Fixed Income
Financial Engineering

Concepts and Buzzwords

- From short rates to bond prices
- The simple Black, Derman, Toy model
- Calibration to current the term structure
- Nonnegativity
- Proportional volatility
- Lognormal limiting distribution
- Independent increments vs. mean reversion

Readings

- Veronesi, Chapters 10-11
- Tuckman, Chapters 11-12
Implementing the No-Arbitrage Derivative Pricing Theory in Practice

1. Start with a model (tree) of one-period rates (short rates) and risk-neutral probabilities.
   For example, Black-Derman-Toy, Ho and Lee, ...

2. Build the tree of bond prices from the tree of short rates using the risk-neutral pricing equation (RNPE)
   \[ \text{price} = \text{discount factor} \times [p \times \text{up payoff} + (1-p) \times \text{down payoff}] \]

3. Build the tree of derivative prices from the tree of bond prices by pricing by replication.
   Replication cost can be also represented as
   \[ \text{price} = \text{discount factor} \times [p \times \text{up payoff} + (1-p) \times \text{down payoff}] \]

4. Calibrate the model parameters (drift, volatility) to make the model match observed bond prices and option prices.

Building the Price Tree from the Rate Tree and Risk-Neutral Probabilities (Step 2)

- Once we have a tree of one-period rates and risk-neutral probabilities, we can price any term structure asset.
- For example, suppose 0.5-year rates and risk-neutral probabilities are as follows:

\[
\begin{align*}
\text{Time 0} & \quad \text{Time 0.5} & \quad \text{Time 1} \\
r_{0.5} &= 5.54\% & \quad r_{0.5} & \quad r_{1.5}^{uu} = 6.915\% \\
0.5r_{1}^{u} &= 6.004\% & \quad 0.5r_{1}^{d} &= 4.721\% \\
0.5r_{1}^{d} &= 4.721\% & \quad r_{1.5}^{ud} &= 5.437\% \\
0.5r_{1}^{u} &= 6.004\% & \quad r_{1.5}^{dd} &= 4.275\%
\end{align*}
\]
Building the Price Tree from the Rate Tree...

- Then we have the prices of bonds for maturities 0.5, 1, and 1.5:
- Time 0 price of the zero maturing at time 0.5
  \[d_{0.5} = \frac{1}{1 + 0.0554/2} = 0.973047\]
- Time 0.5 possible prices of zero maturing at time 1
  \[d_{0.5} = \frac{1}{1 + 0.0554/2} = 0.973047\]
- Time 0 price of the zero maturing at time 1
  \[d_1 = 0.973047 \times 0.9709 + 0.5 \times 0.9769 = 0.9476\]

Time 0

\[r_{0.5} = 5.54\%\]

Time 0.5

\[\begin{align*}
  &\quad 0.5 \quad 0.5 \quad 0.5 \\
  0.5 & r_1^u = 6.004\% \\
  0.5 & r_1^d = 4.721\% \\
\end{align*}\]

Time 1

\[\begin{align*}
  &\quad 0.5 \\
  0.5 & r_{1.5}^{uu} = 6.915\% \\
  0.5 & r_{1.5}^{ud} = 5.437\% \\
  0.5 & r_{1.5}^{dd} = 4.275\% \\
\end{align*}\]

Class Problem: Fill in the tree of prices for the zero maturing at time 1.5

Time 0

\[r_{0.5} = 5.54\%\]

Time 0.5

\[\begin{align*}
  &\quad 0.5 \\
  0.5 & r_1^u = 6.004\% \\
  0.5 & r_1^d = 4.721\% \\
\end{align*}\]

Time 1

\[\begin{align*}
  &\quad 0.5 \\
  0.5 & r_{1.5}^{uu} = 6.915\% \\
  0.5 & r_{1.5}^{ud} = 5.437\% \\
  0.5 & r_{1.5}^{dd} = 4.275\% \\
\end{align*}\]
Modeling the Short Rates

- The goal is to build interest rate models that capture basic properties of interest rates while also fitting the current term structure (and liquid option prices).
- Some basic properties are
  - nonnegative interest rates
  - non-normal distribution
  - mean-reversion
  - stochastic volatility and the level effect.
- We will use a simple version of the Black-Derman-Toy model, which has some of these properties.

Log Model of Interest Rates
(Black-Derman-Toy with Constant Volatility)

The short rate (the rate on $h$-year bonds):
- Each date the short rate changes by a multiplicative factor:
  - up factor = $e^{mh+\alpha \sqrt{h}}$
  - down factor = $e^{mh-\alpha \sqrt{h}}$
- The exponential is always positive, which guarantees that interest rates are always positive in this model.
Description of the Model

• The parameter $h$ is the amount of time between dates in the tree, in years. For example, in a semi-annual tree, $h = 0.5$. In a monthly tree, $h = 1/12 = 0.08333$.

• Each value in the tree represents the short rate or interest rate for a zero with maturity $h$.

• Each date the risk-neutral probability of moving up or down is 0.5.

• The drift parameters $m_1, m_2, \ldots$ are known (nonstochastic) but vary over time – these are calibrated to make the model bond prices match the current term structure.

• The proportional volatility $\sigma$, is constant here – this is typically calibrated to an option price.

• In the full-blown BDT model, $\sigma$ also varies each period to allow the model to fit multiple option prices.

• In the limit, as $h \to 0$, the distribution of the future instantaneous short rate is lognormal, i.e., its log is normally distributed.

Example: Semi-Annual Tree Calibrated to Given Term Structure and Volatility

• Suppose

  • the time steps are 6 months, i.e., $h=0.5$ (typically, the choice of $h$ is a tradeoff between speed and accuracy)

  • the current 6-month rate is 5.54%

  • the drift over the first period is $m_1 = -0.0797$ (this sets the average level of the short rate at time 0.5—it is chosen to make the model’s 1-year zero price match the actual 1-year zero price, i.e., 0.9476 in our case.)

  • the drift over the second period is $m_2 = 0.0422$ (this sets the average level of the short rate at time 1—it is chosen to make the 1.5-year zero price in the model = 0.9222)

  • the proportional volatility $\sigma = 0.17$ (could use historical volatility or volatility implied by the price of liquid option)
Resulting Tree of Short Rates

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
<td>6.915%</td>
</tr>
<tr>
<td>4.721%</td>
<td>5.437%</td>
<td></td>
</tr>
<tr>
<td>4.275%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, at time 0.5, up, the 6-month zero rate is

\[0.0554e^{-0.0797 \times 0.5 + 0.17 \sqrt{0.5}} = 0.06004\]

Class Problems

1) Build a tree of 0.5-year rates out to time 0.5 using 
   \(h=0.5, r_{0.5}=2\%, m_1=0.01, \text{ and } \sigma=0.20\).
Class Problems

2) What is the price of the 0.5-year zero at each node?

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

3) What is the price of the 1-year zero at time 0?

4) What is the 1-year zero rate at time 0?

Extending the Interest Rate Tree

- The tree can be extended, as many periods as necessary by successively fitting drift terms to the prices of longer zeros.
- For example, to extend the tree to time 1.5, set $m_3=0.01686$ to make the tree correctly price the 2-year zero ($d_2=0.8972$).

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
<th>Time 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>0.5</td>
<td>6.004%</td>
<td>6.915%</td>
</tr>
<tr>
<td>4.721%</td>
<td>0.5</td>
<td>5.437%</td>
<td>6.184%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.275%</td>
<td>6.915%</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>4.862%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3.823%</td>
</tr>
</tbody>
</table>
Resulting Zero Price Tree

- At each node, the prevailing prices of outstanding zeros are listed, in ascending order of maturity.
- For instance, the price of a 1-year zero at time 0.5, state up, is $0.5d_{1.5}^u = 0.9418$.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
<th>Time 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.973047</td>
<td>0.970857</td>
<td>0.966581</td>
<td>0.962167</td>
</tr>
<tr>
<td>0.947649</td>
<td>0.941787</td>
<td>0.933802</td>
<td>0.970009</td>
</tr>
<tr>
<td>0.922242</td>
<td>0.913180</td>
<td>0.973533</td>
<td>0.976266</td>
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<tr>
<td>0.897166</td>
<td>0.976941</td>
<td>0.979071</td>
<td>0.981243</td>
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<tr>
<td></td>
<td>0.953790</td>
<td>0.979071</td>
<td></td>
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<tr>
<td></td>
<td>0.930855</td>
<td>0.958270</td>
<td></td>
</tr>
</tbody>
</table>

Resulting Tree of Term Structures

- At each node, the prevailing term structure of zero rates is listed, in ascending order of maturity.
- For instance, the 1-year zero rate at time 1, state up-down, is $1r_{2}^{ud} = 5.479\%$.

<table>
<thead>
<tr>
<th>Time 0</th>
<th>Time 0.5</th>
<th>Time 1</th>
<th>Time 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.54%</td>
<td>6.004%</td>
<td>6.915%</td>
<td>7.864%</td>
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<td>5.45%</td>
<td>6.089%</td>
<td>6.968%</td>
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<td>5.47%</td>
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<td>6.184%</td>
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<td>5.50%</td>
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<td>4.862%</td>
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<td>4.834%</td>
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Limitations of This Model

- Only one volatility parameter
  - The model may not be able to fit the prices of options with different maturities simultaneously.
  - The full-blown Black-Derman-Toy model allows the proportional volatility parameter to vary over time to match prices of options with different maturities, allowing for a term structure of volatilities.
- Independent interest rate change over time.
  - Some feel that rates should be mean reverting. This would mean down moves would be more likely at higher interest rates.
  - The Black-Karasinski Model introduces mean reversion in the interest rate process.

Limitations of This Model...

- Only a One-Factor Model
  - Each period one factor (the short rate) determines the prices of all bonds.
  - This means that each period all bond prices move together. Their returns are perfectly correlated. There is no possibility that some bond yields could rise while others fall.
  - To allow for this possibility the model would require additional factors, or sources of uncertainly, which would expand the dimensions of the state-space. For example, in a two-factor model, each period you could move up or down and right or left, so there would be four possible future states.
  - Large investment banks and derivatives dealers often have their own proprietary models.