Coupon Bonds and Zeroes

Concepts and Buzzwords

- Coupon bonds
- Zero-coupon bonds
- Bond replication
- No-arbitrage price relationships
- Zero rates

- Zeroes
- STRIPS
- Dedication
- Implied zeroes
- Semi-annual compounding

Reading

- Veronesi, Chapters 1 and 2
- Tuckman, Chapters 1 and 2
**Coupon Bonds**

- In practice, the most common form of debt instrument is a coupon bond.
- In the U.S and in many other countries, coupon bonds pay coupons every six months and par value at maturity.
- The quoted coupon rate is annualized. That is, if the quoted coupon rate is $c$, and bond maturity is time $T$, then for each $1$ of par value, the bond cash flows are:

  \[
  \frac{c}{2}, \frac{c}{2}, \frac{c}{2}, \ldots, 1 + \frac{c}{2} \\
  0.5 \text{ years}, 1 \text{ year}, 1.5 \text{ years}, \ldots, T \text{ years}
  \]

  - If the par value is $N$, then the bond cash flows are:

    \[
    N\frac{c}{2}, N\frac{c}{2}, N\frac{c}{2}, \ldots, N(1 + \frac{c}{2}) \\
    0.5 \text{ years}, 1 \text{ year}, 1.5 \text{ years}, \ldots, T \text{ years}
    \]

**U.S. Treasury Notes and Bonds**

- Institutionally speaking, U.S. Treasury “notes” and “bonds” form a basis for the bond markets.
- Non-competitive bidders just submit par amounts, maximum $5$ million, and are filled first. Competitive bidders submit yields and par amounts, and are filled from lowest yield to the “stop” yield. The coupon on the bond, an even eighth of a percent, is set to make the bond price close to par value at the stop yield. All bidders pay this price.
**Class Problem**

- The current “long bond,” the newly issued 30-year Treasury bond, is the 3 7/8’s (3.875%) of August 15, 2040.
- What are the cash flows of $1,000,000 par this bond? (Dates and amounts.)

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**Bond Replication and No Arbitrage Pricing**

- It turns out that it is possible to construct, and thus price, all securities with fixed cash flows from coupon bonds.
- But the easiest way to see the replication and no-arbitrage price relationships is to view all securities as portfolios of “zero-coupon bonds,” securities with just a single cash flow at maturity.
- We can observe the prices of zeroes in the form of Treasury STRIPS, but more typically people infer them from a set of coupon bond prices, because those markets are more active and complete.
- Then we use the prices of these zero-coupon building blocks to price everything else.
Zeroes

- Conceptually, the most basic debt instrument is a zero-coupon bond—a security with a single cash flow equal to face value at maturity.
- Cash flow of $1 par of $t$-year zero:

\[
\text{\$1} \quad \text{Time } t
\]

- It is easy to see that any security with fixed cash flows can be constructed, and thus priced, as a portfolio of these zeroes.

Zero Prices

- Let $d_t$ denote the price today of the $t$-year zero, the asset that pays off $\$1$ in $t$ years.
- I.e., $d_t$ is the price of a $t$-year zero as a fraction of par value.
- This is also sometimes called the $t$-year “discount factor.”
- Because of the time value of money, a dollar today is worth more than a dollar to be received in the future, so the price of a zero must always less than its face value:

\[
d_t < 1
\]

- Similarly, because of the time value of money, longer zeroes must have lower prices.
A Coupon Bond as a Portfolio of Zeroes

Consider: $10,000 par of a one and a half year, 8.5% Treasury bond makes the following payments:

<table>
<thead>
<tr>
<th></th>
<th>$425</th>
<th>$425</th>
<th>$10425</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.5 years</td>
<td>1 year</td>
<td>1.5 years</td>
</tr>
</tbody>
</table>

Note that this is the same as a portfolio of three different zeroes:
- $425 par of a 6-month zero
- $425 par of a 1-year zero
- $10425 par of a 1 1/2-year zero

No Arbitrage and The Law of One Price

- Throughout the course we will assume:

  **The Law of One Price** Two assets which offer exactly the same cash flows must sell for the same price.

  - Why? If not, then one could buy the cheaper asset and sell the more expensive, making a profit today with no cost in the future.

  - This would be an *arbitrage opportunity*, which could not persist in equilibrium (in the absence of market frictions such as transaction costs and capital constraints).
Valuing a Coupon Bond Using Zero Prices

Let’s value $10,000 par of a 1.5-year 8.5% coupon bond based on the zero prices (discount factors) in the table below.

These discount factors come from historical STRIPS prices (from an old WSJ). **We will use these discount factors for most examples throughout the course.**

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Discount Factor</th>
<th>Bond Cash Flow</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.9730</td>
<td>$425</td>
<td>$414</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9476</td>
<td>$425</td>
<td>$403</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9222</td>
<td>$10425</td>
<td>$9614</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Total $10430</td>
</tr>
</tbody>
</table>

On the same day, the WSJ priced a 1.5-year 8.5%-coupon bond at 104 10/32 (=104.3125).

An Arbitrage Opportunity

- What if the 1.5-year 8.5% coupon bond were worth only 104% of par value?
- You could buy, say, $1 million par of the bond for $1,040,000 and sell the cash flows off individually as zeroes for total proceeds of $1,043,000, making $3000 of riskless profit.
- Similarly, if the bond were worth 105% of par, you could buy the portfolio of zeroes, reconstitute them, and sell the bond for riskless profit.
Class Problems

In today’s market, the discount factors are:

\[ d_{0.5} = 0.9991, \quad d_1 = 0.9974, \quad \text{and} \quad d_{1.5} = 0.9940. \]

1) What would be the price of an 8.5%-coupon, 1.5-year bond today? (Say for $100 par.)

2) What would be the price of $100 par of a 2%-coupon, 1-year bond today?

Securities with Fixed Cash Flows as Portfolios of Zeroes

• More generally, if an asset pays cash flows \( K_1, K_2, \ldots, K_n \), at times \( t_1, t_2, \ldots, t_n \), then it is the same as:

\[ K_1 t_j \text{-year zeroes} + K_2 t_2 \text{-year zeroes} + \ldots + K_n t_n \text{-year zeroes} \]

• Therefore no arbitrage requires that the asset’s value \( V \) is

\[ V = K_1 \times d_{t_1} + K_2 \times d_{t_2} + \ldots + K_n \times d_{t_n} \]

or

\[ V = \sum_{j=1}^{n} K_j \times d_{t_j} \]
Coupon Bond Prices in Terms of Zero Prices

For example, if a bond has coupon $c$ and maturity $T$, then in terms of the zero prices $d_r$, its price per $1$ par must be

$$P(c,T) = \left( c / 2 \right) \times \left( d_{0.5} + d_{1} + d_{1.5} + ... + d_T \right) + d_T$$

or

$$P(c,T) = \left( c / 2 \right) \sum_{s=1}^{2T} d_{s/2} + d_T$$

Constructing Zeroes from Coupon Bonds

- Often people would rather work with Treasury coupon bonds than with STRIPS, because the market is more active.

- They can imply zero prices from Treasury bond prices instead of STRIPs and use these to value more complex securities.

- In other words, not only can we construct bonds from zeroes, we can also go the other way.

- Example: Constructing a 1-year zero from 6-month and 1-year coupon bonds.

- Coupon Bonds:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Coupon</th>
<th>Price in 32nds</th>
<th>Price in Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.250%</td>
<td>99-13</td>
<td>99.40625</td>
</tr>
<tr>
<td>1.0</td>
<td>4.375%</td>
<td>98-31</td>
<td>98.96875</td>
</tr>
</tbody>
</table>
**Constructing the One-Year Zero**

- Find portfolio of bonds 1 and 2 that replicates 1-year zero.
- Let $N_{0.5}$ be the par amount of the 0.5-year bond and $N_1$ be the par amount of the 1-year bond in the portfolio.
- At time 0.5, the portfolio will have a cash flow of $N_{0.5} x (1+0.0425/2) + N_1 x 0.04375/2$
- At time 1, the portfolio will have a cash flow of $N_{0.5} x 0 + N_1 x (1+0.04375/2)$
- We need $N_{0.5}$ and $N_1$ to solve

\[
\begin{align*}
(1) \quad &N_{0.5} x (1+0.0425/2) + N_1 x 0.04375/2 = 0 \\
(2) \quad &N_{0.5} x 0 + N_1 x (1+0.04375/2) = 100 \\
\Rightarrow \quad &N_1 = 97.86 \text{ and } N_{0.5} = -2.10
\end{align*}
\]

**Implied Zero Price**

- So the replicating portfolio consists of
  - long 97.86 par value of the 1-year bond
  - short 2.10 par value of the 0.5-year bond.
- **Class Problem:** Given the prices of these bonds below,

<table>
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what is the no-arbitrage price of $100 par of the 1-year zero?
Inferring Zero Prices from Bond Prices: Short Cut

- The last example showed how to construct a portfolio of bonds that synthesized (had the same cash flows as) a zero.
- We concluded that the zero price had to be the same as the price of the replicating portfolio (no arbitrage).
- If we don't need to know the replicating portfolio, we can solve for the implied zero prices more directly:

\[
\text{Price of bond 1} = (100 + 4.25/2) \times d_{0.5} = 99.40625 \\
\text{Price of bond 2} = (4.375/2) \times d_{0.5} + (100 + 4.375/2) \times d_1 = 98.96875 \\
\Rightarrow d_{0.5} = 0.973, \quad d_1 = 0.948
\]

Class Problems

1) Suppose the price of the 4.25%-coupon, 0.5-year bond is 99.50. What is the implied price of a 0.5-year zero per $1 par?

2) Suppose the price of the 4.375%-coupon, 1-year bond is 99. What is the implied price of a 1-year zero per $1 par?
Replication Possibilities

- Since we can construct zeroes from coupon bonds, we can construct any stream of cash flows from coupon bonds.
- Uses:
  - Bond portfolio dedication--creating a bond portfolio that has a desired stream of cash flows
  - funding a liability
  - defeasing an existing bond issue
  - Taking advantage of arbitrage opportunities

Market Frictions

- In practice, prices of Treasury STRIPS and Treasury bonds don't fit the pricing relationship exactly
  - transaction costs and search costs in stripping and reconstituting
  - bid/ask spreads
- Note: The terms “bid” and “ask” are from the viewpoint of the dealer.
  - The dealer buys at the bid and sells at the ask, so the bid price is always less than the ask.
  - The customer sells at the bid and buys at the ask.
**Interest Rates**

- People try to summarize information about bond prices and cash flows by quoting interest rates.

- Buying a zero is lending money—you pay money now and get money later.

- Selling a zero is borrowing money—you get money now and pay later.

- A bond transaction can be described as:
  - buying or selling at a given price, or
  - lending or borrowing at a given rate.

- The convention in U.S. bond markets is to use **semi-annually compounded interest rates**.

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**Annual vs. Semi-Annual Compounding**

At 10% per year, *annually* compounded, $100 grows to $110 after 1 year, and $121 after 2 years:

\[
100 \times 1.10 = 110 \\
100 \times (1.10)^2 = 121
\]

10% per year *semi-annually* compounded really means 5% every 6 months. At 10% per year, *semi-annually* compounded, $100 grows to $110.25 after 1 year, and $121.55 after 2 years:

\[
100 \times (1.05)^2 = 110.25 \\
100 \times (1.05)^4 = 121.55
\]
Annual vs. Semi-Annual Compounding...

After $T$ years, at annually compounded rate $r_A$, $P$ grows to

$$F = P (1 + r_A)^T$$

Present value of $F$ to be received in $T$ years with annually compounded rate $r_A$ is

$$P = \frac{F}{(1 + r_A)^T}$$

In terms of the semi-annually compounded rate $r$, the formulas become

$$F = P (1 + r/2)^{2T}$$

$$P = \frac{F}{(1 + r/2)^{2T}}$$

The key: $(1 + r/2)^2 = 1 + r_A$

An (annualized) semi-annually compounded rate of $r$ per year really means $r/2$ every six months.

Zero Rates

- If you buy a $t$-year zero and hold it to maturity, you lend at rate $r_t$ where $r_t$ is defined by

  $$d_t \times (1 + r_t/2)^{2t} = 1,$$

  or

  $$d_t = \frac{1}{(1 + r_t/2)^{2t}}.$$

  or $r_t = 2 \times \left(\frac{1}{d_t^{2t}} - 1\right)$

- Call $r_t$ the $t$-year zero rate or $t$-year discount rate.
Class Problems: Rate to Price

- According to market convention, zero prices are quoted using rates. Sample STRIPS rates from our historic WSJ:
  - 0.5-year rate: 5.54%
  - 1-year rate: 5.45%

1) What is the 0.5-year zero price?

2) What is the 1-year zero price?

Class Problems: Price to Rate

1) The 1-year zero price implied from coupon bond prices was 0.947665. What was the “implied zero rate?”

2) In today’s market, the 5-year zero price is 0.9075. What is the 5-year zero rate?
Recall that any asset with fixed cash flows can be viewed as a portfolio of zeroes.

So its price must be the sum of its cash flows multiplied by the relevant zero prices:

$$V = \sum_{j=1}^{n} K_j \times d_j$$

Equivalently, the price is the sum of the present values of the cash flows, discounted at the zero rates for the cash flow dates:

$$V = \sum_{j=1}^{n} \frac{K_j}{(1 + r_j / 2)^{2t_j}}$$

**Example**

$10,000 par of a one and a half year, 8.5% Treasury bond makes the following payments:

<table>
<thead>
<tr>
<th></th>
<th>$425</th>
<th>$425</th>
<th>$10,425</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>1 year</td>
<td>1.5 years</td>
<td></td>
</tr>
</tbody>
</table>

Using STRIPS rates from the WSJ to value these cash flows:

$$V = \frac{\$425}{(1 + 0.0554/ 2)^1} + \frac{\$425}{(1 + 0.0545/ 2)^2} + \frac{\$10425}{(1 + 0.0547/ 2)^3}$$

$$= \$10,430$$