Preliminaries

Chapters 1-4

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Bayes’ Law

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{if } P(B) \neq 0 \]

\[ f(A|B) = \frac{f(B|A)f(A)}{f(B)} \quad \text{if } f(B) \neq 0. \]

Definition of a conditional probability:

\[ P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{1} \]

Likewise, \( P(B|A) = P(A \cap B)/P(A) \). Rearranging both of these expressions yields

\[ P(A|B)P(B) = P(A \cap B) = P(B|A)P(A). \quad \tag{2} \]

Dividing through by \( P(B) \) delivers Bayes’ law.
Bayes’ Law with Normal Random Variables

Prior: \( x \sim N(A, \alpha^{-1}) \).

Signal: \( B = x + e \), where \( e \sim N(0, \beta^{-1}) \).

Posterior belief:
\[
E[x|B] = \frac{\alpha A + \beta B}{\alpha + \beta} \tag{3}
\]

Posterior variance:
\[
V[x|B] = (\alpha + \beta)^{-1} \tag{4}
\]
Bayes’ Law with Multivariate Normal

Prior: $x \sim N(A, \alpha^{-1})$ is an $N \times 1$ vector.

Signal: $B = x + e$, where $e \sim N(0, \beta^{-1})$ is an $N \times 1$ vector.

Posterior belief:

$$E[x|B] = (\alpha + \beta)^{-1}(\alpha A + \beta B)$$  \hspace{1cm} (5)

Posterior variance:

$$V[x|B] = (\alpha + \beta)^{-1}$$  \hspace{1cm} (6)

where $V[x|B]$, $\alpha^{-1}$ and $\beta^{-1}$ are $N \times N$ variance-covariance matrices.
Kalman Filter

State equation:

\[ x_{t+1} = Dx_t + Fe_{t+1}. \] (7)

In each period \( t \), there is a signal \( y_t \). Observation equation:

\[ y_t = Gx_t + H\eta_t. \] (8)

e\(_{t}\) and \( \eta_t \) are mutually independent, i.i.d., standard normal.

Posterior beliefs:

\[ \hat{x}_{t+1} = (D - K_tG)\hat{x}_t + K_t y_t \] (9)

\[ \Sigma_{t+1} = D\Sigma_tD' + FF' - D\Sigma_tG'(G\Sigma_tG' + HH')^{-1}G\Sigma_tD. \] (10)

Kalman gain:

\[ K_t = D\Sigma_tG'(G\Sigma_tG' + HH')^{-1} \] (11)
Entropy

How much information is required, on average, to describe $x$ with probability density function $p(\cdot)$?

$$H(x) = -E[\ln(p(x))]$$

$$= - \sum_x [p(x) \ln(p(x))] \quad \text{if } p \text{ discrete}$$

For a multivariate continuous distribution $f$,

$$H(x) = - \int f(x_1, \ldots, x_n) \ln(f(x_1, \ldots, x_n)) dx_1, \ldots, dx_n$$
Mutual Information

Conditional entropy: How much information is required to describe $x$ if $y$ is already known?

$$H(x|y) = H(x, y) - H(y)$$

Mutual Information: How much does knowing one reduce the entropy of the other?

$$I(x, y) = H(x) - H(x|y)$$  \hspace{1cm} (12)
Mutual Information Constraints for Normal Variables

If $x \sim N(\mu, \Sigma)$ is $n \times 1$, then it has entropy

$$H(x) = \frac{1}{2} \ln[(2\pi e)^n |\Sigma|]$$  \hspace{1cm} (13)

If $y$ is a normal, unbiased signal and $V[x|y] = \hat{\Sigma}$, then mutual information is

$$I(x, y) = \frac{1}{2} \left( \ln[(2\pi e)^n |\Sigma|] - \ln[(2\pi e)^n |\hat{\Sigma}|] \right)$$  \hspace{1cm} (14)

A typical constraint: $I(x, y) \leq K$ or equivalently,

$$|\hat{\Sigma}| \geq \exp(-2K)|\Sigma|$$  \hspace{1cm} (15)
Complements and Substitutes

Definition 1 An action $a_i$ by agent $i$ is a strategic complement if the optimal choice of $a_i$ is increasing in $a = \int a_j dj$.

Definition 2 An action $a_i$ by agent $i$ is a strategic substitute if the optimal choice of $a_i$ is decreasing in $a = \int a_j dj$. 
Currency Speculation Model

• Unobserved state: $\theta \sim \text{unif}[0, 1]$.

• If successful, speculator earns utility

$$e^* - f(\theta) - t.$$

If unsuccessful, he earns $-t$.

• The government devalues if

$$\alpha > a(\theta).$$
Currency Speculation Model Timing

1. Nature draws a $\theta$ from $[0, 1]$. This prior distribution of $\theta$ is common knowledge.

2. Each investor $i$ draws signal $x_i$ from an uniform distribution on $[\theta - \epsilon, \theta + \epsilon]$. Based on $x_i$, investors decide whether to speculate.

3. The government observes the attack size $\alpha$ and the true state $\theta$ and devalues if $\alpha > a(\theta)$. 
Define a strategy function $\pi(x)$ such that

$$\pi(x) = 1 \quad \text{if} \quad E \left[ 1_{c(\alpha, \theta) > v} (e^* - f(\theta)) \mid x_i \right] > t$$

$$\pi(x) = 0 \quad \text{otherwise}$$

(16)

The set of states $A$ where devaluation occurs:

$$A(\pi) = \{ \theta : \alpha(\theta, \pi) \geq a(\theta) \}.$$

Expected utility:

$$u(x, \pi) = \frac{1}{2\epsilon} \left[ \int_{\theta \in A(\pi) \cap [x-\epsilon, x+\epsilon]} (e^* - f(\theta)) d\theta \right] - t.$$
Let $\theta^*$ be the highest state that produces a successful attack. It solves

$$\frac{1}{2} + \frac{k - \theta^*}{2\epsilon} = a(\theta^*)$$

(17)

$k$ is the signal of a speculator who is indifferent:

$$P[\theta < \theta^*| x_i = k] E[e^* - f(\theta)| x_i = k, \theta < \theta^*] = t$$

(18)

Apply Bayes’ Law to get

$$\int_{k-\epsilon}^{\theta^*} (e^* - f(\theta)) \frac{1}{2\epsilon} d\theta = t.$$  

(19)

(17) and (19) pin down $k$ and $\theta^*$. 

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Information Choice

Currency Speculation Model: Solution

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Unknown exogenous state variable $s \sim N(y, \tau_y^{-1})$.

- Expected loss:
  $$EL(a_i, a, s) = E \left[ (1 - r)(a_i - s)^2 + r(a_i - a)^2 \right].$$  \hspace{1cm} (20)

- Public and private signals: $z|s \sim N(s, \tau_z^{-1})$ and $w_i|s \sim N(s, \tau_w^{-1})$.

- Solution is a Nash equilibrium in $a_i$.  

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**Beauty Contest Model**

- Unknown exogenous state variable $s \sim N(y, \tau_y^{-1})$.

- Expected loss:
  $$EL(a_i, a, s) = E \left[ (1 - r)(a_i - s)^2 + r(a_i - a)^2 \right].$$  \hspace{1cm} (20)

- Public and private signals: $z|s \sim N(s, \tau_z^{-1})$ and $w_i|s \sim N(s, \tau_w^{-1})$.

- Solution is a Nash equilibrium in $a_i$.  

Apply Bayes’ Law:

\[ E_i[s] = \frac{\tau_y y + \tau_w w_i + \tau_z z}{\tau_y + \tau_w + \tau_z} \] (21)

Define \( \alpha_s = \tau_y/(\tau_y + \tau_w + \tau_z) \), \( \alpha_w = \tau_w/(\tau_y + \tau_w + \tau_z) \), and \( \alpha_z = \tau_z/(\tau_y + \tau_w + \tau_z) \).

First order condition is

\[ a_i = (1 - r)E_i[s] + rE_i[a] \] (22)
Beauty Contest: Solution

Guess that

\[ a_i = y + \gamma_w (w_i - y) + \gamma_z (z - y). \]  
(23)

The guess implies that the average action is

\[ a = y + \gamma_w (s - y) + \gamma_z (z - y). \]  
(24)

\( i \)'s expectation of the average action is

\[ E_i[a] = y + \gamma_w (E_i[s] - y) + \gamma_z (z - y). \]  
(25)

Combining these and matching coefficients yields

\[ \gamma_w = \frac{\alpha_w (1 - r)}{1 - \alpha_w r}, \]  
(26)

\[ \gamma_z = \frac{\alpha_z}{1 - \alpha_w r}. \]  
(27)
Expected Utility in a Beauty Contest

Expected utility from optimal action in beauty contest is

\[ E[(a_i - s)^2] = E[((1 - \tilde{\gamma}_w - \tilde{\gamma}_z)y + \tilde{\gamma}_w w_i + \tilde{\gamma}_z z - s)^2] \]  \hspace{1cm} (28)

Since \((y - s), (w_i - s),\) and \((z - s)\) have zero mean and zero covariance,

\[ E[(a_i - s)^2] = (1 - \tilde{\gamma}_w - \tilde{\gamma}_z)^2 E[(y - s)^2] + \tilde{\gamma}_w^2 E[(w_i - s)^2] + \tilde{\gamma}_z^2 E[(z - s)^2]. \]  \hspace{1cm} (29)

\[ E[(a_i - s)^2] = (1 - \tilde{\gamma}_w - \tilde{\gamma}_z)^2 \tau_y^{-1} + \tilde{\gamma}_w^2 \tau_w^{-1} + \tilde{\gamma}_z^2 \tau_z^{-1}. \]  \hspace{1cm} (30)
Cost of private and public information increases in precision: $C(\tau_w, \tau_z)$.

If $\tau_z \leq \tau_z^*$,

$$(a_i - a) = \tilde{\gamma}_w (w_i - y) - \gamma_w^*(s - y) + \tilde{\gamma}_z (z - y) - \gamma_z^*(z^* - y).$$

$$E[(a_i - a)^2] = \tilde{\gamma}_w^2 \tau_w^{-1} + (-\tilde{\gamma}_w + \gamma_w^* - \tilde{\gamma}_z + \gamma_z^*)^2 \tau_y^{-1} + E[(\tilde{\gamma}_z (z - s) - \gamma_z^*(z^* - s))^2].$$

Thus, $E[(a_i - a)^2] =

\tilde{\gamma}_w^2 \tau_w^{-1} + (-\tilde{\gamma}_w + \gamma_w^* - \tilde{\gamma}_z + \gamma_z^*)^2 \tau_y^{-1} + (\tilde{\gamma}_z - \gamma_z^*)^2 \tau_z^{-1} + (\gamma_z^*)^2 (\tau_z^* - \tau_z)^{-1}.$
If $\tau_z \geq \tau_z^*$,

$$E[(a_i-a)^2] = \dot{\gamma}_w^2 (\tau_w + \tau_z - \tau_z^*)^{-1} + (-\ddot{\gamma}_w + \dot{\gamma}_w^* - \dot{\gamma}_z + \gamma_z^*)^2 \tau_y^{-1} + (\dot{\gamma}_z - \gamma_z^*)^2 (\tau_z^*)^{-1}$$
The marginal value of private information is

\[ B(\tau_w) = -\frac{\partial}{\partial \tau_w} EL(\tau_w, \tau_z; \tau_w^*, \tau_z^*) \].

The marginal value of public information is

\[ B(\tau_z) = -\frac{\partial}{\partial \tau_z} EL(\tau_w, \tau_z; \tau_w^*, \tau_z^*) \].

**Proposition:**

\[ r > 0 \iff \frac{\partial}{\partial \tau_w^*} B(\tau_w), \frac{\partial}{\partial \tau_z^*} B(\tau_z) > 0 \]

\[ r = 0 \iff \frac{\partial}{\partial \tau_w^*} B(\tau_w), \frac{\partial}{\partial \tau_z^*} B(\tau_z) = 0 \]

\[ r < 0 \iff \frac{\partial}{\partial \tau_w^*} B(\tau_w), \frac{\partial}{\partial \tau_z^*} B(\tau_z) < 0 \]
Information Choice with Complementarity in Actions

Chapters 5-6

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Utility

\[ U_i = -(1 - r)(a_i - s)^2 - r(a_i - \bar{a})^2 + r \int_0^1 (a_j - \bar{a})^2 dj. \] (31)

Social welfare:

\[ W \equiv \int_0^1 U_i di = -(1 - r) \int_0^1 (a_i - s)^2 di. \] (32)

Substitute in optimal actions

\[ W = -\frac{1 - r}{(1 - \alpha_w r)^2} \left\{ (1 - \alpha_w - \alpha_z)^2 (y - s)^2 + \alpha_w^2 (1 - r)^2 \int_0^1 (w_i - s)^2 di + \alpha_z^2 (z - s)^2 \right\}. \]
Social Value of Public Information

Take expectations

\[ E[W] = -\frac{1-r}{(1-\alpha_w r)^2} \left\{ (1-\alpha_w - \alpha_z)^2 \tau_y^{-1} + \alpha_w^2 (1-r)^2 \int_0^1 (w_i - s)^2 di \right. \\
+ \alpha_z^2 \tau_z^{-1} \right\}. \]

Substitute in for \( \alpha \)'s

\[ E[W] = -(1-r) \left( \frac{1}{\tau_y + (1-r)\tau_w + \tau_z} \right)^2 \left\{ \tau_y^2 \tau_y^{-1} + \tau_w^2 (1-r)^2 \tau_w^{-1} + \tau_z^2 \tau_z^{-1} \right\}. \]

\[ E[W] = (1-r) \left[ -\frac{\tau_y + (1-r)^2 \tau_w + \tau_z}{(\tau_y + (1-r)\tau_w + \tau_z)^2} \right]. \]
When is welfare increasing in public information precision?

\[
\frac{\partial E[W]}{\partial \tau_z} = -(1 - r) \left[ \frac{-\tau_y - (1 - 2r)(1 - r)\tau_w - \tau_z}{(\tau_y + (1 - r)\tau_w + \tau_z)^3} \right].
\]  

(33)

When \(0 < r < 1\), public information is welfare-enhancing if and only if

\[
\frac{\partial E[W]}{\partial \tau_z} > 0 \quad \iff \quad \frac{\tau_y + \tau_z}{\tau_w} > (2r - 1)(1 - r)
\]

(34)
Utility:

\[ U_i = -(1 - r)(a_i - s)^2 - r \int_0^1 (a_j - \bar{a})^2 dj \]  
\hspace{1cm} (35)

Welfare:

\[ \int_0^1 U_i di = -(1 - r) \frac{1}{\tau_x + \tau_y} - r \frac{\tau_x}{(\tau_x + \tau_y)^2}. \]  
\hspace{1cm} (36)

More precise private information is welfare-increasing if

\[ \partial (\int U_i di) / \partial \tau_x > 0, \]  
which holds if

\[ \tau_x < (2r - 1)\tau_y. \]
Model of Amador and Weill (2009)
Utility depends on agent’s actions in period 1 and 2, $a_{1i}$ and $a_{2i}$, and an unknown state $x$:

$$U = -(a_{1i} - x)^2 - (a_{2i} - x)^2.$$ 

Private and public signals:

$$z_{1i} = x + w_{1i} \quad w_{1i} \sim iidN(0, \pi_1^{-1}) \quad (37)$$

$$Z_1 = x + W_1 \quad W_1 \sim N(0, \Pi_1^{-1}) \quad (38)$$

Then choose $a_{1i}$. Average action: $A \equiv \int a_{1i}di$. 
Information Crowd-Out: Period 2

Observe second-period private and public signals:

\[ z_{2i} = A + w_{2i} \quad w_{2i} \sim iidN(0, p^{-1}) \]  

(39)

\[ Z_2 = A + W_2 \quad W_2 \sim N(0, P^{-1}) \]  

(40)

Then, choose \( a_{2i} \).
First-order condition with respect to $a_{1i}$

$$a_{1i}^* = E[x|z_{1i}, Z_1] = \frac{\pi_1 z_{1i} + \Pi_1 Z_1}{\pi_1 + \Pi_1}. \hspace{1cm} (41)$$

Integrating over $a_i$ and using $\int z_{1i}di = x$:

$$A = \frac{\pi_1 x + \Pi_1 Z_1}{\pi_1 + \Pi_1} \hspace{1cm} (42)$$

Endogenous private signal is $z_{2i} = (\pi_1 x + \Pi_1 Z_1)/(\pi_1 + \Pi_1) + w_{2i}$. Transform into a signal with mean $x$.

$$\tilde{z}_{2i} \equiv z_{2i} + \frac{\Pi_1}{\pi_1} (z_{2i} - Z_1) = x + \frac{\pi_1 + \Pi_1}{\pi_1} w_{2i}. \hspace{1cm} (43)$$

Signal precision is

$$\pi_2 = \frac{p\pi_1^2}{(\pi_1 + \Pi_1)^2}. \hspace{1cm} (44)$$
Information Crowd-Out: Solution

Unbiased public signal:

\[ \tilde{Z}_2 \equiv Z_2 + (\Pi_1/\pi_1)(Z_2 - Z_1). \]

Its precision is

\[ \Pi_2 = \frac{P\pi_1^2}{(\pi_1 + \Pi_1)^2}. \]

(45)

Key result: \( \Pi_2 \) and \( \pi_2 \) are decreasing in \( \Pi_1 \).

More precise initial public information lowers the precision of subsequent signals. This is information crowd-out.
Central bank chooses actions $a_1$ and $a_2$ to maximize

$$U = U_1 + U_2 \quad \text{where} \quad U_1 = -(a_1 - \bar{E}[a_2] - \bar{E}[s])^2 \quad \text{and} \quad U_2 = -(a_2 - s)^2,$$

(46)

where $\bar{E}[a_2]$ and $\bar{E}[s]$ are expected future interest rate and inflation and $\bar{E}$ is average expectation.

Other agents do not act. Only form expectations.
Central bank’s (exogenous) signal:

\[ x_{CB} = s + \eta_{CB} \text{ where } \eta_{CB} \sim N(0, \sigma^2) \]

\( \eta_{CB} \) and \( \eta_i \) are independent.

Agents’ (exogenous, heterogenous) signals:

\[ x_i = s + \eta_i \text{ where } \eta_i \sim i.i.d. N(0, \sigma^2) \]

Agents’ second (common) signal:

\[ y = \eta_{CB} + \eta_y \text{ where } \eta_y \sim N(0, \sigma_y^2) \]

Endogenous signal: After the bank chooses \( a_1 \), all observe \( U_1 \).
First order conditions:

\[ a_2 = E_{CB}[s] \]  
\[ a_1 = E[\bar{E}[a_2] + \bar{E}[s]|x_{CB}] \].

Bank chooses \( a_2 = s \) and gets \( U_2 = 0 \).

Expected utility is

\[ E[U] = -(a_1 - \bar{E}[a_2] - \bar{E}[s])^2. \]  
\[ (48) \]

Substitute into FOC to get

\[ a_1 = E_{CB}^1[\bar{E}[a_2] + \bar{E}[s]] = 2x_{CB}. \]

\[ E[U_1] = -(2x_{CB} - 2s)^2 = -4\sigma^2. \]  
\[ (49) \]

First period utility reveals the expectation error \( x_{CB} \).

Main result: With disclosure, central bank learns true state \( s \).
Solution with No Disclosure

Same first-order conditions as before.

Conjecture:

\[ a_2 = \alpha_1 s + \alpha_2 x_{CB} + (1 - \alpha_1 - \alpha_2)y. \]  

(50)

with unknown coefficients \( \alpha_1 \), \( \alpha_2 \).

Agent \( i \)'s expectation of \( a_2 \) is

\[ E[a_2|x_i, y] = \alpha_1 x_i + \alpha_2 (x_i + y\sigma_y^{-2}/(\sigma_y^{-2} + \sigma^{-2})) + (1 - \alpha_1 - \alpha_2)y. \]

Average expectation is

\[ \bar{E}[a_2] = \alpha_1 s + \alpha_2 (s + y\sigma_y^{-2}/(\sigma_y^{-2} + \sigma^{-2}))) + (1 - \alpha_1 - \alpha_2)y. \]
Substitute $\bar{E}[a_2]$ into the first order condition

$$a_1 = (1 + \alpha_1 + \alpha_2)x_{CB}$$

Utility

$$U_1 = - \left( \frac{\alpha_2 \sigma_y^{-2}}{\sigma_y^{-2} + \sigma^{-2}} \eta_{CB} + \left( 1 - \alpha_1 - \frac{\alpha_2 \sigma^{-2}}{\sigma_y^{-2} + \sigma^{-2}} \right) \eta_y \right)^2.$$ 

Unbiased signal from $U_1$ is

$$-U_1^{1/2} (\sigma_y^{-2} + \sigma^{-2})/\left( \alpha_2 \sigma^{-2} \right)$$

Central bank cannot decompose this and observe $\eta_{CB}$ without $\eta_y$. 

Main result: Without disclosure, central bank cannot learn true state $s$. 

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Information Choice with Substitutability in Actions

Chapters 7-9

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Measurement

Chapter 10

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