Risk-Neutral Probabilities

Concepts

- Risk-neutral probabilities
- Risk-neutral pricing
- Expected returns
- True probabilities

Reading

- Veronesi, Chapter 9
- Tuckman, Chapter 9

No Arbitrage Derivative Pricing

▪Last lecture, we priced a derivative by constructing a replicating portfolio from the underlying zeroes:

–We started with a derivative with a payoff at time 0.5. The payoff depended on the time 0.5 price of the zero maturing at time 1.

–We modeled the random future price of the zero and the future payoff of the derivative.

–We constructed a portfolio of 0.5-year and 1-year zeroes with the same payoff of the derivative by solving simultaneous equations.

–We then set the price of the derivative equal to the value of the replicating portfolio.

General Bond Derivative

• Any security whose time 0.5 payoff is a function of the time 0.5 price of the zero maturing at time 1 can be priced by no arbitrage.

• Suppose its payoff is K_u in the up state, and K_d in the down state:

	Time 0	Time 0.5
0.5-year zero	0.973047	1
1-year zero	0.947649	0.972290
		$1N_{0.5} + 0.97229N_1$
General portfolio	$0.973047\,N_{0.5}$ $+\ 0.947649\,N_1$	K_u
General derivative	?	1
		0.976086
		$1N_{0.5} + 0.976086N_1$
		K_d

Replicating and Pricing the General Derivative

1) Determine the replicating portfolio by solving the equations

$$1N_{0.5} + 0.97229N_1 = K_u$$

$$1N_{0.5} + 0.96086N_1 = K_d$$

for the unknown N's. (The two possible K's are known.)

2) Price the replicating portfolio as

$$0.973047N_{0.5} + 0.947649N_1$$

This is the no arbitrage price of the derivative.

Risk-Neutral Probabilities

• Finance: The no arbitrage price of the derivative is its replication cost.

• We know that's some function of the prices and payoffs of the basic underlying assets.

• Math: We can use a mathematical device, risk-neutral probabilities, to compute that replication cost more directly.

• That's useful when we only need to know the price of the replicating portfolio, but not the holdings.

Start with the Prices and Payoffs of the Underlying Assets

- In our example, the derivative payoffs were functions of the time 0.5 price the zero maturing at time 1.
- So the underlying asset is the zero maturing at time 1 and the riskless asset is the zero maturing at time 0.5.
- The prices and payoffs are, in general terms:

Time 0   Time 0.5

$$d_{0.5}$$
$$d_1$$

$$1$$
$$_{0.5}d_1^u$$

$$1$$
$$_{0.5}d_1^d$$
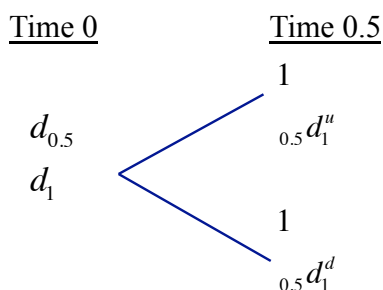
Find the "Probabilities" that "Risk-Neutrally" Price the Underlying Risky Asset

- Find the "probabilities" of the up and down states, p and $1\text{-}p$, that make the price of the underlying asset equal to its "expected" future payoff, discounted back at the riskless rate.
- I.e., find the p that solves "Risk-Neutral Pricing Equation" (RNPE)

Price = discounted "expected" future payoff

for the underlying risky asset.

- In our example, this is the zero maturing at time 1, so

$$d_1 = d_{0.5}[p \times _{0.5}d_1^u + (1-p) \times _{0.5}d_1^d]$$

$$\Rightarrow p = \frac{\dfrac{d_1}{d_{0.5}} - _{0.5}d_1^d}{_{0.5}d_1^u - _{0.5}d_1^d}$$

Example of p

In our example,

$$p = \frac{\dfrac{0.947649}{0.973047} - 0.976086}{0.972290 - 0.976086} = 0.576$$

$$1 - p = 0.424$$

Result

- The *same p* prices all the derivatives of the underlying risk-neutrally.
- I.e., if a derivative has payoffs K_u in the up state and K_d in the down state, its replication cost turns out to be equal to

$$\text{Derivative price} = d_{0.5}[p \times K_u + (1 - p) \times K_d]$$

using the same p that made this risk-neutral pricing equation (RNPE) hold for the underlying asset.

- That is, a single p makes the RNPE

$$\text{price} = \text{discounted "expected" future payoff}$$

hold for the underlying and all its derivatives.

Examples of Risk-Neutral Pricing

With the risk-neutral probabilities, the price of an asset is its expected payoff multiplied by the riskless zero price, i.e., discounted at the riskless rate:

<u>call option:</u> $(0.576 \times 0 + 0.424 \times 1.086) \times 0.9730 = 0.448$

$$\text{or,} \quad \frac{0.576 \times 0 + 0.424 \times 1.086}{1.0277} = 0.448$$

Class Problem:

Price the <u>put option</u> with payoffs $K_u = 2.71$ and $K_d = 0$ using the risk-neutral probabilities.

Examples of Risk-Neutral Pricing...

<u>1-year zero:</u>

$(0.576 \times 0.9723 + 0.424 \times 0.9761) \times 0.9730 = 0.9476$

$$\text{or,} \quad \frac{0.576 \times 0.9723 + 0.424 \times 0.9761}{1.0277} = 0.9476$$

<u>0.5-year zero (riskless asset):</u>

$(0.576 \times 1 + 0.424 \times 1) \times 0.9730 = 0.9730$

$$\text{or,} \quad \frac{0.576 \times 1 + 0.424 \times 1}{1.0277} = 0.9730$$

Why does the *p* that makes the RNP Equation hold for the underlying also make the RNPE work for all its derivatives?

1) By construction, *p* makes

price of underlying risky asset = discount factor x
[*p* x underlying's up payoff + (1-*p*) x underlying's down payoff].

2) It's also always true for any *p* that

price of riskless asset = discount factor
= discount factor x [*p* x 1 + (1-*p*) x 1]

so the *p* that works for the underlying also works for the riskless asset, because any *p* does.

3) Therefore, this *p* also works for any portfolio of these two assets. I.e., for any portfolio with holdings $N_{0.5}$ and N_1:

$N_{0.5}$ x price of riskless asset + N_1 x price of underlying =
disc.factor x
[p x ($N_{0.5}$ x 1 + N_1 x underlying's up payoff) +
(1-*p*) x ($N_{0.5}$ x 1 + N_1 x underlying's down payoff)]

i.e.,   portfolio price = disc.factor x
[p x portfolio's up payoff + (1-*p*) x portfolio's down payoff]

Since every derivative of the underlying is one of these portfolios, the RNPE, using the same *p*, holds for all of them too.

Class Problem

• Suppose the time 0 price of the zero maturing at time 1 is slightly lower:

Time 0	Time 0.5
	1
	0.972290
0.973047	
0.947007	1
	0.976086
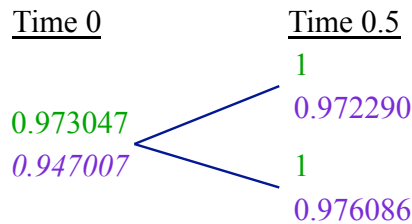
• What would be the risk-neutral probabilities *p* and 1-*p* of the up and down states?

Class Problem

1) Price the call using these new RN probs:

Time 0	Time 0.5
	1
	0.972290
0.973047	0
0.947007	
Call?	1
	0.976086
	1.086

2) Calculate the replication cost of the call the old way and verify that it matches the price above. Hint: the payoffs are unchanged so the replicating portfolio is still $N_{0.5} = -278.163$ and $N_1 = 286.091$.

Expected Returns with RN Probs

- Note that we can rearrange the risk-neutral pricing equation, price = discounted "expected" payoff, as

$$V = d_{0.5}[p \times K_u + (1-p) \times K_d], \quad \text{or}$$

$$V = \frac{p \times K_u + (1-p) \times K_d}{1 + r_{0.5}/2}$$

$$\Leftrightarrow \frac{p \times K_u + (1-p) \times K_d}{V} = 1 + r_{0.5}/2$$

- I.e., "expected" return = the riskless rate.

  (Here return is un-annualized.)

- Thus, with the risk-neutral probabilities, all assets have the same expected return--equal to the riskless rate.

- This is why we call them "risk-neutral" probabilities.

True Probabilities

- The risk-neutral probabilities are not the same as the **true probabilities** of the future states.
- Notice that pricing contingent claims did not involve the true probabilities of the up or down state actually occurring.
- Let's suppose that the true probabilities are 0.5 chance the up state occurs and 0.5 chance the down state occurs.
- What could we do with this information?
- For one, we could compute the *true* expected returns of the different securities over the next 6 months.

True Expected Returns

Recall that the unannualized return on an asset over a given horizon is

$$\frac{\text{future value}}{\text{initial value}} - 1$$

For the 6-month zero the unannualized return over the next 6 months is

$$\frac{1}{0.973047} - 1 = 2.77\%$$

with *certainty*. This will be the return regardless of which state occurs. That's why this asset is *riskless* for this horizon.

Of course, the annualized semi-annually compounded ROR is 5.54%, the quoted zero rate.

True Expected Returns...

The return on the 1-year zero over the next 6 months will be either

$$\frac{0.972290}{0.947649} - 1 = 2.60\% \text{ with probability 0.5, or}$$

$$\frac{0.976086}{0.947649} - 1 = 3.00\% \text{ with probability 0.5.}$$

The expected return on the 1-year zero over the next 6 months is 2.80%.

Notice that it is higher than the return of 2.77% on the riskless asset.

True Expected Returns...

▪Why might the longer zero have a higher expected return?

–Investors have short-term horizons, and dislike the price risk of the longer zero.

–Investors require a premium to hold securities that covary positively with long bonds (bullish securities) because government bonds are in positive net supply.

▪Sometimes the reverse could be true.

▪In general, assets with different risk characteristics have different expected returns. Their expected returns also depend on how their payoffs covary with other assets.

True Expected Returns...

What is the expected rate of return on the call over the next 6 months?

The possible returns are:

$$\frac{0}{0.448} - 1 = -100\% \text{ with probability } 0.5, \text{ or}$$

$$\frac{1.086}{0.448} - 1 = 142\% \text{ with probability } 0.5.$$

The expected return on the call is 21%.

True Expected Returns...

What is the expected rate of return on the put over the next 6 months?

The possible returns are:

$$\frac{2.71}{1.52} - 1 = 78\% \text{ with probability } 0.5, \text{ or}$$

$$\frac{0}{1.52} - 1 = -100\% \text{ with probability } 0.5.$$

The expected return on the put is -11%.

The put is bearish--it insures (hedges) the risk of bullish positions.

By no arbitrage, if bullish assets have positive risk premia, bearish assets must have negative risk premia.

Intuitively, investors must pay up for this insurance.

Risk-Neutral Expected Returns

Using the risk-neutral probabilities to compute expected (unannualized) returns sets all expected returns equal to the riskless rate.

Asset	Unannualized Up Return ("prob"=0.576)	Unannualized Down Payoff ("prob"=0.424)	"Expected" Unannualized Return
0.5-Year Zero	1/0.9730 - 1 = 2.77%	1/0.9730 - 1 = 2.77%	2.77%
1-Year Zero	0.97229/0.947649 - 1 = 2.60%	0.976086/0.947649 - 1 = 3.00%	2.77%
Call	0/0.448 - 1 = -100%	1.0859/0.448 - 1 = 142.39%	2.77%
Put	2.7103/1.519 - 1 = 78.42%	0/1.519 - 1 = -100%	2.77%

Why Does the *p* that Works for the Underlying Asset Also Work for All Its Derivatives?

Reprise:

1) The expected return on a portfolio is the average of the expected returns of the individual assets.

2) The risk-neutral probabilities are constructed to make the expected return on the underlying risky asset equal to the riskless asset return.

3) So under the risk-neutral probabilities, the expected return on every portfolio of the underlying and riskless assets is also that same riskless return.

4) Every derivative of the underlying can be viewed as a portfolio of the underlying asset and the riskless asset.

5) So the derivative's expected return must also equal the riskless return under the risk-neutral probabilities.

6) So the derivative's price must equal its expected payoff, using the risk-neutral probabilities, discounted back at the riskless rate.