Duration

Outline and Reading

- Outline
  - Interest Rate Sensitivity
  - Dollar Duration
  - Duration
- Buzzwords
  - Parallel shift
  - Basis points
  - Modified duration
  - Macaulay duration
- Reading
  - Veronesi, Chapter 3
  - Tuckman, Chapters 5 and 6

Duration

- The *duration* of a bond is a linear approximation of minus the *percent* change in its price given a 100 basis point change in interest rates. (100 basis points = 1% = 0.01)
- For example, a bond with a duration of 7 will gain about 7% in value if interest rates fall 100 bp.
- For zeroes, duration is easy to define and compute with a formula.
- For securities or portfolios with multiple fixed cash flows, we must make assumptions about how rates shift together. We will assume all zero rates move by the same amount.
- To compute duration for other instruments requires further assumptions and numerical estimation.

Other Duration Concepts

- Concept 1: Percent change in the bond's price given 100 bp change in rates
- Concept 2: Average maturity of the bond's cash flows, weighted by present value.
- Concept 3: Holding period over which the return from investing in the bond is riskless, or immunized from immediate parallel shifts in interest rates.
- Math fact: For a security with fixed cash flows, these turn out to be the same.
- For securities with random cash flows, such as options and callable bonds, concept 2 doesn't apply.
- We'll focus on concept 1.

Dollar Duration

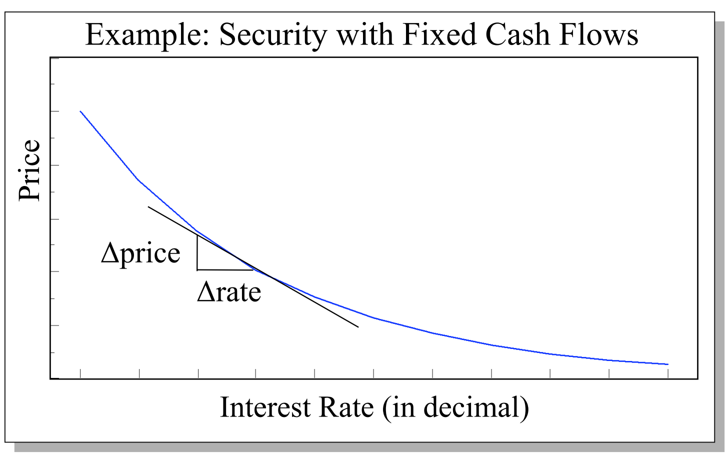
Start with the notion of dollar duration:

Concept: $\text{dollar duration} \approx -\dfrac{\text{change in dollar value}}{\text{change in interest rates (in decimal)}}$

Application:
change in value ≈ -dollar duration x change in rates in decimal

Class Problem: Suppose a bond portfolio has a dollar duration of 10,000,000. Approximately how much will value change if rates rise 20 basis points?

Dollar Duration ≈ $-\Delta p/\Delta r$ = - Slope of Price Rate Function



Dollar Duration vs. DV01, DVBP, BPV

In practice people use

DV01 = DVBP = Dollar Value of a Basis Point

How much will a bond value change if rates change 1 bp?

Approx. change in value = -$dur x change in rates

DV01 = $dur x 0.0001

Change in value ≈ - DV01 x change in rate in basis points

<u>Example:</u>

Bond with $dur = 10,000,000 has DV01 = 1000.

20 bp rate rise causes -1000 x 20 = - $20,000 price change.

Duration

Duration approximates the *percent* change in price for a 100 basis point change in rates:

Duration ≈

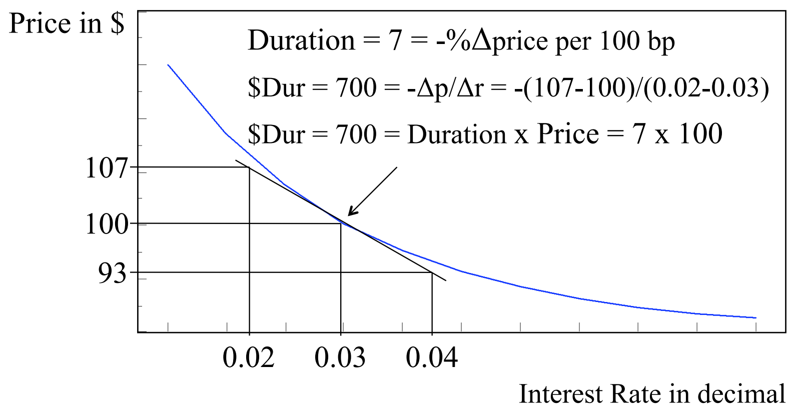
Percent change in price per 100 bp changes in rates

$$= \frac{\text{Dollar change in price per 100bp}}{\text{price}} \times 100$$

$$= \frac{\text{Dollar duration} \times 0.01}{\text{price}} \times 100$$

$$= \frac{\text{Dollar duration}}{\text{price}}$$

Example: Security with Duration 7, Price 100, Dollar Duration 700



Portfolio Dollar Duration

- The dollar duration of a portfolio is the **sum** of the dollar durations of the securities in the portfolio.

  Sketch of proof:

$$\text{Portfolio price} = \sum_i \text{price of security i}$$

$$\Delta\text{Portfolio price} = \sum_i \Delta\text{price of security i}$$

$$\frac{\Delta\text{Portfolio price}}{\Delta\text{rate}} = \sum_i \frac{\Delta\text{price of security i}}{\Delta\text{rate}}$$

$$\text{Portfolio \$duration} = \sum_i \text{\$duration of security i}$$

Portfolio Duration

- The duration of a portfolio is the **average** of the durations of its securities, weighted by price (pv).

Sketch of proof:

$$\text{Portfolio duration} = \frac{\text{portfolio \$duration}}{\text{portfolio price}}$$

$$= \frac{\sum \text{\$duration of security i}}{\sum \text{price of security i}}$$

$$= \frac{\sum \text{price of security i} \times \text{duration of security i}}{\sum \text{price of security i}}$$

$$= \sum w_i \times \text{duration of security i}$$

where w_i is proportion of portfolio present value in i

Class Problems

Consider two securities:

A bond with a duration of 8

A CMO with a duration of 4

1) Each unit of the bond costs \$110. What is its \$duration?

2) Each unit of the CMO costs \$70. What is its \$duration?

3) Portfolio A has 1000 units of the bond and 2000 units of the CMO. What is its \$duration? What is its duration?

4) Portfolio B has \$7.5M (pv) invested in the bond and \$2.5M (pv) invested in the CMO. What is its duration?

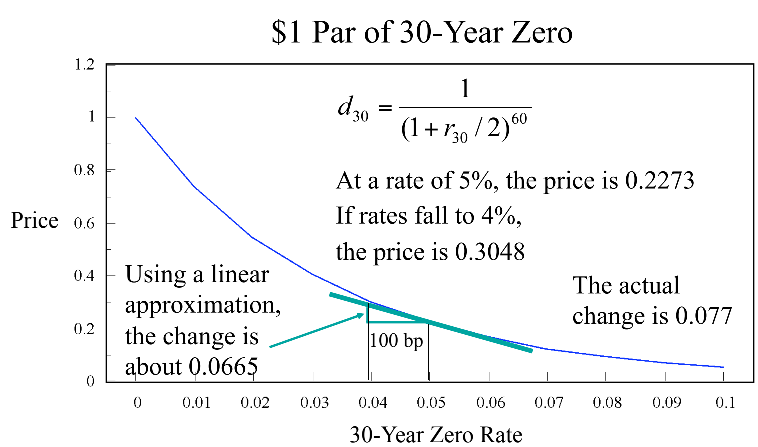
5) Approximate the dollar change in the value of A if rates rise 50 bp.

6) Approximate the percent change in B if rates rise 50 bp.

Formulas:
Dollar Duration for a Zero

- For zero-coupon bonds, there is an explicit formula relating the zero *price* to the zero *rate*.
- We use this price-rate formula to get a formula for dollar duration.
- Of course, with a zero, the ability to approximate price change is not so important, because it's easy to do the exact calculation.
- However, with more complicated securities and portfolios, the exact calculations can be difficult. Seeing how the duration approximation works with the zero makes it easier to understand in less transparent cases.

The Price-Rate Function for a Zero



Dollar Duration of $1 Par of a Zero

Dollar duration is minus the slope of price-rate function.

For zeroes, we can use calculus to get the formula:

$$d_t(r_t) = \frac{1}{(1 + r_t/2)^{2t}}$$

$$d_t^{'}(r_t) = \frac{-t}{(1 + r_t/2)^{2t+1}}$$

To avoid quoting a negative number, change the sign.

The dollar duration of $1 par of a *t*-year zero is

$$\$\mathrm{dur}_t = -d_t^{'}(r_t) = \frac{t}{(1 + r_t/2)^{2t+1}}$$

★ For $*N* par of the zero, both the price and the dollar duration would be *N* times as much.

Class Problems

1) What is the dollar duration of $1 par of a 30-year zero if the 30-year zero rate is 5%?
2) What is the dollar duration of $1 par of a 10-year zero if the 10-year zero rate is 6%?
3) If the 10-year zero rate falls to 5%, how much will $1000 par value of the 10-year zero rise in price?
   a) Using the dollar duration approximation?
   b) Using exact calculations?

Duration for a Zero

Recall that duration = dollar duration/price.

So the duration of a t-year zero is:

$$\text{dur}_t = \frac{\dfrac{t}{(1+r_t/2)^{2t+1}}}{\dfrac{1}{(1+r_t/2)^{2t}}} = \frac{t}{(1+r_t/2)}$$

- Notice that the duration of a zero is approximately equal to its maturity.
- This is its *modified duration*—that is, w.r.t. the semi-annually compounded rate.
- If we used the continuously compounded rate, i.e., $d_t=e^{-rt}$, then we would get *Macaulay duration*, which for a zero is exactly its maturity t.

Class Problems

1) If the 30-year zero rate is 5%, what is the duration of the 30-year zero?

2) If the 10-year zero rate is 6%, what is the duration of a 10-year zero?

Dollar Duration of a Portfolio of Fixed Cash Flows (using zero rates)

- Remember that the portfolio value is a function of all of the different zero rates associated with its cash flows.
- We can approximate the change in the portfolio value assuming *all zero rates change by the same amount.*
- In other words, we can measure the sensitivity of the portfolio value to a *parallel shift* in the zero rate curve.
- How useful will this measure be?
- Of course, rates do not always change by exactly the same amount, but they do tend to move together.

Dollar Duration for a Portfolio of Fixed Cash Flows…

Suppose a portfolio (or bond) has cash flows K_1, K_2, ... at times t_1, t_2,....

Its value is the sum of the values of the components:

$$V = K_1 \times d_{t_1} + K_2 \times d_{t_2} + \ldots$$

If rates change, its value will change by the sum of the changes in value of the components:

$$\Delta V = K_1 \times \Delta d_{t_1} + K_2 \times \Delta d_{t_2} + \ldots$$

We can approximate the change in each zero price using its dollar duration: $\Delta d_t \approx -\$\text{dur}_t \times \Delta r_t$

Portfolio Dollar Duration = Sum of Dollar Durations

The approximate change in the portfolio value is:

$$\Delta V \approx -(K_1 \times \$\mathrm{dur}_{t_1} \times \Delta r_{t_1} + K_2 \times \$\mathrm{dur}_{t_2} \times \Delta r_{t_2} + \ldots)$$

Suppose all rate changes are the same. That is, the yield curve makes a parallel shift:

$$\Delta r_{t_1} = \Delta r_{t_2} = \Delta r_{t_3} = \ldots = \Delta r$$

Then the portfolio value change is

$$\Delta V \approx -(K_1 \times \$\mathrm{dur}_{t_1} + K_2 \times \$\mathrm{dur}_{t_2} + \ldots) \times \Delta r$$

and

$$\text{portfolio \$dur} \approx -\frac{\Delta V}{\Delta r} \approx K_1 \times \$\mathrm{dur}_{t_1} + K_2 \times \$\mathrm{dur}_{t_2} + \ldots$$

$=$ the sum of the \$durations of the cash flows

Bottom Line: Zero-Rate Dollar Duration of a Portfolio or Security with Fixed Cash Flows

- A portfolio or security with fixed cash flows can be viewed as a package of zeroes.
- Assuming all zero rates change by the same amount, i.e., the zero yield curve makes parallel shifts, the dollar duration of a portfolio or security with fixed cash flows is the sum of the dollar durations of the individual zeroes:

$$\sum_{j=1}^{n} K_j \frac{t_j}{(1 + r_{t_j}/2)^{2t_j+1}} \quad \text{or} \quad \sum_{j=1}^{n} \frac{K_j}{(1 + r_{t_j}/2)^{2t_j}} \times \frac{t_j}{(1 + r_{t_j}/2)}$$

Example: Zero-Rate Dollar Duration of a Coupon Bond

The zero-rate dollar duration of $1 par of a *T*-year bond with coupon rate *c* is

$$\frac{c}{2}\left[\frac{0.5}{(1+r_{0.5}/2)^2}+\frac{1}{(1+r_1/2)^3}+\frac{1.5}{(1+r_{1.5}/2)^4}+\ldots+\frac{T}{(1+r_T/2)^{2T+1}}\right]+\frac{T}{(1+r_T/2)^{2T+1}}$$

$$=\frac{c}{2}\left(\sum_{s=1}^{2T}\frac{s/2}{(1+r_{s/2}/2)^{s+1}}\right)+\frac{T}{(1+r_T/2)^{2T+1}}\ .$$

This is the dollar price sensitivity to a parallel shift in the zero yield curve.

Example: dollar duration of $1 par of a 1-year 6%-coupon bond:

$$\frac{0.06}{2}\times\frac{0.5}{(1+0.0554/2)^2}+\left(1+\frac{0.06}{2}\right)\frac{1}{(1+0.0545/2)^3}=0.964389$$

Zero-Rate Duration of a Portfolio or Security with Fixed Cash Flows

- Recall: The duration of a portfolio is the average of the durations of its pieces, weighted by present value.
- So for a portfolio or security with fixed cash flows, its duration is roughly the average maturity of its cash flows—this gives an intuitive way to estimate interest rate sensitivity.
- Using zero rates, this is:

$$\text{Portfolio duration}=\frac{\sum_{j=1}^{n}\frac{K_j}{(1+r_{t_j}/2)^{2t_j}}\times\frac{t_j}{(1+r_{t_j}/2)}}{\sum_{j=1}^{n}\frac{K_j}{(1+r_{t_j}/2)^{2t_j}}}=\sum_{j=1}^{n}w_j\times\frac{t_j}{(1+r_{t_j}/2)}$$

where the $w_j=\frac{K_j}{(1+r_{t_j}/2)^{2t_j}}\Big/\sum_{j=1}^{n}\frac{K_j}{(1+r_{t_j}/2)^{2t_j}}$ are the pv weights.

Class Problem

What is the duration of a portfolio invested

50% in 10-year zeroes and

50% in 30-year zeroes?

Yield Duration and Dollar Duration

- In practice, for securities with fixed cash flows, people compute the duration of a security using the security's yield instead of the individual zero rates for with each cash flow.
- If we work with price as a function of yield instead of individual zero rates, $p(y)$, we would compute
- dollar duration = $-p'(y) \approx$ -change in price/change in yield
- duration = dollar duration/price = $-p'(y) / p(y) \approx$ - percent change in price for 100 bp change in bond yield.
- This gives the similar formulas as before, except that the security's yield replaces the zero rates.

Yield Duration…

$$\text{yield duration} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}} \times \frac{t_j}{(1+y/2)}}{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}}}$$

- Yield duration gives the percent change in price for a 100 bp change in the bond's yield.
- Zero rate duration gives the percent change in price for a 100 bp change in all zero rates.

(These are slightly different. When all zero rates change by 100 bp, the bond's yield change is close to, but not exactly 100 bp.)

Example

Zero rate duration

Par	Coupon (%)	Maturity	Yield (%)	Market Val	Duration	$Duration
1	0	0.5	5.54	0.973047	0.486523	0.473410
1	0	1.0	5.45	0.947649	0.973473	0.922511
100	6.5	1.0	5.451431	101.0072	0.958227	96.787842

Yield duration

Par	Coupon (%)	Maturity	Yield (%)	Market Val	Duration	$Duration
1	0	0.5	5.451431	0.973466	0.486733	0.473818
1	0	1.0	5.451431	0.947636	0.973466	0.922492
100	6.5	1.0	5.451431	101.0072	0.958221	96.787179

Yield Duration vs. Zero Rate Duration? Inconsistency vs. Practicality

- If the yield curve is not flat, then two portfolios with identical cash flows could have different yield durations. For example, think of a coupon bond and the corresponding portfolio of zeroes.
  - In the previous example, the coupon bond had a yield duration of 0.958221.
  - The replicating portfolio of zeroes would have a yield duration of 0.958227.
- So with yield duration the packaging matters. ☹
- However, if zero rates are not readily available, yield duration is easier to compute. ☺
- Also, with yield duration, we can get closed-form solutions.

Yield Duration and Dollar Duration for a Coupon-Bond

- For a coupon bond we can get a closed-form expression:

$$p(y) = \frac{c}{y}[1-\frac{1}{(1+y/2)^{2T}}]+\frac{1}{(1+y/2)^{2T}}$$

$$\$\text{dur} = -p'(y) = \frac{c}{y^2}[1-\frac{1}{(1+y/2)^{2T}}]+(1-\frac{c}{y})\frac{T}{(1+y/2)^{2T+1}}$$

$$\text{duration} = \frac{\$\text{dur}}{\text{price}} = \frac{\frac{c}{y^2}[1-\frac{1}{(1+y/2)^{2T}}]+(1-\frac{c}{y})\frac{T}{(1+y/2)^{2T+1}}}{\frac{c}{y}[1-\frac{1}{(1+y/2)^{2T}}]+\frac{1}{(1+y/2)^{2T}}}$$

- For a par bond, $c=y$, and $p=1$, so this simplifies to:

$$\text{duration} = [1-\frac{1}{(1+y/2)^{2T}}]/\,y$$

Durations of Benchmark Coupon Bonds

6%-Coupon Bonds:

	Maturity in Years						
Yield	1	2	5	7	10	20	30
2%	0.98	1.90	4.41	5.91	7.97	13.67	18.29
4%	0.97	1.88	4.34	5.78	7.70	12.62	16.02
6%	0.96	1.86	4.27	5.65	7.44	11.56	13.84
8%	0.95	1.84	4.19	5.52	7.17	10.50	11.85
10%	0.94	1.82	4.12	5.38	6.89	9.49	10.12
12%	0.93	1.80	4.05	5.25	6.61	8.53	8.68

Par Bonds:

	Maturity in Years						
Yield	1	2	5	7	10	20	30
2%	0.99	1.95	4.74	6.50	9.02	16.42	22.48
4%	0.97	1.90	4.49	6.05	8.18	13.68	17.38
6%	0.96	1.86	4.27	5.65	7.44	11.56	13.84
8%	0.94	1.81	4.06	5.28	6.80	9.90	11.31
10%	0.93	1.77	3.86	4.95	6.23	8.58	9.46
12%	0.92	1.73	3.68	4.65	5.73	7.52	8.08

Summary

For any fixed income security:

duration $\approx$ (minus the) percent change in price per 100 bp change in rates

$$\text{dollar duration} \approx -\frac{\text{change in value}}{\text{change in rates (in decimal)}}$$

$$\text{duration} = \frac{\text{dollar duration}}{\text{value}}$$

dollar duration of a portfolio = sum of dollar durations

$$\text{duration of a portfolio} = \frac{\text{dollar duration}}{\text{value}} = \text{average duration weighted by pv}$$

For securities with fixed cash flows:

$$\text{dollar duration of \$1 par of a zero} = \$\text{dur}_t = \frac{t}{(1+r_t/2)^{2t+1}}$$

$$\text{duration of a zero} = \text{dur}_t = \frac{t}{1+r_t/2}$$

duration $\approx$ average maturity of cash flows, weighted by pv

zero-rate duration (and dollar duration) -- uses zero rates in the formulas

yield duration (and dollar duration) -- uses security yield in the formulas

$$\text{yield duration of a par bond} = [1-\frac{1}{(1+y/2)^{2T}}]/y$$

Portfolio Yield ≈ Dollar-Duration-Weighted Average of Security Yields

- The price of portfolio, P, is the sum of the prices of its individual securities (or zeroes), $p_1, p_2, \ldots, p_n$.
- The yield of the portfolio, y, is the single discount rate that gives the portfolio the same value as the individual security yields (or zero rates), call them $y_1, y_2, \ldots, y_n$.
- That is, $P = \sum p_i(y_i) = \sum p_i(y)$
- Therefore, $0 = \sum p_i(y) - p_i(y_i) \approx \sum p'_i(y_i)(y - y_i)$

$$\Rightarrow y \approx \frac{\sum p'_i(y_i) y_i}{\sum p'_i(y_i)}$$

- I.e., the portfolio yield is the dollar-duration weighted average of the individual security (or zero) yields.

Honorable Mention: Macaulay Duration

- The first measure of duration was developed by Frederick Macaulay in 1938:

$$\text{Macaulay duration} = \frac{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}} \times t_j}{\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}}}$$

- The Macaulay duration of a security is the average maturity of its cash flows weighted by their present value.
- You could call this is Macaulay yield duration if you wanted to put a finer point on it. It approximates minus the percent change in price per 100 bp change in the continuously compounded yield.
- You could define Macaulay zero rate duration, too, but let's stop here.

	Zero Rates	Yield
Semi-annual compounding	This class	In practice
Continuous compounding		Macaulay