Convexity

Concepts and Buzzwords
- Dollar Convexity
- Convexity
- Curvature
- Taylor series
- Barbell, Bullet

Readings
- Veronesi, Chapter 4
- Tuckman, Chapters 5 and 6
Convexity

- Convexity is a measure of the curvature of the value of a security or portfolio as a function of interest rates.
- Duration is related to the slope, i.e., the first derivative.
- Convexity is related to the curvature, i.e., the second derivative of the price function.
- Using convexity together with duration gives a better approximation of the change in value given a change in interest rates than using duration alone.

Price-Rate Function

Example: Security with Positive Convexity
Correcting the Duration Error

• The price-rate function is nonlinear.
• Duration and dollar duration use a linear approximation to the price rate function to measure the change in price given a change in rates.
• The error in the approximation can be substantially reduced by making a convexity correction.

Taylor Series

• The Taylor Theorem from calculus says that the value of a function can be approximated near a given point using its “Taylor series” around that point.
• Using only the first two derivatives, the Taylor series approximation is:

\[ f(x) \approx f(x_0) + f'(x_0) \times (x - x_0) + \frac{1}{2} f''(x_0) \times (x - x_0)^2 \]

Or, \( f(x) - f(x_0) \approx f'(x_0) \times (x - x_0) + \frac{1}{2} f''(x_0) \times (x - x_0)^2 \)
Dollar Convexity

• Think of bond prices, or bond portfolio values, as functions of interest rates.

• The Taylor Theorem says that if we know the first and second derivatives of the price function (at current rates), then we can approximate the price impact of a given change in rates.

\[ f(x) - f(x_0) \approx f'(x_0) \times (x - x_0) + 0.5 \times f''(x_0) \times (x - x_0)^2 \]

★ The first derivative is minus dollar duration.

★ Call the second derivative dollar convexity.

• Then change in price \( \approx -\text{duration} \times \text{change in rates} \)

\[ + 0.5 \times \text{convexity} \times \text{change in rates squared} \]

Dollar Convexity of a Portfolio

If we assume all rates change by the same amount, then

The dollar convexity of a portfolio is the sum of the dollar convexities of its securities.

Sketch of proof:

\[ \sum f_i(x) - f_i(x_0) \approx \left( \sum f_i'(x_0) \right) \times (x - x_0) + 0.5 \times \left( \sum f_i''(x_0) \right) \times (x - x_0)^2 \]

I.e., \( \Delta \) portfolio value \( \approx - (\text{sum of dollar durations}) \times \Delta r \)

\[ + 0.5 \times (\text{sum of dollar convexities}) \times (\Delta rates)^2 \]

⇒ Portfolio dollar duration = sum of dollar durations

⇒ Portfolio dollar convexity = sum of dollar convexities
Convexity

• Just as dollar duration describes dollar price sensitivity, dollar convexity describes curvature in dollar performance.

• To get a scale-free curvature measure, i.e., curvature per dollar invested, we define

\[
\text{convexity} = \frac{\text{dollar convexity}}{\text{price}}
\]

\[\Rightarrow \text{The convexity of a portfolio is the average convexity of its securities, weighted by present value:} \]

\[
\text{convexity} = \frac{\sum \text{price}_i \times \text{convexity}_i}{\sum \text{price}_i} = \text{pv wtd average convexity}
\]

• Just like dollar duration and duration, dollar convexities add, convexities average.

Dollar Formulas for $1$ Par of a Zero

For $1$ par of a $t$-year zero-coupon bond

\[
\text{price} = d_t(r_t) = \frac{1}{(1 + r_t/2)^{2t}}
\]

\[
\text{dollar duration} = -d_t'(r_t) = \frac{t}{(1 + r_t/2)^{2t+1}}
\]

\[
\text{dollar convexity} = d_t''(r_t) = \frac{t^2 + t/2}{(1 + r_t/2)^{2t+2}}
\]

For $N$ par, these would be multiplied by $N$. 
### Percent Formulas for Any Amount of a Zero

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>[ \text{duration} = \frac{\text{dollar duration}}{\text{price}} = \frac{N \times t / (1 + r/2)^{2t+1}}{N \times 1/(1 + r/2)^t} = \frac{t}{1 + r/2} ]</td>
<td>Dollar duration divided by price for any par amount of the zero.</td>
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<tr>
<td>[ \text{convexity} = \frac{\text{dollar convexity}}{\text{price}} = \frac{N \times (t^2 + t/2)/(1 + r/2)^{2t+2}}{N \times 1/(1 + r/2)^t} = \frac{t^2 + t/2}{(1 + r/2)^2} ]</td>
<td>Dollar convexity divided by price for any par amount of the zero.</td>
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- These formulas hold for any par amount of the zero – they are scale-free.
- The duration of the \( t \)-year zero is approximately \( t \).
- The convexity of the \( t \)-year zero is approximately \( t^2 \).

(If we defined price as \( d_t = e^{-rt} \), and differentiated w.r.t. this \( r \), then the duration of the \( t \)-year zero would be exactly \( t \) and the convexity of the \( t \)-year zero would be exactly \( t^2 \).)

### Class Problems

Calculate the price, dollar duration, and dollar convexity of $1 par of the 20-year zero if \( r_{20} = 6.50\% \).
Class Problems
Suppose \( r_{20} \) rises to 7.50%.
1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 using both dollar duration and dollar convexity.

3) What is the exact price change?

Class Problems
Suppose \( r_{20} \) falls to 5.50%.
1) Approximate the price change of $1,000,000 par using only dollar duration.

2) Approximate the price change of $1,000,000 using both dollar duration and dollar convexity.

3) What is the exact price change?
**Sample Risk Measures**

Duration and convexity for $1 par of a 10-year, 20-year, and 30-year zero.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Rate</th>
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<td>0.553676</td>
<td>5.375493</td>
<td>9.7087</td>
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For zeroes,
- duration is roughly equal to maturity,
- convexity is roughly equal to maturity squared.

**Dollar Convexity of a Portfolio of Zeroes**

- Consider a portfolio with fixed cash flows at different points in time ($K_1, K_2, \ldots, K_n$ at times $t_1, t_2, \ldots, t_n$).
- Just as with dollar duration, the dollar convexity of the portfolio is the sum of the dollar convexities of the component zeroes.
- The dollar convexity of the portfolio gives the correction to make to the duration approximation of the change in portfolio value given a change in zero rates, assuming all zero rates change by the same amount.

portfolio dollar convexity = \[ \sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j / 2}{(1 + r_j / 2)^{2t_j + 2}} \]
Class Problem:

**Dollar Convexity of a Portfolio of Zeroes**

Consider a portfolio consisting of

- $25,174 par value of the 10-year zero,
- $91,898 par value of the 30-year zero.

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What is the dollar convexity of the portfolio?

**Convexity of a Portfolio of Zeroes**

\[
\text{convexity} = \frac{\sum_{j=1}^{n} K_j \times \frac{t_j^2 + t_j / 2}{(1 + r_{t_j} / 2)^{2t_j}}}{\sum_{j=1}^{n} K_j / (1 + r_{t_j} / 2)^{2t_j}}
\]

\[
= \frac{\sum_{j=1}^{n} K_j / (1 + r_{t_j} / 2)^{2t_j} \times \frac{t_j^2 + t_j / 2}{(1 + r_{t_j} / 2)^2}}{\sum_{j=1}^{n} K_j / (1 + r_{t_j} / 2)^{2t_j}}
\]

= average convexity weighted by present value

\approx \text{average maturity}^2 \text{ weighted by present value}
Class Problem

Consider the portfolio of 10- and 30-year zeroes.

• The 10-year zeroes have market value
  $25,174 \times 0.553676 = $13,938.

• The 30-year zeroes have market value
  $91,898 \times 0.151084 = $13,884.

• The market value of the portfolio is $27,822.

• What is the convexity of the portfolio?

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Barbells and Bullets

• Consider two portfolios with the same duration:
  
  • A barbell consisting of a long-term zero and a short-term zero
  
  • A bullet consisting of an intermediate-term zero
  
  • The barbell will have more convexity.
Example

- Bullet portfolio: $100,000 par of 20-year zeroes
  market value = $100,000 x 0.27822 = 27,822
  duration = 19.37
- Barbell portfolio: from previous example
  $25,174 par value of the 10-year zero
  $91,898 par value of the 30-year zero.
  market value = 27,822

  \[
  \text{duration} = \frac{(13,938 \times 9.70874) + (13,884 \times 29.06977)}{13,938 + 13,884} = 19.37
  \]

  - The convexity of the bullet is 385.
  - The convexity of the barbell is 478.

Securities with Fixed Cash Flows:
More disperse cash flows, more convexity

- In the previous example,
  the duration of the bullet is about 20 and
  the convexity of the bullet is about \(20^2 = 400\).
- The duration of the barbell is about
  \[0.5 \times 10 + 0.5 \times 30 = 20\]
  but the convexity is about
  \[0.5 \times 10^2 + 0.5 \times 30^2 = 500 > 400 = (0.5 \times 10 + 0.5 \times 30)^2\]
  - I.e., the average squared maturity is greater than the average maturity squared.
- Indeed, recall \(\text{Var}(X) = E(X^2) - (E(X))^2\)
- Think of cash flow maturity \(t\) as the variable and \(pv\) weights as probabilities. Duration is like \(E(t)\) and convexity is like \(E(t^2)\).
- So convexity \(= \text{duration}^2 + \text{dispersion (variance)}\) of maturity.
Value of Barbell and Bullet

Barbell: \( V_2(s) = \frac{25,174}{(1 + (0.06 + \Delta r)/2)^30} + \frac{91,898}{(1 + (0.064 + \Delta r)/2)^60} \)

Bullet: \( V_1(s) = \frac{100,000}{(1 + (0.065 + \Delta r)/2)^40} \)

At current rates, they have the same value and the same slope (duration). But the barbell has more curvature (convexity).

Does the Barbell Always Outperform the Bullet?

- If there is an immediate parallel shift in interest rates, either up or down, then the barbell will outperform the bullet.
- If the shift is not parallel, anything could happen.
- If the rates on the bonds stay exactly the same, then as time passes the bullet will actually outperform the barbell:
  - the bullet will return 6.5%
  - the barbell will return about 6.2%, the market value-weighted average of the 6% and 6.4% on the 10- and 30-year zeroes.
Value of Barbell and Bullet: One Year Later

If the rates don't change, the bullet will be worth more.
If there is a large enough parallel yield curve shift, the barbell will be worth more.

Convexity and the Shape of the Yield Curve?

- If the yield curve were flat and made parallel shifts, more convex portfolios would always outperform less convex portfolios, and there would be arbitrage.

- So to the extent that market movement is described by parallel shifts, bullets must have higher yield to start with, to compensate for lower convexity.

- This would explain why the term structure is often hump-shaped, dipping down at very long maturities where convexity is greatest relative to duration—investors may give yield to buy convexity.

- Some evidence suggests that the yield curve is more curved when volatility is higher and convexity is worth more.