explore the non-linear dynamics of a New Keynesian model with a ZLB. My solution method for the infinite horizon model is a global non-linear projection and builds on theirs.

Second, the present paper relates to the small, but fast-growing literature that investigates ZLB-consistent term structure models. Most papers in this literature propose shadow rate models for imposing the zero lower bound for nominal interest rates following Black (1995)’s simple, yet theoretically appealing, insight that the nominal short rate can be thought of as an option. The nominal short rate is modeled as the maximum of zero and a process, the so-called shadow rate, which can go negative. The vast majority of these papers are characterized by a lack of closed-form solutions, a continu-
ous time framework that requires computationally heavy techniques and/or latent state variables. Recent papers include Kim and Singleton (2012), Krippner (2012), Bauer and Rudebusch (2013), Christensen and Rudebusch (2013), Ichiue and Ueno (2013), and Wu and Xia (2016). The model developed by Monfort et al. (2017) boils down to an Affine Term Structure Model. These models are all informative, but are based on a reduced-form approach which limits insights into the underlying drivers of bond risks, and how these change at the lower bound. Moreover, none of these papers consider stocks.

A third related line of research studies asset pricing in endowment models. Papers that examine bond pricing in a representative agent model with exogenous consumption growth and inflation but with non-standard preferences include Piazzesi and Schneider (2006), Burkhardt and Hasseltoft (2012), and Bansal and Shaliastovich (2013) (Epstein-Zin preferences) and Wachter (2006) (external habit). In all these papers the risk compensation for (expected) inflation shocks plays a crucial role to generate an upward sloping nominal term structure. This is achieved through a negative correlation between (expected or current) consumption growth and (expected or current) inflation. The variable rare disasters model of Gabaix (2012) links consumption disasters to periods of high inflation. Albuquerque et al. (2016) argue that demand shocks, arising from stochastic changes in agents’ rate of time preference, play a central role in the determination of asset prices. Branger et al. (2016) study the term structure of interest rates with a ZLB by incorporating Black (1995)’s insights into a long-run risks model in the spirit of Bansal and Shaliastovich (2013). In contrast, my model is a production economy and allows me to study both stocks as well as bonds. Moreover, I endogenize consumption and inflation. In fact, all of the aforementioned papers model inflation dynamics exogenously rather than being the outcome of monetary policy interacting with (or reacting to) agents’ optimizing behavior. Gallmeyer et al. (2007) endogenize inflation but monetary policy in their model still plays no role beyond determining the inflation rate.

In Real Business Cycle (RBC) models monetary policy plays a restricted role as it only influences the nominal side of the economy through inflation but it has no impact on the real side of the economy. A second subset of the equilibrium asset pricing literature examines the asset pricing implications of New Keynesian models where price stickiness enables monetary policy to have a real bite. Important contributions include van Binsbergen et al. (2012), Kung (2015), Li and Palomino (2014), and Rudebusch and Swanson (2012). None of these papers include the ZLB nor do they model stocks. Corhay et al. (2017) study how nominal government debt maturity operations impact the nominal term structure.
Closer in spirit to my paper is Campbell et al. (2015) who argue that different monetary policy regimes can account for the time-variation in the stock-bond return correlation. Relative to their paper, my model incorporates the ZLB and provides a complementary mechanism that is able to generate correlation sign-switches. My paper points to the risk of the ZLB binding as an alternative mechanism to generate correlation sign-switches in a fully endogenous manner without having to appeal to regime-switches. In contemporaneous work, Gourio and Ngo (2017) empirically document a change in the correlation between stock returns and inflation and interpret this as a change in the correlation between consumption growth and inflation. They do not look at equity risks nor the stock-bond return correlation. In other concurrent work, Howard (2016) studies the stock - real bond correlation at the ZLB in a model with two regimes. All action comes from an exogenous regime-change in the correlation between productivity shocks and inflation level shocks depending on whether the economy is at the ZLB or not. In contrast, in my model the correlation between inflation and consumption growth endogenously changes without the need for an exogenous change in the correlation structure of the shocks. Moreover, my paper shows that, first, investors’ long-term expectations for short-term rates matter and, second, that a change in this expectation (e.g., the prospect of faster interest-rate rises) can have a large impact on today’s (asset) correlations. Finally, Nakata and Tanaka (2016) study the term structure of nominal interest rates at and away from the ZLB. They do not look at stocks nor any of the correlations my paper focuses on.

Finally, the interaction of monetary policy and the financial market has also been studied empirically. Bernanke and Kuttner (2005) and Rigobon and Sack (2003) study the impact of monetary policy surprises on stock prices. Gorodnichenko and Weber (2016) document that after monetary policy announcements, stock returns are more volatile for firms with a higher degree of price stickiness. Swanson and Williams (2014) measure the effect of the ZLB on longer term interest rates.

The remainder of the paper is organized as follows. Section 2 describes the stylized three-period model and the main economic forces at work. Section 3 presents the quantitative infinite-horizon model. Section 4 describes the calibration and explores the quantitative implications of the model. Using the quantitative model, Section 5 conducts a monetary policy experiment inspired by the Taper Tantrum of June 2013. Finally, Section 6 concludes.


2 Stylized Model

In this section, I start with a stylized model of asset pricing with imperfect price adjustment and a zero lower bound on nominal interest rates. It cleanly illustrates the economic mechanism at work.

Time is discrete and there are three periods: \( t \in \{0, 1, 2\} \). The producers are active in period 0 and period 1, whereas period 2 is an endowment economy. Hence, output is exogenously determined in period 2 but is endogenous in the first two periods. There is no capital and no government spending. As there is no investment, output equals consumption.

2.1 Households

The representative household consumes the final good \( C_t \) and supplies labor \( L_t \). The representative household seeks to maximize its utility over streams of consumption and labor:

\[
\max E_0 \left[ \frac{C_0^{1-\gamma}}{1-\gamma} - \frac{L_0^\psi}{\psi} + \beta_0 \frac{C_1^{1-\gamma}}{1-\gamma} - \beta_0 \frac{L_1^\psi}{\psi} + \beta_0 \beta_1 \frac{C_2^{1-\gamma}}{1-\gamma} \right]
\]

The risk aversion is given by \( \gamma > 1 \) and \( \beta_t \) is the subjective discount factor at time \( t \). As I discuss below, there is uncertainty about the time 1 discount factor. There are also one-period nominal bonds \( B_t \) available in zero net supply. The budget constraints for the households are:

\[
p_0 C_0 + B_0 = w_0 L_0 + F_0
\]
\[
p_1 C_1 + B_1 = w_1 L_1 + F_1 + B_0 (1 + i_0)
\]
\[
p_2 C_2 = p_2 Y_2 + B_1 (1 + i_1)
\]

where \( p_t \) is the nominal price of the final goods at time \( t \) and \( i_t \) is the one-period nominal interest rate set at time \( t \) by the monetary authority. \( F_t \) is the nominal profit received from the intermediate firms and \( w_t \) is the nominal wage rate. \( Y_2 \) is the endowment in period 2.

The first order conditions (intertemporal respectively intratemporal condition) of the households’ problem are:

\[ C_t^{-\gamma} = E_t \left[ \beta_t (1 + i_t) \frac{p_t}{p_{t+1}} C_{t+1}^{-\gamma} \right] \quad t = 0, 1 \]
\[ C_t^\gamma L_t^{\psi-1} = \frac{\omega_t}{p_t} \quad t = 0, 1 \]

**Demand shocks**

The economy has uncertainty about the discount rate at time 1. The discount rate at time 0 is known but there is uncertainty about the discount rate at time 1, modeled as a shock to the rate of time preference \( \rho_t = -\ln(\beta_t) \). For tractability and without loss of generality, assume that the rate of time preference of the agent is governed by the following binomial random walk model with fixed up and down step sizes:

\[
\rho_1 = \begin{cases} 
\rho_0 + u & \text{w.p. } q, \\
\rho_0 - d & \text{w.p. } 1 - q.
\end{cases}
\]

Note that conditionally this is a two-point distribution. In the Online Appendix I show that this shock-distribution can be generalized to an AR(1) process with Gaussian shocks and all the results still hold. As I describe below, a positive shock to the discount rate (negative shock to the rate of time preference) induces agents to increase saving and cut spending, which would lower output. This negative demand shock can push the economy to the ZLB as it can drive nominal rates to zero.

### 2.2 The Final Good Producer

The representative firm produces the final consumption good \( Y_t \) in a perfectly competitive market at times \( t \in \{0, 1\} \). The firm only uses a continuum of differentiated intermediate goods \( X_{i,t} \) as input in a constant elasticity of substitution (CES) production function:

\[
Y_t = \left( \int_0^1 X_{i,t}^{-\frac{1}{\epsilon}} \, di \right)^{\frac{\epsilon}{\epsilon-1}}
\]

where \( \epsilon \) is the elasticity of substitution between intermediate goods and \( X_{i,t} \) is the input of intermediate good \( i \) at time \( t \). The firm operates in a perfectly competitive environment and maximizes profits subject to the production technology. It takes as given the intermediate goods’ prices \( p_{i,t} \) and the final goods’ price \( p_t \). The profit maximization problem of the firm gives the following input demand \( X_{i,t} \) and the price level \( p_t \):

\[
X_{i,t} = \left( \frac{p_{i,t}}{p_t} \right)^{-\epsilon} Y_t \quad \forall i,
\]

8
\[ p_t = \left( \int_0^1 p_{i,t}^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}} \]

### 2.3 Intermediate Goods Producers

Each intermediate firm produces differentiated goods using labor at times \( t \in \{0, 1\} \). There is a continuum of monopolistic firms and all have the same linear production technology:

\[ X_{i,t} = A L_{i,t} \quad \forall i \]

where \( A \) is the productivity and \( L_{i,t} \) is labor rented by firm \( i \) at time \( t \). Therefore, the (nominal) marginal cost of all intermediate good producers is

\[ mc_t = \frac{w_t}{A} \]

Nominal firm profits are thus given by

\[ F_{i,t} = \left( p_{i,t} - \frac{w_t}{A} \right) X_{i,t} \]

### Nominal Rigidities

A lower bound on nominal interest rates does not necessarily affect real allocations. A key ingredient is nominal rigidities such that the bound on the nominal rate is translated into a bound on the real rate, which impacts real variables. Thus through a sticky price mechanism, the nominal lower bound will have a real effect. This also implies that the equilibrium allocations of real variables cannot be determined independently of monetary policy. In the stylized model, I capture this by introducing an extreme form of price stickiness, an assumption which is later relaxed in the quantitative model by introducing a quadratic adjustment cost à la Rotemberg (1982).

The monopolistic firms face a nominal price rigidity. Ideally, firms would like to optimize their price each period:

\[ p_{i,t}^* = \arg \max F_{i,t}(p_{i,t}) = \frac{\epsilon}{\epsilon - 1} \frac{w_t}{A_t} \]

optimal markup = \( \mu^* \)
but it is assumed that it has already set a price $p_{i,t}$ and has committed to sell whatever is demanded at that price.\footnote{So instead of $p^*_{i,t} = \mu \frac{w_t}{A_t}$ holding true, we will have a markup equal to: $\mu_i = \frac{p_{i,t}}{w_t A_t} = \frac{1}{\kappa M_{i,t}} = \mu^* \frac{p_{i,t}}{p^*_{i,t}}.$} In this baseline model, I assume fully rigid prices and that all the monopolistic intermediate firms’ preset prices are the same:

$$p_{i,t} = p \quad \forall i, t$$

This implies that the final good price is also constant, $p_t = p$ for all $t$ and that all consumers (or technically, the final good producer) demand the same amount $X_{i,t}$ from each firm. By choosing prices to be fully sticky, real quantities will adjust maximally. This enables me to sharply highlight the effects of the ZLB. In contrast to fixed prices, wages are flexible and are determined by the market-clearing condition for labor:

$$L_t = \int_0^1 L_{i,t} di$$

### 2.4 Monetary Policy and Lower Bound on the Nominal Rate

The final key ingredient is a lower bound on the nominal rate set by the central bank. For tractability and following Eggertsson and Woodford (2003), the central bank pursues a strict inflation target of zero: $\pi^* = 0$. It commits to adjust the nominal interest rate so that

$$\pi_t = \pi^* = 0$$

Hence, by construction, inflation is always on target. The central bank sets the nominal interest rate following a simple interest rate rule, analogous to a Taylor-rule, subject to a lower bound:

$$i_t = \max(0, r^n_t + \pi^*)$$

Here $r^n_t$ denotes the natural real rate of interest which can be thought of as the real rate of interest in the absence of nominal rigidities:

$$r^n_t = \beta_t^{-1} - 1$$
Finally, note that Equations 1 and 2 together imply that the nominal and real rate are the same.\(^4\) Accordingly, the real rate is also bounded from below: \(i_t = r_t \geq 0 \forall t.\)

### 2.5 Main Mechanism

A positive shock to the discount rate induces agents to increase savings and cut spending which would lower output. By construction, prices are fully rigid and bonds are in zero net supply so the quantity saved cannot change in equilibrium. To restore equilibrium following the shock, something else must happen to discourage a rise in the quantity saved. The central bank restores equilibrium and prevents a drop in output by lowering the nominal rate (which also lowers the real rate) which discourages the household to save. So ordinarily a discount rate shock is met by an appropriate drop in the interest rate so that there is no change in output.

Now imagine a shock to the discount rate large enough to render the natural rate negative. Ideally, the central bank would react by also setting the interest rate negative but with the ZLB on the interest rate (and a zero-inflation target), the real interest rate is stuck at zero. Instead, a drop in consumption (and output) restores equilibrium. Given the discount rate is persistent, a large shock to the discount rate results in a similar drop in future output. This future expected drop in output encourages households to save even more, consequently increasing the amount by which today’s output must drop to restore equilibrium. This illustrates the mechanism of a demand-determined output at the ZLB. Note that nominal price rigidities are a crucial ingredient of this mechanism; because prices cannot (fully) adjust, real quantities must adjust.\(^5\)

### 2.6 Allocations

Figure 2a illustrates how the discount factor \(\beta\) and the gross nominal short rate vary as a function of the rate of time preference \(\rho\). The X-axis shows the rate of time preference \(\rho_0 = -\ln \beta_0\), which can also be interpreted as the log interest rate \(\iota_0 = \rho_0\) away from the ZLB. As explained above, initially the central bank reacts perfectly to changes in the

\[^4\] Note that \(1 + r_{t+1} = \frac{1 + i_{t+1}}{1 + \iota_{t+1}} = \frac{p_t}{p_{t+1}} = (1 + i_{t+1}) \frac{p_t}{p_{t+1}} = \frac{1 + \iota_{t+1}}{1 + \iota_{t+1}}\) which gives the Fisher equation: \(r_{t+1} = 1 + \frac{1 + \iota_{t+1}}{1 + \iota_{t+1}} - 1 \approx i_{t+1} - \pi_{t+1} .\)

\[^5\] Even though the baseline model does not explicitly have money, it is still reasonable to assume that the ZLB on nominal interest rates exists. The presence of paper money in a infinitely small amount still sets a lower bound on the nominal rate. This economy can then be thought of as the limit in which the transaction value of money approaches zero. For more details, I refer to Woodford (2003) and Korinek and Simsek (2016).
Figure 2: Stylized model: nominal interest rate, discount factor, and consumption allocations as a function of the rate of time preference $\rho_0 = -\ln \beta_0$.

discount rate by adjusting the nominal short rate. At the ZLB the central bank can no longer change the nominal short rate.

Consumption at time 2 is fixed and equal to the endowment: $C_2 = Y_2$. Proposition 1 determines the consumption path which pins down all the allocations.

Lemma 1. (Consumption and expected consumption)

The consumption in period 0 is given by

$$C_0 = \left\{ \beta_0 (1 + i_0) \left[ (1 - q) \cdot \left( \mathbb{1}_{\{\rho_0 - d_\epsilon < 0\}} \cdot \beta_0 e^{d_\epsilon} + \mathbb{1}_{\{\rho_0 - d_\epsilon > 0\}} \cdot 1 \right) 
+ q \cdot \left( \mathbb{1}_{\{\rho_0 + u_\epsilon < 0\}} \cdot \beta_0 e^{-u_\epsilon} + \mathbb{1}_{\{\rho_0 + u_\epsilon > 0\}} \cdot 1 \right) \right] \right\}^{\frac{1}{7}} Y_2$$

where $\mathbb{1}_{\{\rho_0 - d_\epsilon < 0\}}$ is an indicator function which equals 1 when $\rho_0 - d_\epsilon < 0$, and similar for the other indicator functions.

The time-0 expectation of consumption in period 1 is:

$$\mathbb{E}_0 [C_1] = \left[ (1 - q) \cdot \left( \mathbb{1}_{\{\rho_0 - d_\epsilon < 0\}} \cdot \left( \beta_0 e^{d_\epsilon} \right)^{\frac{1}{7}} + \mathbb{1}_{\{\rho_0 - d_\epsilon > 0\}} \cdot 1 \right) 
+ q \cdot \left( \mathbb{1}_{\{\rho_0 + u_\epsilon < 0\}} \cdot \left( \beta_0 e^{-u_\epsilon} \right)^{\frac{1}{7}} + \mathbb{1}_{\{\rho_0 + u_\epsilon > 0\}} \cdot 1 \right) \right] Y_2$$

Proof. See Appendix. □
Figure 2b plots the consumption allocations as a function of the rate of time preference \( \rho_0 \). The process of the preference shocks is taken to be fully symmetric: \( u_\epsilon = d_\epsilon = 0.02 \) and \( q = 0.5 \). As expected, consumption in period 2 is constant given the endowment setup. If the ZLB binds at time 0 (\( \rho_0 < 0 \)), we see that \( \mathbb{E}_0 [C_1] \) decreases because of the possibility of the ZLB binding in period 1. This also illustrates the demand-driven recession.

Imagine the endowment in period 2 actually represents the agents knowing with certainty that the economy will not be at the ZLB at time 2 (and thus consumption is at its regular level). This shows that as the expected duration of the ZLB increases (\( \rho_1 \) is expected to be negative so \( \mathbb{E}_0 [C_1] \) is below its regular level), current consumption decreases. Even when \( \rho_0 \) is slightly positive and the economy is above the ZLB, consumption can still drop because of the possibility of the ZLB binding in period 1. In short, the risk of being at the ZLB in the next period affects current consumption allocations.

It is important to note that the presence of both the ZLB as well as sticky prices are crucial ingredients to achieve this result. In the absence of sticky prices, output is only a function of the technology \( A \) and monetary policy actions do not have a real effect. This means that in this stylized model output would be constant with flexible prices. I derive this result in the Appendix. In the absence of the ZLB, it follows immediately from the expressions above (only the indicator functions where \( \rho_1 > 0 \) are satisfied) that consumption \( C_0 \) and \( \mathbb{E}[C_1] \) are constant and exactly equal \( Y_2 \).

As the allocations have been pinned down, the stochastic discount factor is also known: \( M_1 = \beta_0 \left( \frac{C_1}{C_0} \right)^{-\gamma} \). This stochastic discount factor can now be used to determine asset prices.

### 2.7 Dividend Strips

Assume that the asset to be priced is a levered claim on time-1 consumption and thus that the asset pays dividends \( D_1 = C_1^\lambda \) where \( \lambda \) can be interpreted as a measure of leverage. This interpretation of dividends is common in the asset pricing literature, see for example Abel (1999), Campbell (1986), and Campbell (2003). In the quantitative model, I relax this assumption and equity becomes a claim on the profits of the intermediate goods producers. The price of the dividend strip is determined by Proposition 1. As an immediate corollary, we get the Price-Dividend ratio.
Proposition 1. (Dividend strip price)
The price at time-0 of the stock which pays a dividend at time-1 is given by:

\[ V_0^{(1)} = \frac{\beta_0 Y_2^{\lambda - \gamma}}{C_0^{-\gamma}} A_0(\rho_0) \]

where:

\[ A_0(\rho_0) = \left[ (1 - q) \cdot \left( \mathbb{1}_{\{\rho_0 - d_\epsilon < 0\}} \cdot \left( \beta_0 e^{d_\epsilon} \right)^{1 - \frac{\lambda}{\gamma}} + \mathbb{1}_{\{\rho_0 - d_\epsilon > 0\}} \cdot 1 \right) 
+ q \cdot \left( \mathbb{1}_{\{\rho_0 + u_\epsilon < 0\}} \cdot \left( \beta_0 e^{-u_\epsilon} \right)^{1 - \frac{\lambda}{\gamma}} + \mathbb{1}_{\{\rho_0 + u_\epsilon > 0\}} \cdot 1 \right) \right] \]

The Price-Dividend ratio is given by:

\[ \frac{V_0^{(1)}}{D_0} = \frac{V_0^{(1)}}{C_0^\lambda} = \frac{\beta_0 Y_2^{\lambda - \gamma}}{C_0^{-\gamma}} A_0(\rho_0) \]

Proof. See Appendix. \(\square\)

Figure 3a plots the value of the dividend strip at time \(t = 0\) as a function of \(\rho_0\) and Figure 3b plots the corresponding price-dividend ratio. The leverage ratio \(\lambda\) is set at 2. When the economy is far away from the ZLB, the dividend strip value is driven exclusively by the change in the interest rate. As the interest rate decreases, the dividend strip price increases given that dividends (i.e., levered consumption) are constant in expectation. Once the economy is at the ZLB, the dividend strip price starts to decline as now the demand-driven recession kicks in. The interest rate is now stuck at zero but the recession drives dividends down.

Away from the ZLB, the price-dividend ratio is increasing as the interest rate decreases. As can be seen on Figure 3b, once the ZLB binds, the increase in the price-dividend ratio slows down as the natural rate decreases. The price-dividend ratio shows that at the ZLB, dividends are depressed, but not as depressed as prices.

From the expressions in Proposition 1, it also immediately follows that in the absence of the ZLB constraint (only the indicator functions where \(\rho_1 > 0\) are ever satisfied and hence \(A_0(\rho_0) = 1\)), both the dividend strip price and the price-dividend ratio are monotonically increasing in \(\beta_0\).
Figure 3: Stylized model: dividend strip value, PD ratio, stock return variance, and equity risk premium as a function of the rate of time preference $\rho_0 = -\ln \beta_0$. 

(a) Dividend strip price 

(b) Price-dividend ratio 

(c) Stock return variance 

(d) Equity risk premium
Proposition 2 derives the expected return on stocks, the variance of stock returns and the equity risk premium.

**Proposition 2.** (Expected stock return, variance of stock return and equity premium)
The risk that the ZLB will bind in the future is priced in stocks. As the probability of the ZLB binding in the future increases, expected dividends decrease which results in a higher equity risk premia. Lower expected dividends and higher equity risk premia lower current stock prices.

The expected log-return on the stock is:

$$\mathbb{E}_0 \left[ r^i_1 \right] = \mathbb{E}_0 \left[ \ln \left( \frac{D_1}{V_0^{(1)}} \right) \right]$$

$$= \ln \left( C_0^{-\gamma} \right) - \ln \left( \beta_0 A_0 (\rho_0) Y_2^{-\gamma} \right)$$

$$- \frac{\lambda}{\gamma} \left[ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} \cdot (- (\rho_0 - d_e)) + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (- (\rho_0 + u_e)) \right]$$

The Equity Risk Premium is defined as $\text{ERP} = - \text{Cov}_0 \left[ m_1, r^i_1 \right]$ and is:

$$\text{ERP} = \frac{\lambda}{\gamma} \left\{ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} \cdot (\rho_0 - d_e)^2 + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (\rho_0 + u_e)^2 \right\}$$

$$- \left[ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} \cdot (- (\rho_0 - d_e)) + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (- (\rho_0 + u_e)) \right]^2$$

where $m_1$ denotes the log stochastic discount factor: $\log(M_1)$ where $M_1 = \beta_0 \frac{C_1^{-\gamma}}{C_0^{-\gamma}}$.

Finally, the variance of the log-return is:

$$\mathbb{V}_0 \left[ r^i_1 \right] = \frac{\lambda^2}{\gamma^2} \left\{ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} \cdot (\rho_0 - d_e)^2 + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (\rho_0 + u_e)^2 \right\}$$

$$- \left[ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} \cdot (- (\rho_0 - d_e)) + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (- (\rho_0 + u_e)) \right]^2$$

**Proof.** See Appendix.

Figure 3c plots the variance of the stock return as a function of $\rho_0$ and Figure 3d plots the equity risk premium. It is important to note that the conditional variance of the log dividend strip return essentially represents the variance of the dividend yield. When the economy is far above the ZLB, dividends are expected to remain constant and hence the variance is essentially 0. As the economy approaches the ZLB, there is some uncertainty
about next period’s dividends which results in an increasing variance. The variance flattens out again when the economy is so far in the ZLB that the possibility of exiting the ZLB in the next period is non-existent.

Figure 3d plots the equity risk premium. If the nominal short rate is high, there is no risk of entering the ZLB in the next period. Dividend strips behave as a risk-free asset and the equity risk premium is zero. Once the nominal rate is low enough, there is a chance of hitting the lower bound. The macroeconomic risk of hitting the ZLB demands a positive risk premium because the dividend is low in exactly these high marginal utility states (the ZLB-states).

2.8 Bonds

In the stylized model, the price level is taken to be fixed which means there is no inflation. This also implies that there is no distinction between nominal and real bonds. However, I still assume that the bonds in this section are nominal bonds.

In this case, the real stochastic discount factor (SDF) $M_{t+1}$ is identical to the nominal stochastic discount factor (SDF):

$$M_{t+1} = M_t^S = \beta_t \frac{C_{t+1}^{\gamma}}{C_t^{\gamma}}$$

The next proposition derives the price of a two-period nominal bond and the two-period yield.

**Proposition 3.** (Price and yield of a two-period nominal bond)

*The price of a two-period bond is given by:*

$$p_0^{(2),S} = \frac{\beta_0 Y_2^{-\gamma}}{C_0^{-\gamma}} \left( q \cdot (\beta_0 e^{-u_c}) + (1 - q) \cdot (\beta_0 e^{d_c}) \right)$$

*The 2-period yield also follows directly:*

$$y_0^{(2),S} = \left( \frac{1}{p_0^{(2),S}} \right)^{\frac{1}{2}} - 1$$

*Proof. See Appendix.*
Figure 4a shows that bond prices increase as $\rho_0$ decreases. As $\rho_0$ decreases (or alternatively, as the expected duration of the ZLB increases), forward rates (and future short rates) are expected to be lower (or zero), and hence the prices of zero-coupon bonds converge to 1. This, in turn, decreases bond price volatility. The pattern exhibited here is very similar to what is known as pull-to-par: as the bond reaches its maturity date, the price of a bond converges to its par value or face value, thereby decreasing its volatility. In fact, the classic pull-to-par effect is amplified as follows. Assume that from time 0 to time 1 the rate of time preference stays constant: $\rho_1 = \rho_0$. A two-period bond at time 0 will become a one-period bond at time 1 and hence the price is simply determined on Figure 4a by moving vertically straight up from the dotted line to the solid line. However, if $\rho_1 < \rho_0$ then not only does the price of the bond increase due to the pull-to-par effect (straight vertical movement), the bond experiences an extra capital gain due to the negative shock to $\rho$ (a horizontal move to the left). This shows that when a bad shock hits the economy (i.e., when the marginal utility is high), the bond holding period return is actually higher than expected (bond price increases so the holding period return increases). This immediately tells us that bonds are a good hedge in this economy, an insight that is more formally verified later by calculating the bond risk premium.

Figure 4b displays the one- and two-period yields at time-0. Close to $\rho_0 = 0$, the 2-period yield displays the option effect. Even if $\rho_0 < 0$ but close to 0, there is a positive probability that the next period’s $\rho_1$ is positive which would yield a positive future short rate. This implies that the current two-period yield is also positive. Similarly, if $\rho_0 << 0$, there is practically zero probability that next period’s $\rho_1$ (and thus next period’s short rate) is positive, which implies a current two-period yield of 0. When $\rho_0 >> 0$, there is zero chance that the ZLB will bind in the next period, so in expectation the short rate does not move (due to the random walk assumption there is no mean reversion in the $\rho$-process). This implies that at this point the term structure of interest rates is flat. However, if there is mean-reversion in the $\rho$-process (e.g., $\rho_1 = \phi \rho_0 + \epsilon$ where $0 < \phi < 1$), the term structure of interest rates would be downward sloping if $\rho_0 >> 0$ due to the implied mean-reversion in the interest rate.

Define the gross holding period return on the bond as the return when you buy a two-period bond today and sell it when it becomes a one-period bond tomorrow:
Figure 4: Stylized model: bond prices and yields, bond return variance, bond risk premium, and correlation between stock and bond returns as a function of the rate of time preference $\rho_0 = -\ln \beta_0$
Then the log holding-period return is \( r^{(2),s}_1 = \log \left( R^{(2),s}_1 \right) \). Proposition 4 derives expressions for the expected holding period return, the variance of this return and the bond risk premium.

**Proposition 4.** (Expected bond return, variance of bond return and bond risk premium)

*The risk that the ZLB will bind in the future is priced in bonds. As the probability of the ZLB binding in the future increases, investors expect future short rates to be lower. This raises long-term bond prices and lowers bond risk premia.*

The expected holding-period log-return on the 2-period bond is given by:

\[
E_0 \left[ r^{(2),s}_1 \right] = - \log \left( P^{(2),s}_0 \right) - \left[ (1 - q) \cdot \mathbb{1}_{\{\rho_0 - d_e > 0\}} \left( \rho_0 - d_e \right) + q \cdot \mathbb{1}_{\{\rho_0 + u_e > 0\}} \cdot (\rho_0 + u_e) \right]
\]

The Bond Risk Premium is defined as \( \text{BRP}^{s}_0 = - \text{Cov}_0 \left( m^{s}_t, r^{(2),s}_1 \right) \) and is given by:

\[
\text{BRP}^{s}_0 = - \left[ (1 - q) \cdot \mathbb{1}_{\{\rho_0 - d_e > 0\}} \left( \rho_0 - d_e \right) + q \cdot \mathbb{1}_{\{\rho_0 + u_e > 0\}} \cdot (\rho_0 + u_e) \right]
\cdot \left[ (1 - q) \cdot \mathbb{1}_{\{\rho_0 - d_e < 0\}} \left( - (\rho_0 - d_e) \right) + q \cdot \mathbb{1}_{\{\rho_0 + u_e < 0\}} \cdot (- (\rho_0 + u_e)) \right]
\]

The variance of the log-return on the 2-period bond is given by:

\[
\text{Var}_0 \left[ r^{(2),s}_1 \right] = \left[ (1 - q) \cdot \mathbb{1}_{\{\rho_0 - d_e > 0\}} \left( \rho_0 - d_e \right)^2 + q \cdot \mathbb{1}_{\{\rho_0 + u_e > 0\}} \cdot (\rho_0 + u_e)^2 \right]
\]

\[
- \left[ \left( (1 - q) \cdot \mathbb{1}_{\{\rho_0 - d_e > 0\}} \left( \rho_0 - d_e \right) + q \cdot \mathbb{1}_{\{\rho_0 + u_e > 0\}} \cdot (\rho_0 + u_e) \right)^2 \right]
\]

*Proof.* See Appendix.

Figure 4c shows the variance of the bond return. As expected, if the economy is stuck at the ZLB \( \rho_0 \ll 0 \), bond prices are steady at par which results in a zero variance of bond holding period returns. As \( \rho_0 \) increases, the economy’s bond prices move when the interest rate moves which results in a positive variance of the bond holding period return. The variance flattens out when the probability of hitting the ZLB in the next period becomes zero.

Figure 4d plots the bond risk premium of the two-period bond as a function of the state variable \( \rho_0 \). As discussed before, when the marginal utility of the household is high, the return on the bond is also high, which means the bond is a good hedge. This is why the bond risk premium is negative. More specifically, assume \( \rho_0 = 0 \) and a negative shock hits the economy. At that point, the marginal utility is at a high level. However,
the negative shock results in an interest rate of 0 (whereas in expectation the interest rate was positive due to the option effect). Because the interest rate is low, the price increases, resulting in a higher holding period return. If the economy is expected to stay at the ZLB ($\rho_0 \ll 0$), the price of the bond is arbitrarily close to 1 (recall the enhanced pull-to-par effect) and the holding period return will be steady which results in a less negative bond risk premium. Similarly, if a positive shock hits the economy when $\rho_0 = 0$, the opposite happens. The marginal utility is now low, and the shock increases the short rate which brings about a lower resale price of the bond. The lower price results in a lower holding period return. Finally note that the bond risk premium becomes zero when $\rho_0 \gg 0$ because at that point the marginal utility is very low. Remember that consumption in period 1 is expected to be at its regular level when $\rho_0 \gg 0$. The near-zero risk in these states of the world, pushes the bond risk premium to 0.

### 2.9 Correlation between Dividend Strip Returns and Bond Returns

In the previous sections I discussed the properties of dividend strip returns and bond returns. Proposition 5 investigates the correlation between dividend strip returns and bond returns.

**Proposition 5.** (Covariance and Correlation between dividend strip and bond returns)

The covariance between dividend strip returns and bond returns is given by:

$$
\text{Cov}_0 \left[ r_{1,1}^i, r_{1}^{(2),S} \right] = -\frac{\lambda}{\gamma} \left[ (1 - q) \cdot 1_{\{\rho_0 - d_e > 0\}} (\rho_0 - d_e) + q \cdot 1_{\{\rho_0 + u_e > 0\}} \cdot (\rho_0 + u_e) \right]
\cdot \left[ (1 - q) \cdot 1_{\{\rho_0 - d_e < 0\}} (-\rho_e - d_e) + q \cdot 1_{\{\rho_0 + u_e < 0\}} \cdot (-\rho_0 - u_e) \right]
$$

The correlation between dividend strip returns and bond returns then immediately follows as:

$$
\text{Corr}_0 \left[ r_{1,1}^i, r_{1}^{(2),S} \right] = \frac{\text{Cov}_0 \left[ r_{1,1}^i, r_{1}^{(2),S} \right]}{\sqrt{\text{V}_0 \left[ r_{1,1}^i \right] \text{V}_0 \left[ r_{1}^{(2),S} \right]}}
$$

**Proof.** See Appendix. \(\Box\)

Figure 4e documents that dividend strip and bond returns de-couple as the expected duration of staying at the ZLB increases ($\rho_0 \ll 0$) as well as when the economy moves
far away from the ZLB ($\rho_0 >> 0$). Bond returns become steady due to the pull-to-par effect which results in a zero correlation when $\rho_0 << 0$. When $\rho_0 \approx 0$, the correlation is strongly negative. This again reflects some of the earlier intuition we gained; bonds are a hedge, whereas dividend strips are risky which gives a negative correlation between their returns. Note that when the economy is in a high-rate environment ($\rho_0 >> 0$), the conditional dividend strip return is mainly driven by the dividend process (i.e., levered consumption) and consumption will be constant. This means the conditional stock return will be constant, consequently driving down the correlation between stock and bond returns to 0.

3 Quantitative Model

My quantitative model augments a New Keynesian macroeconomic model featuring monopolistic competition in the goods market, sticky prices, and an interest rate rule with a ZLB in two ways to enable me to match asset pricing quantities. First, households are assumed to have Epstein-Zin preferences, which allow for a separation between the intertemporal elasticity of substitution (IES) and households’ risk aversion. Second, the agents in the model face a small probability of a productivity disaster occurring. These low probability events, such as economic depressions, wars or natural disasters, have a large negative impact on the economy and help account for asset prices puzzles, as in Rietz (1988), Barro (2006), Gabaix (2012), and Gourio (2012).

3.1 Households

The economy is set in infinite discrete time with dates $t \in \{0, 1, \ldots\}$. There is a representative household with recursive utility over streams of consumption $C_t$ and labor $L_t$. As in Rudebusch and Swanson (2012), the recursive utility follows a generalized Epstein-Zin specification

$$V_t = (1 - \bar{\beta})u(C_t, L_t) + \beta_t \left( E_t \left[ V_{t+1}^{1-\alpha} \right] \right)^{1/(1-\alpha)}$$

To prevent cases where the correlation would be undefined due to division by a standard deviation of zero, I add a machine epsilon of 1e-16 to the standard deviations of both returns.

Note that as in Rudebusch and Swanson (2012), if $u \leq 0$ everywhere, like it is here, then one can define $V$ to be negative everywhere and switch the signs:

$$V_t = (1 - \bar{\beta})u(C_t, L_t) - \beta_t \left( E_t \left[ (-V_{t+1})^{1-\alpha} \right] \right)^{1/(1-\alpha)}$$
where the parameter $\alpha$ can take on any real value and $u(C_t, L_t)$ is the per-period utility function. $\beta_t$ is the stochastic discount rate which mean-reverts to $\bar{\beta}$ and its dynamics are specified below in Section 3.4. The per-period utility, consistent with a balanced growth path, is given by the following additively separable utility specification:

$$u(C_t, L_t) = \frac{C_t^{1-\gamma}}{1-\gamma} + \chi_0 Z_t^{1-\gamma} (1-L_t)^{1-\psi}$$

where $\gamma, \psi \geq 1$. The households are endowed with a unit of time and $Z_t$ is an aggregate productivity trend. The scaling of the labor component by $Z_t$ is needed to ensure the existence of a balanced growth path.

The household supplies $L_t$ units of labor for which they receive a nominal wage $W_t$. The labor market is assumed to be perfectly competitive and the total supply of labor is the integral of labor $L_{i,t}$ supplied to each intermediate good producer: $L_t = \int_0^1 L_{i,t} \, di$. The household can consume an amount of $C_t$ and invest in financial assets which are, a share $\theta_t^S$ of an equity index $S_t$ which pays out real dividends $D_t$, nominal as well as real zero-coupon bonds of maturities ranging from 1 to $N$ periods. This implies that the period-by-period nominal budget constraint of the household is given by:

$$P_t C_t + \theta_t^S P_t S_t + \sum_{n=1}^{N} B^S_t(n) P_t^S(n) + P_t \sum_{n=1}^{N} B_t(n) P_t(n) =$$

$$W_t L_t + \theta_{t-1}^S P_t (S_t + D_t) + \sum_{n=1}^{N} B_{t-1}^S(n) P_t(n-1) + P_t \sum_{n=1}^{N} B_{t-1}(n) P_t(n-1) + T_t$$

where $P_t$ is the nominal price of the final good (or in other words, the aggregate price level), $S_t$ is the price of a share in the equity index which is a claim on (the portion that is paid out) all firms’ profits, $B_t^S(n)$ is holdings of an $n$-period nominal bond, $B_t(n)$ is holdings of an $n$-period real bond. $T_t$ is a lump-sum transfer to the household sector. Note that the household owns the intermediate firms and that equity shares sum to one, $\theta_t^S = 1 \forall t$. All bonds are in zero net supply, $B_t^S(n) = B_t(n) = 0 \forall t, \forall n$.

The household’s intertemporal Euler equation is

$$1 = \mathbb{E}_t \left[ \frac{M_{t,t+1} I_t}{\Pi_{t+1}} \right]$$

where $I_t = \frac{1}{P^S_t(0)}$ denotes the gross nominal one-period interest rate (recall $P^S_t(0) = 1$), and $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ denotes the inflation rate. $M_{t,t+1}$ denotes the real stochastic discount factor.
between time $t$ and $t + 1$:

$$M_{t,t+1} = \beta_t \left( \frac{(-V_{t+1})}{(E_{t} [(-V_{t+1})^{1-a}])^{1/(1-a)}} \right)^{-\alpha} \frac{U'_C(C_{t+1}, L_{t+1})}{U'_C(C, L)}$$

The intratemporal condition is (denote the real wage by $\tilde{W}_t$):

$$\tilde{W}_t = \frac{W_t}{P_t} = Z_t^{1-\gamma} C_t^\gamma X_0 (1 - L_t)^{-\psi}$$

### 3.2 Firms

Just as in the stylized model, there are two types of producers, a final good producer and an intermediate goods sector.

#### 3.2.1 Final good producer

A representative firm produces the final consumption good $Y_t$ in a perfectly competitive market. The firm uses a continuum of differentiated intermediate goods $X_{i,t}$ as input in a constant elasticity of substitution (CES) production function:

$$Y_t = \left( \int_0^1 X_{i,t}^{\frac{\epsilon}{1-\epsilon}} di \right)^{\frac{1}{\epsilon-1}}$$

where $\epsilon$ is the elasticity of substitution between intermediate goods and $X_{i,t}$ is the input of intermediate good $i$ at time $t$. The firm maximizes profits subject to the production technology. It takes as given the intermediate goods nominal prices $P_{i,t}$ and the final goods nominal price $P_t$. The profit maximization problem of the firm gives the following input demand:

$$X_{i,t} = \left( \frac{P_{i,t}}{P_t} \right)^{-\epsilon} Y_t \quad \forall i,$$

and the price level

$$P_t = \left( \int_0^1 P_{i,t}^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$$
3.2.2 Intermediate goods producers

There are a continuum of monopolistic firms. Each intermediate firm produces a differentiated good $X_{i,t}$ using labor $L_{i,t}$:

$$X_{i,t} = A_t Z_t L_{i,t}$$

where $A_t$ is a transitory disturbance to productivity that affects all firms and $Z_t$ is the aggregate productivity trend. Both productivity components follow exogenous stochastic processes which are described in Section 3.4.

Each intermediate firm faces a cost of adjusting its nominal price. This quadratic adjustment cost is measured in terms of the final good as in Rotemberg (1982) and has the following form:

$$G(P_{i,t}, P_{i,t-1}) = \frac{\phi_R}{2} \left( \frac{P_{i,t}}{\Pi_{ss}P_{i,t-1}} - 1 \right)^2 Y_t$$

where $\Pi_{ss}$ is the gross steady-state inflation and $\phi_R$ is the magnitude of the cost. The aggregate resource constraint takes this quadratic adjustment cost into account:

$$Y_t = C_t + \frac{\phi_R}{2} \left( \frac{\Pi_t}{\Pi_{ss}} - 1 \right)^2 Y_t.$$ Each period firms set their price to maximize profits:

$$\max_{\{P_{i,t}\}} \sum_{k=0}^{\infty} M_{t,t+k} \left\{ \frac{P_{i,t+k}}{P_{i,t+k}} X_{i,t+k} - MC_{t+k} X_{i,t+k} - G(P_{i,t+k}, P_{i,t+k-1}) - \nu_{D,t+k} Z_{t+k} \right\}$$

subject to the demand curve and the production function, where $MC_{t+k}$ is the marginal cost which is given by: $MC_{t+k} = \frac{W_{t+k}}{Z_{t+k} A_{t+k} P_{i,t+k}}$ and $M_{t,t+k}$ is the real pricing kernel between time $t$ and time $t+k$. $\nu_{D,t+k}$ is an operating cost which I describe below. All firms face an identical problem and thus choose the same price producing the same quantity. This implies the following forward-looking non-linear Phillips curve:

$$\left[ (1 - \epsilon) + \epsilon MC_t - \phi_R \left( \frac{\Pi_t}{\Pi_{ss}} - 1 \right) \frac{\Pi_t}{\Pi_{ss}} \right] Y_t + \phi_R E_t \left[ M_{t,t+1} \left( \frac{\Pi_{t+1}}{\Pi_{ss}} - 1 \right) \frac{\Pi_{t+1}}{\Pi_{ss}} Y_{t+1} \right] = 0$$

---

8It costs resources to change goods’ prices, possibly due to the difficulties changing prices impose on consumers. This cost is a function of the magnitude of the price change. It is costless to have an inflation-rate of $\Pi_{ss}$. Hence, consumers prefer small and recurrent price changes to occasional large ones. I use Rotemberg (1982) rather than Calvo (1983) pricing due to its computational advantage; the Calvo model would contain one additional state variable that tracks firm price dispersion.
This shows that inflation dynamics mainly depend on real marginal costs and expected inflation. It is also clear that future inflation terms can be recursively substituted out, such that inflation is driven by current as well as future (discounted) marginal costs.

### 3.3 Central Bank

The central bank sets the nominal one-period interest rate, $I_t$, following a Taylor rule with an occasionally binding ZLB constraint. The Taylor rule depends on the lagged interest rate, inflation deviations, and a measure of the output gap. The max-operator represents the ZLB and $I^*_t$ is the shadow rate:

$$I_t = \max (1, I^*_t)$$

$$I^*_t = I_{ss}^{1-\rho_{I}} I_{t-1}^{\rho_{I}} \left[ \frac{\Pi_t}{\Pi_{ss}} \left( \frac{Y_t}{Y_{t-1}} \right) \right]^{\phi_Y} \exp(v_{M,t})$$

where $I_{ss}$ is the steady state of the gross nominal rate, $v_{M,t}$ is a mean-0 Gaussian shock with standard deviation $\sigma_{v_{M}}$, and $Y_{t-1}^*$ is defined as in Rudebusch and Swanson (2012), namely it is the trend level of output $yZ_{t-1}$ where $y$ denotes the steady-state level of detrended output ($y = Y/Z$).

### 3.4 Exogenous Stochastic Processes

The discount factor $\beta_t$ fluctuates around its mean $\beta$ with a persistence $\rho_b$ and follows an AR(1) process in logs:

$$\ln(\beta_t) = (1-\rho_b) \ln(\bar{\beta}) + \rho_b \ln(\beta_{t-1}) + \sigma_b \epsilon_b,t$$

with $|\rho_b| < 1$ and $\epsilon_b,t \sim$ i.i.d. $\mathcal{N}(0, 1)$.

The aggregate productivity trend $Z_t$ follows an exogenous process:

$$\ln Z_t = \ln Z_{t-1} + \ln \Lambda_Z - d_t \bar{\theta}_Z$$

where $d_t$ is a rare disaster indicator function. A disaster hits the economy with a fixed probability $p_d$, i.e. $d_t = 1$ with probability $p_d$. If a disaster occurs then the permanent part

---

9Note that the advantage of this formulation is that in normal non-disaster states, the output gap is simply measured as the standard $\tilde{y}_{ss}$ as in Fernández-Villaverde et al. (2015), Kung (2015) and many other papers. During a disaster the output gap measure boils down to $\frac{\tilde{y}_t}{\tilde{y}_{ss}} \exp(-d_t \bar{\theta}_Z)$. 

---
of productivity drops by an amount $\theta_Z$. Thus, the productivity trend is deterministic, except for when a disaster occurs (so $\ln Z_t$ resembles a sawtooth wave.

The stationary aggregate productivity shock $A_t$ evolves according to:

$$\ln A_t = \rho_a \ln A_{t-1} + \sigma_a \epsilon_{a,t} - d_t \bar{A}$$

with $|\rho_a| < 1$ and $\epsilon_{a,t} \sim \text{i.i.d. } N(0, 1)$.

Even though disasters could be modeled as purely instantaneous and permanent (i.e., they only affect the permanent part of productivity), empirical research has shown that disasters typically include a short-run shock, which represents a temporary economic or financial weakness, from which the economy can partially recover in a few years (e.g., Nakamura et al. 2013). This feature is captured by a drop in the stationary part of productivity when a disaster takes place. If a disaster occurs then the transitory part of productivity drops by an amount $\theta_A$.

### 3.5 Pricing of Bonds

The price of an n-period nominal zero coupon bond that pays one dollar at maturity $P^{(n)}_t$ follows from the optimality condition of the households and can be derived recursively as:

$$P^{(n)}_t = E_t \left[ M^{\$}_{t+1} \cdot P^{(n-1)}_{t+1} \right]$$

where $M^{\$}_{t+1} = \frac{M_{t+1}}{\Pi_{t+1}}$ is the nominal stochastic discount factor and $P^{(0)}_t = 1$ $\forall t$. The yield-to-maturity on the n-period bond is defined as $y^{(n)}_t = -\frac{1}{n} \ln(P^{(n)}_t)$. Similarly, the price and yield-to-maturity of an n-period real bond follow from:

$$P^{(n)}_t = E_t \left[ M_{t+1} \cdot P^{(n-1)}_{t+1} \right]$$

where $M_{t+1}$ is the real stochastic discount factor and $P^{(0)}_t = 1$ $\forall t$ and the yield-to-maturity is $y^{(n)}_t = -\frac{1}{n} \ln(P^{(n)}_t)$

### 3.6 Pricing of Stocks

Equity is a claim on profits earned by the intermediate goods producers. The aggregate real profits of the firms are $F_t$. In the data not all firms’ profits are paid out in dividends
and a recent literature has also pointed out the importance of shocks to firm profits other than productivity (e.g. Bianchi et al. 2017). To capture these notions, firms only pay out a portion of their profits due to an operating cost \( \nu_{D,t} \). The net dividend payout is then:

\[
D_t = F_t - \nu_{D,t} Z_t
\]

where the operating cost is modeled as a Gaussian AR(1):

\[
\ln(\nu_{D,t}) = (1 - \rho_D) \ln(\bar{\nu}_D) + \rho_D \ln(\nu_{D,t-1}) + \sigma_D \epsilon_{D,t}
\]

The shock to the fixed operating cost directly affects corporate earnings but does not scale with production. The operating cost captures expenditures in the corporate sector that redistribute resources away from the shareholders. The equilibrium pricing equation for stocks implies that the stock price \( S_t \) in real terms equals:

\[
S_t = \mathbb{E}_t [M_{t+1} (S_{t+1} + D_{t+1})]
\]

Finally, note that the operating cost is transferred as a lump-sum to the household sector and hence it represents a source of household income. The implies that in equilibrium \( T_t = \nu_{D,t} Z_t \).

### 3.7 Equilibrium

Given a sequence of shocks \( \{\beta_t, A_t, d_t, v_{M,t}, v_{D,t}\}_{t=0}^\infty \) and the initial condition, a competitive equilibrium consists of sequences of quantities, \( \{C_t, L_t, Y_t, B_t^{S(n)}, B_t^{(n)}, \theta_t\}_{t=0}^\infty \), and prices, \( \{\bar{W}_t, I_t, \Pi_t, p_t^{S(n)}, p_t^{(n)}, S_t\}_{t=0}^\infty \), such that the household and firms optimize, the monetary policy rule is satisfied and the goods, labor, and asset markets clear.

### 3.8 Numerical Procedure

Appendix B.3 presents the key optimality equations and shows how to stationarize the model. The presence of an occasionally binding constraint (the ZLB) and the disaster shocks make this a challenging problem to solve. Due to these features, and the fact that asset prices are especially sensitive to non-linearities, I carefully solve the model using a global projection method that builds on Fernández-Villaverde et al. (2015). A key feature of the procedure is that I solve non-linearly for consumption and inflation and solve for the other variables by exploiting the equilibrium conditions. This means that the method
fully respects the non-linearity induced by the ZLB because the interest rate is determined by directly applying the Taylor rule Equation 3.

4 Quantitative Results

This section discusses the key results of the model. First, there is a discussion of the calibration of the model followed by a discussion of the results.

4.1 Calibration

The model is calibrated at quarterly frequency. Table 2 summarizes the parameter values for the calibration. The parameter values are fairly standard in the literature and are chosen such that macro, stock, and term structure moments away from the ZLB are in line with those in the data. $\chi_0$ is set to imply a steady-state number of hours worked, $l$, of $\frac{1}{3}$ of the total time endowment. The curvature of utility with respect to labor $\psi$ is set to imply a Frisch elasticity of labor supply of $\frac{2}{3}$ which is consistent with micro-estimates from Pistaferri (2003) and is also used by Rudebusch and Swanson (2012). The curvature of household utility with respect to consumption is chosen by setting the intertemporal elasticity of substitution (IES) in consumption to 0.49. This value is consistent with estimates using micro data (e.g., Vissing-Jørgensen 2002 finds that the wealthiest stockholders have an IES of 0.486) and a similar value of 0.5 is used in Rudebusch and Swanson (2012).

The Epstein-Zin parameter $\alpha$ is set to imply a relatively low coefficient of relative risk aversion of 5.0 to help match a realized excess stock return of 7% in normal times. This value is also consistent with calibrations in the rare disaster literature (Gourio 2012 has a risk aversion of 3.8 whereas Nakamura et al. 2013 use a risk aversion of 6.4). The price elasticity of demand $\epsilon$ is set to 10 (corresponds to an average markup of 11%) which is a common value, see for example Fernández-Villaverde et al. (2015b). The price adjustment cost parameter $\phi_R$ is set to 316, which corresponds to the Calvo sticky price probability of 0.85. Although this value is somewhat higher than what is often used in the pre-crisis New Keynesian literature, it is in line with models that are calibrated with the recent crisis in mind (Del Negro et al. 2015 estimate a Calvo parameter of 0.87).

The Taylor-rule parameter determining the sensitivity of the nominal short rate to inflation $\phi_{\pi}$ is set to be 3.5. This value is chosen to keep the response of inflation under the ZLB from spiraling out of control. The policy rule parameter governing the sensi-

\footnote{Gust et al. (2012) estimate a Taylor-rule inflation coefficient of 5 which allows their model to fit the
tivity of the nominal short rate to output $\phi_y$ is set to 0.5 which is in line with the original Taylor (1993) rule. The persistence of the policy rule $\rho_I$ is calibrated to 0.76 to match the volatility of the nominal short rate (2.35% in the data). These numbers are consistent with calibrations and estimates in the literature. For example, Clarida et al. (2000) obtain reduced-form estimates for the same parameter values of respectively 2.15, 0.23, and 0.79 in the Volcker and Greenspan era. Finally, the volatility of the monetary policy shock is 0.15%, which is consistent with Gust et al. (2012) who find that the standard deviation of innovations in monetary policy are about six times smaller than innovations in productivity.

The preference shock parameters $\rho_b$ and $\sigma_b$ are calibrated to generate persistent ZLB episodes as well as to match the overall frequency with which the ZLB is (expected to be) observed. Following Fernández-Villaverde et al. (2015), the preference shock persistence $\rho_b$ is set to 0.8. The unconditional standard deviation of the shock is about 0.25% (corresponding to $\sigma_b = 0.002$). This results in the economy hitting the ZLB with a frequency consistent with values calculated in the literature and a correlation between consumption growth and inflation consistent with the data. In long simulations my economy is at the ZLB during 9.7% of the quarters. This percentage is in line with the findings of Reifschneider and Williams (1999) and Chung et al. (2011). The discount factor $\beta = 0.988$ is set to match the average nominal short-term rate of about 5%.

For the technology process, I set $\rho_A = 0.95$ and the standard deviation $\sigma_a = 0.011$, standard values in the literature (see e.g. Hansen 1985). The TFP trend growth rate (in logs) is set to 0.0045. This value gives me a consumption growth average comparable to the data. The probability of a disaster is taken from Barro (2006) and corresponds to an annual probability of 1.7%. The permanent TFP drop $\theta_Z$ is set to 0.23 and the temporary TFP drop $\theta_A$ is calibrated to 0.06. The temporary TFP drop is chosen to match the average slope of the nominal term structure observed in the data before the ZLB. Overall these disaster size parameter values are in line with Barro (2006).

### 4.2 Results

The ability of my model to fit the data is now described by looking at model-implied moments and comparing them to their data counterparts. The following tables report various model-implied moments, along with their empirical counterparts for quarterly US data from 1984Q1 to 2015Q3. The data part is split into two samples, the “Pre-ZLB” sample from 1984Q1 until 2008Q3 and the “At ZLB” sample from 2008Q4 until 2015Q3. relatively modest decline in inflation during the ZLB.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Name</th>
<th>Value</th>
<th>Target/Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taylor rule (inflation)</td>
<td>φ&lt;sub&gt;π&lt;/sub&gt;</td>
<td>3.5</td>
<td>Low volatility of inflation of 1.1%</td>
</tr>
<tr>
<td>Taylor rule (output)</td>
<td>φ&lt;sub&gt;y&lt;/sub&gt;</td>
<td>0.5</td>
<td>Taylor (1993)</td>
</tr>
<tr>
<td>Taylor rule (smoothing)</td>
<td>ρ&lt;sub&gt;I&lt;/sub&gt;</td>
<td>0.76</td>
<td>Volatility of nominal short-rate of 2.35%</td>
</tr>
<tr>
<td>Demand shock persistence</td>
<td>ρ&lt;sub&gt;b&lt;/sub&gt;</td>
<td>0.80</td>
<td>Fernández-Villaverde et al. (2015)</td>
</tr>
<tr>
<td>Demand shock volatility</td>
<td>σ&lt;sub&gt;b&lt;/sub&gt;</td>
<td>0.002</td>
<td>Correlation inflation-consumption growth</td>
</tr>
<tr>
<td>Productivity shock persistence</td>
<td>ρ&lt;sub&gt;A&lt;/sub&gt;</td>
<td>0.95</td>
<td>Standard (e.g. Hansen 1985)</td>
</tr>
<tr>
<td>Productivity shock volatility</td>
<td>σ&lt;sub&gt;A&lt;/sub&gt;</td>
<td>0.011</td>
<td>Standard (e.g. Hansen 1985)</td>
</tr>
<tr>
<td>Rotemberg cost</td>
<td>φ&lt;sub&gt;R&lt;/sub&gt;</td>
<td>316</td>
<td>Calvo ≈ 0.85</td>
</tr>
<tr>
<td>Risk Aversion</td>
<td>CRRA</td>
<td>5.0</td>
<td>Excess stock return of 7%</td>
</tr>
<tr>
<td>Discount factor (mean)</td>
<td>β</td>
<td>0.988</td>
<td>Average nominal short rate of 4.9%</td>
</tr>
<tr>
<td>Demand elasticity</td>
<td>ε</td>
<td>10</td>
<td>Standard (e.g. Fernández-Villaverde et al. 2015b)</td>
</tr>
<tr>
<td>IES</td>
<td>IES</td>
<td>0.49</td>
<td>Vissing-Jørgensen (2002)</td>
</tr>
<tr>
<td>Curvature Labor utility</td>
<td>ψ</td>
<td>3</td>
<td>Rudebusch and Swanson (2012)</td>
</tr>
<tr>
<td>Frisch elasticity of labor</td>
<td>Frisch</td>
<td>0.67</td>
<td>Rudebusch and Swanson (2012)</td>
</tr>
<tr>
<td>Operating cost persistence</td>
<td>ρ&lt;sub&gt;D&lt;/sub&gt;</td>
<td>0.99</td>
<td>Bianchi et al. (2017)</td>
</tr>
<tr>
<td>Operating cost volatility</td>
<td>σ&lt;sub&gt;D&lt;/sub&gt;</td>
<td>0.03</td>
<td>Volatility of dividend-output ratio of 0.6% (BEA)</td>
</tr>
<tr>
<td>Operating cost average</td>
<td>ν&lt;sub&gt;D&lt;/sub&gt;</td>
<td>0.03</td>
<td>Average dividend-output ratio of 2% (BEA)</td>
</tr>
<tr>
<td>Prob of a disaster</td>
<td>p&lt;sub&gt;d&lt;/sub&gt;</td>
<td>0.0043</td>
<td>Barro (2006)</td>
</tr>
<tr>
<td>Permanent TFP drop</td>
<td>θ&lt;sub&gt;Z&lt;/sub&gt;</td>
<td>0.23</td>
<td>Barro (2006)</td>
</tr>
<tr>
<td>Temporary TFP drop</td>
<td>θ&lt;sub&gt;A&lt;/sub&gt;</td>
<td>0.06</td>
<td>Average yield curve slope of 1.3%</td>
</tr>
</tbody>
</table>

Table 2: Parameter values for the Quantitative Model calibrated at the quarterly level

The model-implied moments are calculated as follows. I run a long simulation of the model and condition on the model being away from the ZLB or at the ZLB, these two samples correspond to the “No ZLB” and “ZLB” columns in the tables below. The ZLB sample is further split into multiple sub-samples. In the “≥ 2Y” sample, I select only those ZLB episodes in the model simulation where the ZLB binds for at least 8 consecutive quarters. Similarly, in the “≥ 4Y” sample, the selection criterion is that the ZLB binds for at least 16 consecutive quarters. This distinction in ZLB subsamples is made because the severity of a ZLB episode, of which one measure is the overall duration of that particular ZLB episode, plays a role in how asset pricing moments change. Moreover, the “ZLB” sample is contaminated with episodes where the ZLB only binds for very short periods of time (e.g., 1 quarter ZLB episodes) where the economy essentially has a touch and go with the lower bound. This is a theoretically interesting situation on its own, but is not representative of the recent ZLB episode in the US. Hence, in the asset pricing implications below my focus is mostly on the “≥ 4Y” sample as it is most comparable to the empirical episode we observed.
4.2.1 Macro Moments

Table 3 presents the moments of consumption growth, inflation, the shadow rate $I^*$ and the amount of time spent in each subsample in the model simulation. As discussed earlier, the severity of the ZLB episode can be measured using the overall duration of the ZLB or alternatively the shadow rate $I^*$. In Table 3 we indeed see that a more negative average shadow rate corresponds to a longer ZLB episode. On average the ZLB binds 9.7% of the time, which is consistent with Reifschneider and Williams (1999) and Chung et al. (2011). About two thirds of the ZLB episodes have a duration of at least 2 years (8 quarters in the model) and only about a third of the ZLB episodes last at least 4 years. The average consumption growth in the model is 1.8% which is in line with the data. Note that the model economy grows at the certain rate of $\Lambda_Z$ (bar any disaster shocks) which means that the volatility of consumption growth is low in the model by construction. The average inflation away from the ZLB in the model is quantitatively close to that in the data. At the ZLB there is on average modest deflation in the model, whereas in the data the drop in inflation was less severe. In the macroliterature this is referred to as the puzzle of the missing disinflation. For example, Coibion and Gorodnichenko (2015) argue that a sharp increase in household inflation expectations between 2009 and 2013 can account for the absence of strong disinflationary pressures in the data. This rise in household inflation expectations is primarily driven by the increase in oil prices from 2009 to 2011. For future extensions we could imagine introducing features to the model which would lower the disinflationary pressures such as an aversion to nominal wage cuts, an aversion to price cuts, or inattention to macro conditions (Gabaix 2016). Finally, we see that inflation in the model is more volatile at the ZLB than away from the ZLB, just as in the data.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Model Above ZLB</th>
<th>ZLB</th>
<th>≥ 2Y</th>
<th>≥ 4Y</th>
<th>Data Pre-ZLB</th>
<th>At ZLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆ Consumption Mean</td>
<td>1.78</td>
<td>1.92</td>
<td>1.78</td>
<td>1.78</td>
<td>1.90</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>[0.13]</td>
<td>[0.10]</td>
<td>[0.12]</td>
<td>[0.13]</td>
<td>[0.73 ]</td>
<td>[0.74]</td>
</tr>
<tr>
<td>Volatility</td>
<td>0.13</td>
<td>0.10</td>
<td>0.12</td>
<td>0.13</td>
<td>0.73</td>
<td>0.74</td>
</tr>
<tr>
<td>Inflation Mean</td>
<td>3.41</td>
<td>-2.08</td>
<td>-2.79</td>
<td>-3.42</td>
<td>3.09</td>
<td>1.34</td>
</tr>
<tr>
<td>Volatility</td>
<td>[1.81]</td>
<td>[2.06]</td>
<td>[2.17]</td>
<td>[2.33]</td>
<td>[1.06 ]</td>
<td>[1.32]</td>
</tr>
<tr>
<td>Shadow Rate $I^*$</td>
<td>5.17</td>
<td>-0.84</td>
<td>-1.12</td>
<td>-1.48</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Time spent (in %)</td>
<td>90.3</td>
<td>9.7</td>
<td>6.1</td>
<td>3.5</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3: Annualized Macro Moments in %.
4.2.2 Bond Yields

Table 4 shows nominal yield moments in both the model and the data. The model is able to match the features of the nominal term structure both away from the ZLB as well as at the ZLB. As expected, the average nominal short rate (which in the model is the 3-month yield) is nearly identical in the model and the data. The yield curve is steeper on average at the ZLB than away from the ZLB in the data as well as in the model. The model replicates the average slope successfully. The model shows that the average slope depends on the severity of the ZLB episode, the more severe the episode the lower the average slope. The term structure of yield volatility is downward sloping on average when the short rate is above the ZLB and upward sloping at the ZLB. The model again captures this feature well.

4.2.3 Stock and Bond Returns

My model is able to generate a realistic average equity excess return of about 7% during normal times. Moreover, as shown in the first and second row of Table 5, both the equity risk premium as well as the realized excess stock return are higher on average during the long ZLB episodes than away from the ZLB. This is consistent with the realized real equity return pattern in the data. The reasoning for an increase in the equity risk premium is the same as in the stylized model. Expected dividends are lower at the ZLB which increases the equity risk premium. The real rate is high at the ZLB due to the deflation which is why the equity risk premium only increases moderately at the ZLB.

In the third row of Table 5, we see that the nominal bond risk premium is higher away from the ZLB than at the ZLB. However, the realized excess bond returns are higher at the


ZLB than away from the ZLB. The bond risk premium drops at the ZLB because bonds become a better hedge. However, the difference in magnitudes between the bond risk premium and the realized excess bond return at the ZLB is striking. At the ZLB, realized returns are significantly higher than the expected returns because investors keep expecting that bond yields will go up but they keep falling. Nominal bonds keep surprising on the upside because investors do not expect the ZLB to last as long as it does. The economy keeps getting hit with shocks that prolong the ZLB and keep future short rates low. On top of this, an unexpected longer ZLB spell is typically accompanied by unexpected lower inflation and hence nominal bonds perform even better. In the data bonds also performed better at the ZLB (3.78%) than pre-ZLB (3%). Similarly, stocks only perform slightly better than expected. The same shocks are only moderately good news for stocks.

### 4.2.4 Correlation Moments

Table 6 shows the average correlation between inflation and consumption growth, dividend growth as well as real stock returns, and the correlation between bond returns and stock returns. In both the data and the model, inflation is countercyclical away from the ZLB. Higher inflation implies lower consumption growth, lower dividend growth and lower real stock returns. The model matches the magnitudes of the data-implied inflation correlations very well. In the long ZLB spells we see that the cyclicality of inflation changes in both the model and the data; inflation becomes procyclical and is now good news for the real economy. This illustrates that the correlation between economic activity and inflation changes sign endogenously. This is driven by both supply shocks being weakened and demand shocks being amplified at the ZLB compared to normal times.

The fourth row shows that in normal times, Treasury bonds are typically risky as on average they move in the same direction as the stock market. At the ZLB, Treasury bonds typically serve to hedge stock market returns; on average their correlation is negative.
My model is again able to replicate this feature successfully and the correlation between stocks and bonds turns negative in long ZLB episodes. Moreover, the drop in this correlation in the data is comparable to the drop in the model (a drop of 0.6 versus 0.49 percentage points). The intuition is as follows. In ZLB times, the stock market performs well (stock return is high) when there is good news for the overall economy. Good news here means that the economy is growing which comes hand in hand with higher inflation. The economy is slowly growing out of the ZLB and hence future short rates increase, which increases (decreases) the current long term nominal bond yield (price). This results in a negative correlation because exactly when stock returns are high, nominal bonds are performing poorly.

5 Monetary Policy Experiment: Taper Tantrum

Using the quantitative model from the previous section, I now conduct a policy experiment. As described in Section 1, the taper tantrum in the summer of 2013 resulted in an unexpected shock to market participants’ expectations about the future path of short-term interest rates; the expected duration of the ZLB decreased. As shown on Figure 1, the drop in expected duration of the ZLB (rise in FF futures rate) coincided with a sharp increase in the stock-bond return correlation. This section helps to shed light on whether this sudden increase in the correlation can be accounted for by the change in the expected duration of the ZLB or whether other forces are responsible. My model provides the ideal framework for this experiment.

I examine in the model how the stock-bond return correlation changes in response to a sudden decrease in the expected duration of the ZLB. Starting from a baseline where the 2-year forward rate is comparable to what we saw pre-Taper Tantrum (≈ 20 bps), I run a large Monte Carlo simulation of the model. In period 6 of each simulation, I let the

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Model Above ZLB</th>
<th>Model At ZLB ≥ 4Y</th>
<th>Data Pre-ZLB</th>
<th>Data At ZLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation - ΔC</td>
<td>-0.22</td>
<td>0.24</td>
<td>-0.15</td>
<td>0.27</td>
</tr>
<tr>
<td>Inflation - Dividend Growth</td>
<td>-0.05</td>
<td>0.28</td>
<td>-0.05</td>
<td>0.53</td>
</tr>
<tr>
<td>Inflation - Real Stock Return</td>
<td>-0.21</td>
<td>0.08</td>
<td>-0.14</td>
<td>0.56</td>
</tr>
<tr>
<td>Stock - 5Y Bond Return</td>
<td>0.54</td>
<td>-0.06</td>
<td>0.14</td>
<td>-0.35</td>
</tr>
</tbody>
</table>

Table 6: Correlation Moments
model draw random supply and demand shocks. Conditioning on those sample paths that generate the same increase in the 2-year forward rate as we saw in the data (i.e., the forward rate jumps from about 20 basis points to at least 50), Figure 5 illustrates the average outcome. Despite the economy still being at the ZLB, the figure illustrates that a large jump in the 2 year forward rate, or alternatively a decrease in the expected duration of the ZLB, causes the stock-bond return correlation to jump up. Before the large shock the correlation is around -0.25 but soon after the shock the correlation jumps up to about +0.25. We observed similar changes in the magnitude in Figure 1. A drop in the expected duration of the ZLB can indeed rationalize how financial markets, and more specifically the stock-bond return correlation, reacted after the taper tantrum in June 2013. First, it illustrates that investors’ long-term expectations for short-term rates matter and, second, that a change in this expectation (e.g., the prospect of faster interest-rate rises) can have a large impact on today’s asset correlations.

6 Conclusion

This paper analyzes whether the behavior of asset prices, risk premia, and asset correlations is fundamentally different at and near the ZLB. I find that the presence of the ZLB indeed generates a new source of macroeconomic risk: the risk that the ZLB will be binding in the future. At the ZLB, stock market risk increases but bond risk decreases. When the probability of the ZLB binding in the near future increases, investors cut spending to increase savings. This lowers current and future output and dividends. Lower expected
dividends and higher equity risk premia lower current stock prices. Simultaneously, investors expect future short rates and bond risk premia to drop which raise long-term bond prices. These opposite exposures to the same ZLB risk sharply lower the correlation between stock and bond returns. In fact, the stock-bond correlation turns negative. I illustrated the economic mechanism at work first in a stylized three-period New Keynesian economy, before developing and calibrating an infinite horizon model that endogenously generates these observed changes. My model points to the risk of the ZLB binding as a mechanism to generate correlation sign-switches in a fully endogenous way without having to appeal to regime-switches. Finally, I use the quantitative model to conduct an experiment and show that a sudden decrease in the expected duration of the ZLB results in a sharp increase in the stock-bond return correlation. It illustrates that investors’ long-term expectations about the future short-term nominal interest rate path matter.

This work opens up avenues for future research. For example, in Bilal (2017) I find that the risk of a binding ZLB constraint also has important implications for the dynamics of exchange rates. Near the ZLB, the correlation between currency and stock returns increases sharply while the correlation between currency and bond returns decreases. In that paper, I propose an open economy New Keynesian framework with nominal rigidities to explain these findings, alongside the properties of carry trade returns and exchange dynamics near the ZLB. An increase in the probability of the ZLB binding in the domestic country lowers current and future domestic output. It increases the value of domestic goods relative to foreign goods, which depreciates the foreign currency. The domestic carry-trade investor who is long the foreign bond, expects a positive risk premium. Simultaneously, the investor expects future short rates and dividends to drop which raises bond prices and lowers stock prices. Hence, currency and stock returns have the same exposure to the risk of a binding ZLB constraint while currency and bonds have opposite exposures.

Other fruitful questions relate to the effect of the ZLB on the cross-section of asset returns, as well as its implications for wealth inequality.
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A Appendix

A.1 Stock-bond return correlation in Japan

Figure 6: Stock-Bond return correlation and target call rate in Japan
A.2 Stock-bond return correlation in the US

Figure 7: Stock-Bond return correlation and the Federal Funds rate
A.3 Producers in the three period model

A.3.1 Intermediate Firm profit maximization

Assume there is a benevolent government which pays each intermediate firm a production subsidy $\tau_p$ (for simplicity, after the current section I assume that $\tau_p = 0$). Nominal firm profits are given by

$$F_{i,t} = \left(1 + \tau_p\right)p_{i,t} - \frac{w_t}{A_t} \right) X_{i,t}$$

$$= \left(1 + \tau_p\right)p_{i,t} - \frac{w_t}{A_t} \right) \frac{p_{i,t}}{p_t} \right)^{-1} Y_t$$

where $\tau_p$ denotes a production subsidy paid by the benevolent government.

The monopolistic firms face a nominal price rigidity. Ideally, firms would like to optimize their price each period:

$$p^{*}_{i,t} = \arg\max F_{i,t}(p_{i,t})$$

The first order condition is:

$$0 = \frac{(1 + \tau_p)(1 - \epsilon)p_{i,t}^{-\epsilon}Y_t}{p_t^{-\epsilon}} + \epsilon \frac{p_{i,t}^{-\epsilon-1}}{A_t} \frac{w_t}{p_t^{-\epsilon}} Y_t$$

$$= (1 + \tau_p)(1 - \epsilon) + \epsilon \frac{w_t}{A_t} p_{i,t}^{-1}$$

Solving this for the price of good $i$ gives:

$$p_{i,t} = \frac{\epsilon}{\epsilon - 1} \cdot \frac{1}{1 + \tau_p} \frac{w_t}{A_t}$$

A.3.2 Final Good Producer optimal choice

The representative firm produces the final consumption good $Y_t$ in a perfectly competitive market. The firm only uses a continuum of differentiated intermediate goods $X_{i,t}$ as input in a constant elasticity of substitution (CES) production function:

$$Y_t = \left(\int_0^1 X_{i,t}^{\frac{\epsilon - 1}{\epsilon}} \, di\right)^{\frac{\epsilon}{\epsilon - 1}}$$
where $\epsilon$ is the elasticity of substitution between intermediate goods and $X_{i,t}$ is the input of intermediate good $i$ at time $t$. The firm operates in a perfectly competitive environment and maximizes profits subject to the production technology. The profit maximization problem of the firm can alternatively be written as the minimization of expenditure given the production constraint. If we do that, the Lagrangian of the firm becomes:

$$
L = \int_0^1 p_{i,t}X_{i,t}di + p_t \left( Y_t - \left( \int_0^1 \frac{\epsilon-1}{\epsilon} X_{i,t}^{\frac{1}{\epsilon}} di \right)^\frac{\epsilon}{\epsilon-1} \right)
$$

The first order condition $\frac{\partial L}{\partial X_{i,t}} = 0$ gives:

$$
p_{i,t} = p_t \frac{\partial Y_t}{\partial X_{i,t}}
$$

$$
= p_t \frac{\epsilon}{\epsilon-1} \left( \int_0^1 \frac{\epsilon-1}{\epsilon} X_{i,t}^{\frac{1}{\epsilon}} di \right)^\frac{1}{\epsilon-1} \frac{\epsilon-1}{\epsilon} X_{i,t}^{\frac{1}{\epsilon}}
$$

$$
= p_t \left( \frac{Y_t}{X_{i,t}} \right)^{1/\epsilon}
$$

which gives the following input demand:

$$
X_{i,t} = \left( \frac{p_{i,t}}{p_t} \right)^{-\epsilon} Y_t \quad \forall i,
$$

Plugging this demand into the production function allows us to solve for the unknown value of the multiplier $p_t$:

$$
Y_t = \left( \int_0^1 \left( \frac{p_{i,t}}{p_t} - \epsilon \right) Y_t \frac{\epsilon-1}{\epsilon} \right)^\frac{\epsilon}{\epsilon-1}
$$

$$
= Y_t \left( \int_0^1 \left( \frac{p_{i,t}}{p_t} - \epsilon \right)^\frac{\epsilon-1}{\epsilon} di \right)^\frac{\epsilon}{\epsilon-1}
$$

Now $Y_t$ drops from both sides (production function is constant returns to scale) and this allows us to solve for $p_t$:

$$
p_t = \left( \int_0^1 p_{i,t}^{1-\epsilon} di \right)^\frac{1}{\epsilon}\epsilon$
can be interpreted as the minimum cost of producing one unit of the final good $Y_t$ (because of constant returns to scale, this is independent of the quantity produced). That is why the Lagrange multiplier $p_t$ is interpreted as the aggregate price index.

A.4 Consumption in the three period model with flexible prices

A.4.1 Why sticky prices

A lower bound on nominal interest rates does not necessarily affect real allocations. If we want monetary policy actions to have a real effect (and not only a nominal effect), we need nominal rigidities. The presence of sticky prices implies that the equilibrium allocations of real variables cannot be determined independently of monetary policy. In the stylized model, I capture this by introducing an extreme form of price stickiness, an assumption which is relaxed in the quantitative model by introducing Rotemberg pricing. The presence of nominal rigidities will ensure that a change in the nominal short rate is not matched by one-for-one changes in expected inflation. This means that there will be changes in the real interest rate. So nominal price stickiness translates the bound on the nominal rate into a bound on the real interest rate.

Flexible prices derivation From the perspective of the monopolistic producer this can be seen as follows. Assume there is monopolistic competition but flexible prices. Break the overall problem of the producer down into two sub-parts. First look at the cost minimizing problem and then the price setting problem.

Minimize the cost conditional on the output $X_{i,t}$ produced.

$$\min_{L_{i,t}} \frac{w_t}{p_t} L_{i,t} + Z_t (X_{i,t} - A_t L_{i,t})$$

where $Z_t$ is the Lagrange multiplier on the constraint. The FOC with respect to $L_{i,t}$ gives:

$$\frac{w_t}{p_t} = Z_t A_t$$

From this we can derive the labor demand by using $\frac{X_{i,t}}{L_{i,t}} = A_t$. Also note that we can think of $Z_t$ as real marginal cost (this is described in the main text). The inverse of $Z_t$ is then also the markup (this is also described in the main text).

Now the firm will optimize their prices each period, so their markup will always be the optimal markup. Because of symmetry $p_{i,t} = p_t$ and $Z_t = \frac{1}{\mu^\epsilon} = \frac{\epsilon - 1}{\epsilon}$ (where I assumed
that \( \tau_p = 0 \). Combining the previous equation (\( \frac{w_t}{\rho_t} = Z_t A_t \)) with the labor supply optimality condition (\( C_t^\gamma L_t^{\psi - 1} = \frac{w_t}{\rho_t} \) which is derived in the main text) and imposing market clearing (\( Y_t = C_t \)) gives:

\[
Y_t^\gamma L_t^{\psi - 1} = Z_t A_t
\]

and finally noting that \( L_t = \frac{Y_t}{X_t} \) and \( C_t = Y_t \) gives:

\[
C_t = Y_t = \left( A_t^\psi Z_t \right)^{\frac{1}{\gamma + \psi - 1}}
\]

This shows that consumption and output in the model with flexible prices are only a function of technology \( A_t \) and that monetary policy actions do not have a real effect.

### A.5 Consumption in the three period model with uncertainty

This section outlines the proof of Proposition 1.

**Proof.** Period 2 is an endowment period so there is no production: \( C_2 = Y_2 \). From the Euler Equation it follows that

\[
C_1^{-\gamma} = \mathbb{E}_1 \left[ \beta_1 (1 + i_1) Y_2^{-\gamma} \right] \\
C_1 = [\beta_1 (1 + i_1)]^{-\frac{1}{\gamma}} Y_2
\]

In period 0 we get:

\[
C_0^{-\gamma} = \mathbb{E}_0 \left[ \beta_0 (1 + i_0) C_1^{-\gamma} \right] \\
= \beta_0 (1 + i_0) \mathbb{E}_0 [\beta_1 (1 + i_1)] Y_2^{-\gamma}
\]

Recall that the nominal short rate is set by the central bank and is subject to the lower bound constraint: \( i_t = r^n_t + \pi^n \) and \( i_t \geq 0 \). Using this we get:

\[
\mathbb{E}_0 [\beta_1 (1 + i_1)] = \Pr (e^{-\rho_1} \geq 1) \cdot \mathbb{E}_0 [\beta_1 | \beta_1 \geq 1] + \Pr (e^{-\rho_1} < 1) \cdot 1 \\
= \Pr (\phi \rho_0 + \sigma \rho \epsilon_1^\rho \leq 0) \cdot \mathbb{E}_0 [e^{-\rho_1} | e^{-\rho_1} \geq 1] + \Pr (\phi \rho_0 + \sigma \rho \epsilon_1^\rho > 0) \cdot 1 \\
= \Phi (b_0) \cdot \exp \left( -\phi \rho_0 + \frac{\sigma^2 \rho}{2} \right) \frac{\Phi (\sigma - a_0)}{\Phi (-a_0)} + 1 - \Phi (b_0) \\
= \Phi (b_0) \cdot \exp \left( -\phi \rho_0 + \frac{\sigma^2 \rho}{2} \right) \frac{\Phi (\sigma - a_0)}{\Phi (-a_0)} + \Phi (a_0)
\]
where I used Lemma 3 to get the third line and \( a_0 \) and \( b_0 \) are defined as:

\[
a_0 = \frac{\ln(1) + \phi \rho_0}{\sigma_\rho} = \frac{\phi \rho_0}{\sigma_\rho}
\]

\[
b_0 = \frac{-\phi \rho_0}{\sigma_\rho}
\]

where \( \Phi(x) \) denotes the CDF of the standard normal distribution. The expression for \( C_0 \) is then:

\[
C_0 = \left\{ \beta_0 (1 + i_0) \left[ \Phi(b_0) \cdot \exp \left( \frac{-\phi \rho_0 + \sigma_\rho^2}{2} \right) \frac{\Phi(\sigma_\rho - a_0)}{\Phi(-a_0)} + \Phi(a_0) \right] \right\}^{-\frac{1}{7}} Y_2
\]

The time-0 expectation of \( C_1 \) is then:

\[
E_0[C_1] = E_0 \left[ [\beta_1 (1 + i_1)]^{-\frac{1}{7}} Y_2 \right]
\]

\[
= Y_2 E_0 \left[ [\beta_1 (1 + i_1)]^{-\frac{1}{7}} \right]
\]

\[
= Y_2 \left\{ \Pr(e^{-\rho_1} \geq 1) \cdot E_0 \left[ \beta_1^{-\frac{1}{7}} \mid \beta_1 \geq 1 \right] + \Pr(e^{-\rho_1} < 1) \cdot 1 \right\}
\]

\[
= Y_2 \left\{ \Pr(\rho_1 \leq 0) \cdot E_0 \left[ e^{-(-\frac{1}{7})\rho_1} \mid \beta_1 \geq 1 \right] + \Pr(\rho_1 > 0) \cdot 1 \right\}
\]

Using Lemma 3 with \( r = -1/\gamma \) then gives:

\[
= Y_2 \left[ \Phi(b_0) \cdot \exp \left( \frac{\phi \rho_0}{\gamma} + \frac{\sigma_\rho^2}{2\gamma^2} \right) \frac{\Phi\left(\frac{-\sigma_\rho}{\gamma} - a_0\right)}{\Phi(-a_0)} + \Phi(a_0) \right]
\]

which proves Proposition 1.

\[\Box\]

A.6 Stock Value in the three period model with uncertainty

This section outlines the proof of Proposition 1

Proof. The price at time-0 of the time-1 stock. As before, the stock is a levered claim to consumption \( D_1 = C_1^\lambda \). Then the value of the stock is:

\[
V_0^{(1)} = E_0 \left[ \beta_0 \left( \frac{C_1}{C_0} \right)^{-\gamma} D_1 \right]
\]
\[= \frac{\beta_0}{C_0^{-\gamma}} \mathbb{E}_0 \left[ c_{1}^{\lambda-\gamma} \right] \]
\[= \frac{\beta_0}{C_0^{-\gamma}} \mathbb{E}_0 \left[ \left( [\beta_1 (1 + i_1)]^{-\frac{1}{\gamma}} Y_2 \right)^{\lambda-\gamma} \right] \]
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{C_0^{-\gamma}} \mathbb{E}_0 \left[ [\beta_1 (1 + i_1)]^{1 - \frac{1}{\gamma}} \right] \]
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{C_0^{-\gamma}} \left\{ \Pr(e^{-\rho_1} \geq 1) \cdot \mathbb{E}_0 \left[ \beta_1^{1 - \frac{1}{\gamma}} \mid \beta_1 \geq 1 \right] + \Pr(e^{-\rho_1} < 1) \cdot 1 \right\} \]
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{C_0^{-\gamma}} \left\{ \Pr(\rho_1 \leq 0) \cdot \mathbb{E}_0 \left[ e^{-(1 - \frac{1}{\gamma})\rho_1} \mid \beta_1 \geq 1 \right] + \Pr(\rho_1 > 0) \cdot 1 \right\} \]

Now introduce \( \theta = 1 - \lambda/\gamma \) and apply Lemma 3 where \( \theta \) is \( r \):  
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{C_0^{-\gamma}} \left[ \Phi(b_0) \cdot \exp \left( -\theta \phi \rho_0 + \theta^2 \sigma^2_\rho \frac{\sigma^2_\rho}{2} \right) \frac{\Phi(\theta \sigma_\rho - a_0)}{\Phi(-a_0)} + \Phi(a_0) \right] \]

Plugging in the expression for \( C_0 \) yields:  
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{(1 + i_0)} \left[ \Phi(b_0) \cdot \exp \left( -\phi \rho_0 + \sigma^2_\rho \frac{\sigma^2_\rho}{2} \right) \frac{\Phi(\sigma_\rho - a_0)}{\Phi(-a_0)} + \Phi(a_0) \right] Y_2^{-\gamma} \]
\[= \frac{\beta_0 Y_2^{\lambda-\gamma}}{(1 + i_0)} \left[ \Phi(b_0) \cdot \exp \left( -\phi \rho_0 + \sigma^2_\rho \frac{\sigma^2_\rho}{2} \right) \frac{\Phi(\sigma_\rho - a_0)}{\Phi(-a_0)} + \Phi(a_0) \right] \]

and now define the function \( A_0(\rho_0) \) as the ratio of the two expressions in square brackets:  
\[= \frac{Y_2^\lambda A_0(\rho_0)}{(1 + i_0)} \]
which proves Proposition 1  

\[\square\]

A.7 Expected excess stock return in the three period model

This section outlines the proof of Proposition 2
Proof. The log-return on the stock is:

\[ r_1^i = \ln \left( R_1^i \right) = \ln \left( \frac{D_1}{V_0^{(1)}} \right) \]

\[ = \ln \left( \frac{[\beta_1(1 + i_1)]^{-\frac{1}{\gamma}} Y_2^\lambda}{V_0^{(1)}} \right) \]

\[ = \ln \left( \frac{[\beta_1(1 + i_1)]^{-\frac{1}{\gamma}} Y_2^\lambda}{\frac{Y_2^\lambda A_0(\rho_0)}{(1 + i_0)}} \right) \]

\[ = \ln \left( \frac{(1 + i_0) [\beta_1(1 + i_1)]^{-\frac{1}{\gamma}}}{A_0(\rho_0)} \right) \]

First derive some useful results using Lemma 2:

\[ \mathbb{E}_0 [\ln (\beta_1(1 + i_1))] = \Pr(\beta_1 \geq 1) \mathbb{E}_0 [\ln e^{-\rho_1} \mid \beta_1 \geq 1] + \Pr(\beta_1 < 1) \cdot 0 \]

\[ = \Phi \left( \frac{-\phi \rho_0}{\sigma_\rho} \right) \mathbb{E}_0 [-\rho_1 \mid -\rho_1 \geq 0] \]

\[ = \Phi (b_0) \left( -\phi \rho_0 + \sigma_\rho \frac{\phi_N (a_0)}{1 - \Phi (a_0)} \right) \]

\[ \mathbb{E}_0 [\ln (\beta_1(1 + i_1)) \ln (1 + i_1)] = 0 \]

\[ \mathbb{E}_0 [\ln (1 + i_1)] = \Pr(\beta_1 \geq 1) \cdot 0 + \Pr(\beta_1 < 1) \mathbb{E}_0 [\rho_1 \mid \rho_1 \geq 0] \]

\[ = \Phi (a_0) \left( \phi \rho_0 + \sigma_\rho \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right) \]

\[ \mathbb{E}_0 \left[ \left\{ \ln \left( \beta_1(1 + i_1) \right) \right\}^2 \right] = \Pr(\beta_1 \geq 1) \mathbb{E}_0 [\rho_1^2 \mid \rho_1 \leq 0] + \Pr(\beta_1 < 1) \cdot 0 \]

\[ = \Phi (b_0) \left[ (\phi \rho_0)^2 - 2\phi \rho_0 \sigma_\rho \frac{\phi_N (b_0)}{\Phi (b_0)} + \sigma_\rho^2 \left( 1 - b_0 \frac{\phi_N (b_0)}{\Phi (b_0)} \right) \right] \]

The expected log-return is:

\[ \mathbb{E}_0 [r_1^i] = \ln (1 + i_0) - \ln (A_0(\rho_0)) + \mathbb{E}_0 \left[ \ln \left( [\beta_1(1 + i_1)]^{-\frac{1}{\gamma}} \right) \right] \]

\[ = \ln (1 + i_0) - \ln (A_0(\rho_0)) - \frac{\lambda}{\gamma} \mathbb{E}_0 [\ln ([\beta_1(1 + i_1)])] \]

\[ = \ln (1 + i_0) - \ln (A_0(\rho_0)) - \frac{\lambda}{\gamma} \Phi (b_0) \left( -\phi \rho_0 + \sigma_\rho \frac{\phi_N (a_0)}{1 - \Phi (a_0)} \right) \]

where the last line uses one of the useful results from above. Here \( \phi_N (\cdot) \) denotes the standard normal probability density function. The variance of stock returns follows by
applying the law of total variance:

\[
\begin{align*}
V_0 [\ln R_i] &= V_0 \left[ \ln (1 + i_0) - \ln (A_0(\rho_0)) + \ln \left( \frac{[\beta_1(1 + i_1)]^{-\frac{1}{\gamma}}}{1 - \frac{1}{\gamma}} \right) \right] \\
&= \frac{\lambda^2}{\gamma^2} V_0 [\ln (\beta_1(1 + i_1))] \\
&= \frac{\lambda^2}{\gamma^2} \left\{ \Pr(\beta_1 \geq 1) \ V_0 \left[ \ln e^{\rho_1} | \beta_1 \geq 1 \right] + \Pr(\beta_1 < 1) \cdot 0 \\
&\quad + \Pr(\beta_1 \geq 1) \left( E_0 \left[ \ln e^{\rho_1} | \beta_1 \geq 1 \right] \right)^2 + \Pr(\beta_1 < 1) \cdot 0^2 \\
&\quad - \left( E_0 \left[ \ln (\beta_1(1 + i_1)) \right] \right)^2 \right\} \\
&= \frac{\lambda^2}{\gamma^2} \Phi \left( -\frac{\rho_0}{\sigma_\rho} \right) \left\{ V_0 [\rho_1 | \rho_1 \leq 0] + \left( -\rho_0 + \sigma_\rho \frac{\Phi \left( \frac{\rho_0}{\sigma_\rho} \right)}{1 - \Phi \left( \frac{\rho_0}{\sigma_\rho} \right)} \right)^2 \\
&\quad - \Phi \left( -\frac{\rho_0}{\sigma_\rho} \right) \left( -\rho_0 + \sigma_\rho \frac{\Phi \left( \frac{\rho_0}{\sigma_\rho} \right)}{1 - \Phi \left( \frac{\rho_0}{\sigma_\rho} \right)} \right)^2 \right\}
\end{align*}
\]

using Lemma 2 this yields:

\[
\begin{align*}
&= \frac{\lambda^2}{\gamma^2} \Phi \left( b_0 \right) \left\{ \sigma_\rho^2 \left( 1 - \frac{b_0 \Phi \left( b_0 \right)}{\Phi \left( b_0 \right)} - \left[ \frac{\Phi \left( b_0 \right)}{\Phi \left( b_0 \right)} \right]^2 \right) \\
&\quad + \left( -\rho_0 + \sigma_\rho \frac{\Phi \left( a_0 \right)}{1 - \Phi \left( a_0 \right)} \right)^2 - \Phi \left( b_0 \right) \left( -\rho_0 + \sigma_\rho \frac{\Phi \left( a_0 \right)}{1 - \Phi \left( a_0 \right)} \right)^2 \right\}
\end{align*}
\]

The Equity Risk Premium ERP is:

\[
\begin{align*}
ERP &= -\operatorname{Cov}_0 \left[ m_1, r_i \right] \\
&= -\operatorname{Cov}_0 \left[ \ln \beta_0 + \ln C_{1}^{-\gamma} - \ln C_{0}^{-\gamma}, \ln \left( \frac{[\beta_1(1 + i_1)]^{-\frac{1}{\gamma}}}{V_0^{(1)}} \right) \right] \\
&= \frac{\lambda}{\gamma} \operatorname{Cov}_0 \left[ \ln C_{1}^{-\gamma}, \ln (\beta_1(1 + i_1)) \right] \\
&= \frac{\lambda}{\gamma} \operatorname{Cov}_0 \left[ \ln (\beta_1(1 + i_1)), \ln (\beta_1(1 + i_1)) \right] \\
&= \frac{\lambda}{\gamma} \left\{ \Phi \left( b_0 \right) \left[ (\rho_0)^2 - 2\rho_0 \sigma_\rho \frac{\Phi \left( b_0 \right)}{\Phi \left( b_0 \right)} + \sigma_\rho^2 \left( 1 - \frac{b_0 \Phi \left( b_0 \right)}{\Phi \left( b_0 \right)} \right) \right] \right\}
\end{align*}
\]
\[- \Phi (b_0) \left( \phi \rho_0 - \sigma_\rho \frac{\phi_N(b_0)}{\Phi(b_0)} \right) \] 

where on the last line I used the definition of \( \text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}X \mathbb{E}Y \) and some of the useful results that were derived above. This proves Proposition 2.

\[ \square \]

### A.8 Bond price in the three period model

This section outlines the proof of Proposition 3

**Proof.** The price at time-0 of a 2-period nominal bond which pays at time-2. Note that because there is no inflation, a nominal bond is essentially the same as a real bond (which is why I can directly use the real SDF). Derive the price recursively:

\[
P_1^{(1),s} = \mathbb{E}_1 \left[ \beta_1 \left( \frac{C_2}{C_1} \right)^{-\gamma} \cdot 1 \right] \\
= \frac{1}{1 + i_1}
\]

\[
P_0^{(2),s} = \mathbb{E}_0 \left[ \beta_0 \left( \frac{C_1}{C_0} \right)^{-\gamma} \cdot \frac{1}{1 + i_1} \right] \\
= \frac{\beta_0}{C_0^{-\gamma}} \cdot \mathbb{E}_0 \left[ \frac{\beta_1 (1 + i_1) Y_2^{-\gamma}}{1 + i_1} \right] \\
= \frac{\beta_0 Y_2^{-\gamma}}{C_0^{-\gamma}} \cdot \mathbb{E}_0 [\beta_1] \\
= \frac{\beta_0 Y_2^{-\gamma}}{C_0^{-\gamma}} \cdot \exp \left( -\phi \rho_0 + \frac{\sigma_\rho^2}{2} \right) \\
= \exp \left( -\phi \rho_0 + \frac{\sigma_\rho^2}{2} \right) \Phi(b_0) \cdot \exp \left( -\phi \rho_0 + \frac{\sigma_\rho^2}{2} \right) \frac{\Phi(\sigma_\rho - a_0)}{\Phi(-a_0)} + \Phi(a_0)
\]

The 2-period yield (for example, think of this as the 2-month yield). So essentially there is a "yield curve" since we have 1-period and 2-period yields.

\[
Y_0^{(2),s} = \left( \frac{1}{P_0^{(2),s}} \right)^{\frac{1}{2}} - 1
\]
which proves Proposition 3

A.9 Expected excess bond return in the three period model

This section outlines the proof of Proposition 4

Proof.

• The expected return is:

\[
\mathbb{E}_0 \left[ r^{(2),S}_1 \right] = - \log \left( P^{(2),S}_0 \right) - \mathbb{E}_0 \left[ \log \left( 1 + i_1 \right) \right]
\]

\[
= - \log \left( P^{(2),S}_0 \right) - \Phi (a_0) \left( \phi \rho_0 + \sigma_\rho \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right)
\]

where the last line uses one of the useful results from above.

• The variance of the return follows again by applying the law of total variance:

\[
\mathbb{V}_0 \left[ r^{(2),S}_1 \right] = \mathbb{V}_0 \left[ \log \left( P^{(1),S}_1 \right) \right] = \mathbb{V}_0 \left[ \log \left( \frac{1}{1 + i_1} \right) \right] = \mathbb{V}_0 \left[ - \log \left( 1 + i_1 \right) \right]
\]

\[
= \mathbb{V}_0 \left[ \log \left( 1 + i_1 \right) \right]
\]

\[
= \Pr (\beta_1 \leq 1) \mathbb{V}_0 \left[ \rho_1 \mid \rho_1 \geq 0 \right] + 0
\]

\[
+ \Pr (\beta_1 \leq 1) \left( \mathbb{E}_0 \left[ \rho_1 \mid \rho_1 \geq 0 \right] \right)^2 + \Pr (\beta_1 \geq 1) \left( \mathbb{E}_0 \left[ 0 \mid \rho_1 \leq 0 \right] \right)^2
\]

\[
- \left( \mathbb{E}_0 \left[ \log \left( 1 + i_1 \right) \right] \right)^2
\]

\[
= \Phi (a_0) \left\{ \sigma_\rho^2 \left[ \frac{b_0 \phi_N (b_0)}{1 - \Phi (b_0)} - \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right]^2 \right.
\]

\[
+ \left( \phi \rho_0 + \sigma_\rho \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right)^2 - \Phi (a_0) \left( \phi \rho_0 + \sigma_\rho \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right)^2 \right\}
\]

where I used some of the useful results from above and Lemma 2.

• The Bond Risk Premium is defined as \( BRP^S_0 = - \text{Cov}_0 \left( m^{S}_1, r^{(2),S}_1 \right) \) and is given by:

\[
BRP^S_0 = - \text{Cov}_0 \left[ m^{S}_1, r^{(2),S}_1 \right]
\]

\[
= - \text{Cov} \left[ \ln \beta_0 + \ln C_1^{-\gamma} - \ln C_0^{-\gamma}, - \ln P^{(2),S}_0 - \ln (1 + i_1) \right]
\]

\[
= \text{Cov} \left[ \ln C_1^{-\gamma}, \ln (1 + i_1) \right]
\]

\[
= \text{Cov} \left[ \ln (\beta_1 (1 + i_1)), \ln (1 + i_1) \right]
\]

\[
= \Phi (b_0) \left( \phi \rho_0 - \sigma_\rho \frac{\phi_N (a_0)}{1 - \Phi (a_0)} \right) \cdot \Phi (a_0) \left( \phi \rho_0 + \sigma_\rho \frac{\phi_N (b_0)}{1 - \Phi (b_0)} \right)
\]
where on the last line I used the definition of $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}X \mathbb{E}Y$ and some of the useful results that were derived above. This proves Proposition 4.

\[ \square \]

A.10 Correlation between dividend strip returns and bond returns in the three period model

This section outlines the proof of Proposition 5

Proof. The covariance between dividend strip returns and bond returns is given by:

\[
\text{Cov}_0 \left[ r^{(2),s}_1, r^{(2),s}_1 \right] = \text{Cov}_0 \left[ \ln \left( \frac{\beta_1 (1 + i_1)}{V_0^{(1)}} \right), \ln \frac{1}{1 + i_1} \right]
\]

\[
= \frac{\lambda}{\gamma} \text{Cov}_0 \left[ \ln (\beta_1 (1 + i_1)), \ln (1 + i_1) \right]
\]

\[
= \frac{\lambda}{\gamma} \Phi (b_0) \left( \phi \rho_0 - \sigma \frac{\phi N (a_0)}{1 - \Phi (a_0)} \right) \cdot \Phi (a_0) \left( \phi \rho_0 + \sigma \frac{\phi N (b_0)}{1 - \Phi (b_0)} \right)
\]

where on the last line I used the definition of $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}X \mathbb{E}Y$ and some of the useful results that were derived above. The correlation between dividend strip returns and bond returns then immediately follows as:

\[
\text{Corr}_0 \left[ r^{(2),s}_1, r^{(2),s}_1 \right] = \frac{\text{Cov}_0 \left[ r^{(2),s}_1, r^{(2),s}_1 \right]}{\sqrt{V_0 \left[ r^{(2),s}_1 \right] V_0 \left[ r^{(2),s}_1 \right]}}
\]

This proves Proposition 5.

\[ \square \]

A.11 Auxiliary Lemma

Lemma 2. (Properties truncated normal distribution)

Suppose $X$ has a normal distribution with mean $\mu$ and standard deviation $\sigma$, $a$ and $b$ are constants and denote by $\Phi (.)$ the standard normal CDF function. Define:

\[
a_0 = \frac{a - \mu}{\sigma}
\]

\[
b_0 = \frac{b - \mu}{\sigma}
\]
Then the mean and variance of the truncated normal variable are

\[
\begin{align*}
E[X | a < X] &= \mu + \sigma \frac{\phi(a_0)}{1 - \Phi(a_0)} \\
V[X | a < X] &= \sigma^2 \left\{ 1 + \frac{a_0\phi(a_0)}{1 - \Phi(a_0)} - \left[ \frac{\phi(a_0)}{1 - \Phi(a_0)} \right]^2 \right\} \\
E[X | X \leq b] &= \mu - \sigma \frac{\phi(b_0)}{\Phi(b_0)} \\
V[X | X \leq b] &= \sigma^2 \left\{ 1 - \frac{b_0\phi(b_0)}{\Phi(b_0)} - \left[ \frac{\phi(b_0)}{\Phi(b_0)} \right]^2 \right\} \\
E[X^2 | X \leq b] &= \mu^2 - 2\mu \sigma \frac{\phi(b_0)}{\Phi(b_0)} + \sigma^2 \left( 1 - \frac{b_0\phi(b_0)}{\Phi(b_0)} \right)
\end{align*}
\]

More results can be found in e.g., Greene (2011).

**Lemma 3.** (Moments of a truncated lognormal distribution)

If \( x \sim \mathcal{N}(\mu, \sigma^2) \) and \( y = \exp(x) \) then :

\[
\begin{align*}
E[y^r] &= \exp \left( r\mu + r^2\sigma^2 / 2 \right) \\
E[y^r | y > a] &= E[y^r] \frac{\Phi(r\sigma - a_0)}{\Phi(-a_0)} \\
E[y^r | y \leq b] &= E[y^r] \frac{\Phi(-r\sigma + b_0)}{\Phi(b_0)}
\end{align*}
\]

where

\[
\begin{align*}
a_0 &= \frac{\ln a - \mu}{\sigma} \\
b_0 &= \frac{\ln b - \mu}{\sigma}
\end{align*}
\]

Also note that the denominators are respectively:

\[
\begin{align*}
\Pr(y > a) &= \Phi(-a_0) \\
\Pr(y \leq b) &= \Phi(b_0)
\end{align*}
\]
B Appendix Quantitative Model

B.1 Household

- Period-by-period budget constraint for HH in nominal terms. The household supplies \( L_t \) units of labor for which they receive a nominal wage \( W_t \). They can consume an amount of \( C_t \) and invest in financial assets which are, a share \( \theta_t \) of an equity index \( S_t \) which pays out real dividends \( D_t \), nominal and real zero-coupon bonds of maturities ranging from 1 to \( N \)

\[
P_t C_t + \theta_t S_t = W_t L_t + \theta_{t-1} S_t + D_t + \sum_{n=1}^{N} B_{t-1}^{S,n} P_{t-1}^{S,n-1} + P_t \sum_{n=1}^{N} B_t^{n} P_t^{n-1} - P_t \text{ is price level}
\]

- \( P_t \) is price level
- \( C_t \) is consumption at time \( t \)
- \( \theta_t S_t \) is holdings of a share in an equity index
- \( S_t \) is price of a share in an equity index, which is a claim on (at least the portion that is paid out) all firms’ profits
- \( B_t^{S,n} \) is holdings (zero net supply) at time \( t \) of an \( n \)-period nominal bond
- \( B_t^{n} \) is holdings (zero net supply) at time \( t \) of an \( n \)-period real bond
- \( W_t \) is nominal wage
- \( L_t \) is labor
- \( D_t \) is real profits (dividends) from firms
- Note that all bonds are in zero net supply: \( B_t^{S,n} = B_t^{n} = 0 \) \( \forall t, \forall n \)
- Household owns the intermediate firms, equity shares sum to one: \( \theta_t S_t = 1 \) \( \forall t \).

- Now denote \( I_t \) is gross nominal one-period interest rate. This is \( I_t = \frac{P_t^{S,(0)}}{P_t^{S,(1)}} \)

- The following derivation is similar to the derivation in Rudebusch and Swanson (2012). For more details I refer to Rudebusch and Swanson (2012)

- The household’s optimization problem is solved as a Lagrange optimization problem with the states of nature explicitly specified.
• The naming convention is the following (a similar naming convention is followed in Ljungqvist and Sargent (2012)):
  
  - \( s^0 \in S_0 \) is the initial state of the economy at time 0
  
  - \( s_t \in S \) are the realizations of the shocks that hit the economy in period \( t \)
  
  - \( s^t = \{ s^{t-1}, s^t \} \in S_0 \times S^t \) denotes the history of all shocks up to and including time \( t \). So \( s^t \) is the state of the world at time \( t \).
  
  - \( s^t_{t-1} \) is the projection of the history \( s^t \) onto its first \( t \) components. In other words, \( s^t_{t-1} \) is the history \( s^t \) as it would have been viewed at time \( t-1 \) before time-\( t \) shocks have been realized
  
  - If \( s^t \) is an arbitrary element of \( S_0 \times S^t \) and \( s^{t-1} \) an arbitrary element of \( S_0 \times S^{t-1} \), then denote \( s^t \supseteq s^{t-1} \) if \( s^t_{t-1} = s^{t-1} \)
  
  • The Lagrangian is (where \( \mu_t \) is the Lagrange Multiplier on the value function and \( \lambda_t \) is the LM on the budget constraint):

  \[
  \mathcal{L} = V_{s^0} - \sum_{t=0}^{\infty} \sum_{s^t} \mu_{s^t} \left\{ V_{s^t} - (1 - \bar{\beta}) u(C_{s^t}, L_{s^t}) + \beta_s \left( \sum_{s^{t+1}} \pi_{s^{t+1}|s^t} (-V_{s^{t+1}})^{1-\alpha} \right) \right\}^{1/(1-\alpha)} \\
  - \sum_{t=0}^{\infty} \sum_{s^t} \lambda_{s^t} \left\{ P_{s^t} C_{s^t} + \theta_{s^t} S_{s^t} + \sum_{n=1}^{N} B_{s^t}^{(n)} P_{s^t} S_{s^t}^{(n)} + P_{s^t} \sum_{n=1}^{N} B_{s^t}^{(n)} P_{s^t}^{(n)} \\
  - W_{s^t} L_{s^t} - \theta_{s^t}^{S_{s^t-1}} P_{s^t} (S_{s^t} + D_{s^t}) - \sum_{n=1}^{N} B_{s^t}^{(n)} P_{s^t}^{(n-1)} - P_{s^t} \sum_{n=1}^{N} B_{s^t}^{(n)} P_{s^t}^{(n-1)} \right\}
  \]

  • The FOC’s are:

  \[
  [C_{s^t}] = 0 = (1 - \bar{\beta}) \mu_{s^t} u'(C_{s^t}, L_{s^t}) - \lambda_{s^t} P_{s^t} \\
  [L_{s^t}] = 0 = (1 - \bar{\beta}) \mu_{s^t} u'(C_{s^t}, L_{s^t}) + \lambda_{s^t} W_{s^t} \\
  [V_{s^t}] = 0 = -\mu_{s^t} - \pi_{s^t|s^t-1} \bar{\beta}_{s^t-1} \mu_{s^t-1} \left\{ \sum_{s^{t+1} \supseteq s^t} \pi_{s^{t+1}|s^t} (-V_{s^{t+1}})^{1-\alpha} \right\}^{1/(1-\alpha)} (-1)(-V_{s^t})^{-\alpha} \\
  [\theta_{s^t}] = 0 = -\lambda_{s^t} P_{s^t} \bar{S}_{s^t} + \sum_{s^{t+1} \supseteq s^t} \lambda_{s^t+1}^{\bar{S}_{s^t+1}} (\bar{S}_{s^t+1} + D_{s^t+1}) \\
  [B_{s^t}^{S_{s^t}}} = 0 = -\lambda_{s^t} P_{s^t} S_{s^t}^{(n)} + \sum_{s^{t+1} \supseteq s^t} \lambda_{s^t+1} P_{s^t}^{(n)} S_{s^t+1}^{(n)} \\
  [B_{s^t}^{P_{s^t}S_{s^t}^{(n)}}] = 0 = -\lambda_{s^t} P_{s^t} P_{s^t}^{(n)} + \sum_{s^{t+1} \supseteq s^t} \lambda_{s^t+1} P_{s^t}^{(n-1)} P_{s^t+1}^{(n-1)}
  \]
• Define the discounted Lagrange multipliers $\tilde{\lambda}_{st} \equiv (\Pi_{i=0}^t \beta s^{1-t})^{-1} (\pi_{st} | s^0)^{-1} \lambda_{st}$ and $\tilde{\mu}_{st} \equiv (\Pi_{i=0}^t \beta s^{1-t})^{-1} (\pi_{st} | s^0)^{-1} \mu_{st}$. The FOCs become:

\[
\begin{align*}
[C_{st}] & \quad 0 = (1 - \tilde{\beta}) \tilde{\mu}_{st} u_C'(C_{st}, L_{st}) - \tilde{\lambda}_{st} P_{st} \\
[L_{st}] & \quad 0 = (1 - \tilde{\beta}) \tilde{\mu}_{st} u_L'(C_{st}, L_{st}) + \tilde{\lambda}_{st} W_{st} \\
[V_{st}] & \quad 0 = -\tilde{\lambda}_{st} P_{st} S_{st} + \beta_s^{(st+1)} \Pi_{st+1} (S_{st+1} + D_{st+1}) \\
[\theta_{st}] & \quad 0 = -\tilde{\lambda}_{st} P_{st} S_{st} + \beta_s^{(st+1)} \Pi_{st+1} (S_{st+1} + D_{st+1}) \\
[B_{st}^{(n)}] & \quad 0 = -\tilde{\lambda}_{st} P_{st} S_{st} + \beta_s^{(st+1)} \Pi_{st+1} (S_{st+1} + D_{st+1}) \\
[B_{st}^{(n)}] & \quad 0 = -\tilde{\lambda}_{st} P_{st} S_{st} + \beta_s^{(st+1)} \Pi_{st+1} (S_{st+1} + D_{st+1}) \\
\end{align*}
\]

• Now the first FOC for $[C_{st}]$ gives the first two lines below and the third FOC $[V_{st}]$ gives the third line:

\[
\begin{align*}
\tilde{\lambda}_{st} & = \frac{(1 - \tilde{\beta}) \tilde{\mu}_{st} u_C'(C_{st}, L_{st})}{P_{st}} \\
\tilde{\lambda}_{st+1} & = \frac{(1 - \tilde{\beta}) \tilde{\mu}_{st+1} u_C'(C_{st+1}, L_{st+1})}{P_{st+1}} \\
\tilde{\mu}_{st+1} & = \frac{(E_{st} \left[(-V_{st+1})^{1-\alpha}\right])^{\alpha/(1-\alpha)}}{(-V_{st+1})^\alpha} \\
\end{align*}
\]

• Using the nominal bond FOC $[B_{t}^{(n)}]$, plug in $n = 1$, and using the definition of $I_t = \frac{p_{st+1}(0)}{p_{st+1}(1)}$ then gives the Euler Equation (where I simplified the notation):

\[
1 = E_t \left[ \beta_t \frac{\mu_{t+1} u_C'(C_{t+1}, L_{t+1})}{\mu_t u_C'(C_t, L_t)} I_t P_t \right] \\
\]

• From this Euler equation we also immediately get the real SDF as:

\[
M_{t+1} = \beta_t \left( \frac{(-V_{t+1})}{(E_t \left[(-V_{t+1})^{1-\alpha}\right])^{1/(1-\alpha)}} \right)^{-\alpha} \frac{u_C'(C_{t+1}, L_{t+1})}{u_C'(C_t, L_t)} \\
\]

• The nominal SDF is $M_{t+1} = \frac{M_{t+1}}{P_{t+1}}$ where inflation is $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$

• Finally, the intratemporal condition follows from the FOC for labor. Rewrite the
FOC for consumption in function of the Lagrange Multiplier $\lambda$ (as I did above) and plug this into the labor FOC:

$$0 = u'_t(C_t, L_t) + u'_c(C_t, L_t) \frac{W_t}{P_t}$$

which then gives $\frac{W_t}{P_t} = \tilde{W}_t = Z_t^{1-\gamma}C_t^\gamma \chi_0 (1 - L_t)^{-\psi}$

### B.2 Phillips Curve

Each period firms set their price to maximize profits:

$$\max_{P_{i,t}} \mathbb{E}_t \sum_{k=0}^{\infty} M_{t,t+k} \left\{ \frac{P_{i,t+k}}{P_{t+k}} X_{i,t+k} - MC_{t+k} X_{i,t+k} - G (P_{i,t+k}, P_{t+k-1}) - MC_{t+k} \Phi \right\}$$

subject to the demand curve and the production function, where $MC_{t+k}$ is the marginal cost which is given by: $MC_{t+k} = \frac{W_{t+k}}{Z_{t+k} A_{t+k} L_{t+k}}$ and $M_{t,t+k}$ is the real pricing kernel between time $t$ and time $t+k$. All firms face an identical problem and thus choose the same price producing the same quantity. Take the FOC wrt $[P_{i,t}]$ after plugging in for the demand curve $X_{i,t+k} = \left( \frac{P_{i,t+k}}{P_{t+k}} \right)^{-\epsilon} Y_{t+k}$ gives (we only need to look at $k = 0$ and $k = 1$ in the sum):

$$0 = (1 - \epsilon) \frac{Y_t}{P_t^{1-\epsilon} P_{i,t}^{-\epsilon}} + \frac{W_t}{A_t Z_t} \frac{Y_t}{P_t^{1-\epsilon}} e P_{i,t}^{-1-\epsilon} - \phi R Y_t \left( \frac{P_{i,t}}{\Pi_{ss} P_{i,t-1}} - 1 \right) \frac{1}{\Pi_{ss} P_{i,t-1}}$$

$$+ \mathbb{E}_t \left[ M_{t,t+1} \phi R Y_{t+1} \left( \frac{P_{i,t+1}}{\Pi_{ss} P_{t,t}} - 1 \right) \frac{P_{i,t+1}}{\Pi_{ss} (P_{t,t})^2} \right]$$

Now every firm faces the exact same problem so every firm sets the same price. This together with the definition of the price level indicates that $P_{t,i} = P_t$. Using this insight allows us to simplify the expression to:

$$0 = (1 - \epsilon) \frac{Y_t}{P_t^{1-\epsilon} P_{i,t}^{-\epsilon}} + \frac{W_t}{A_t Z_t} \frac{Y_t}{P_t^{1-\epsilon}} e P_{i,t}^{-1-\epsilon} - \phi R Y_t \left( \frac{P_t}{\Pi_{ss} P_{t-1}} - 1 \right) \frac{1}{\Pi_{ss} P_{t-1}}$$

$$+ \mathbb{E}_t \left[ M_{t,t+1} \phi R Y_{t+1} \left( \frac{P_{t+1}}{\Pi_{ss} P_t} - 1 \right) \frac{P_{t+1}}{\Pi_{ss} (P_t)^2} \right]$$

Now multiply both sides by $P_t$ and note the definitions of inflation $\Pi_t = \frac{P_t}{P_{t-1}}$ and that $MC_t$ is denoted in real terms. This implies the following forward-looking non-linear Phillips
curve:
\[ (1 - \epsilon) + \epsilon MC_t - \phi R \left( \frac{\Pi_t}{\Pi_{ss}} - 1 \right) \frac{\Pi_t}{\Pi_{ss}} Y_t + \phi R E_t \left[ M_{t,t+1} \left( \frac{\Pi_{t+1}}{\Pi_{ss}} - 1 \right) \frac{\Pi_{t+1}}{\Pi_{ss}} Y_{t+1} \right] = 0 \]

\section{B.3 Detrended Stationary Model}

The permanent part of technology \( Z_t \) is non-stationary. In the model, consumption, output and \textit{real} wages fluctuate around a stochastic balanced growth path. Note that the nominal wage is not necessarily stationary but the real wage is. Therefore, I define normalized stationary variables as lowercase letters:

\[
\begin{align*}
  y_t &= \frac{Y_t}{Z_t} \\
  c_t &= \frac{C_t}{Z_t} \\
  w_t &= \frac{W_t}{Z_t P_t} \\
  v_t &= \frac{V_t}{Z_t^{1-\gamma}}
\end{align*}
\]

The detrended equations are:

- Per-period utility function:
  \[
  u(c_t, L_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \chi_0 rac{(1 - L_t)^{1-\psi}}{1-\psi}
  \]

- Value function
  \[
  v_t = (1 - \beta) u(c_t, L_t) - \beta_t \left( E_t \left[ (-v_{t+1})^{1-\alpha} \exp(1-\alpha)(1-\gamma)\Delta \log Z_{t+1} \right] \right)^{1/(1-\alpha)}
  \]

- Real Stochastic Discount Factor
  \[
  M_{t+1} = \beta_t \left( \frac{(-v_{t+1}) \exp(1-\gamma)\Delta \log Z_{t+1}}{E_t \left[ (-v_{t+1})^{1-\alpha} \exp(1-\alpha)(1-\gamma)\Delta \log Z_{t+1} \right]^{1/(1-\alpha)}} \right)^{-\alpha} \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} e^{-\gamma \Delta \log Z_{t+1}}
  \]

- Euler Equation
  \[
  1 = E_t \left[ I_t M_{t+1}^S \right]
  \]
• Inflation

\[ \Pi_{t+1} = \frac{P_{t+1}}{P_t} \]

• Wage equations (in real terms)

\[ w_t = c_t^\gamma \chi_0 (1 - L_t)^\psi \]

• Firm production function:

\[ y_{i,t} = A_t L_{i,t} \]

• Aggregate resource constraint

\[ c_t = \left( 1 - \frac{\phi_R}{2} \left( \frac{\Pi_t}{\Pi_{ss}} - 1 \right)^2 \right) y_t \]

• The non-linear Phillips curve:

\[
\left[ (1 - \epsilon) + \epsilon w_t \frac{A_t}{A_t} - \phi_R \left( \frac{\Pi_t}{\Pi_{ss}} - 1 \right) \frac{\Pi_t}{\Pi_{ss}} \right] y_t \\
+ \phi_R \mathbb{E}_t \left[ e^{\Delta \log Z_{t+1} M_{t+1}} \left( \frac{\Pi_{t+1}}{\Pi_{ss}} - 1 \right) \frac{\Pi_{t+1}}{\Pi_{ss}} y_{t+1} \right] = 0
\]

• Monetary policy rule (which is already detrended by definition):

\[ I_t = \max(I_t^*, 1) \]

\[ I_t^* = I^{1-\rho_I} I_{t-1}^{\rho_I} \left[ \left( \frac{\Pi_t}{\Pi_{ss}} \right)^{\phi_I} \left( \frac{y_t}{y} \right)^{\phi_I} \right]^{1-\rho_I} \exp(v_{M,t}) \]

where \( v_{M,t} \) is a mean-0 Gaussian shock with standard deviation \( \sigma_{v_M} \) and where \( I = \frac{\Pi e^{\gamma A}}{\beta} \) is the steady state nominal rate (which e.g. follows from the Euler equation in steady state)

• Bond prices are defined as before because rates and inflation are already stationary. Recursively:

\[ p_t^{(n),S} = \mathbb{E}_t \left[ M_{t+1}^{S} \cdot P_{t+1}^{(n-1),S} \right] \]
where \( M_{t+1}^S = \frac{M_{t+1}}{n_{t+1}} \) is the nominal SDF and \( P_t^{(0),S} = 1 \).

- Dividends paid out by the firms are detrended. Note that only real dividends are stationary: 
  \[
  d_t = \frac{f_t}{z_t} - v_{D,t} = \frac{p_{C,t} - w_{i,L_t}}{z_t p_t} - v_{D,t} = c_t - w_t L_t - v_{D,t}
  \]
- The detrended stock price \( s_t \) then satisfies the recursion:
  \[
  s_t = \mathbb{E}_t \left[ e^{\Delta \log z_{t+1}} M_{t+1} (s_{t+1} + d_{t+1}) \right]
  \]