American Options

Detailed outline

1. Payoff process
2. Exercise policy (stopping time)
3. Optimal stopping problem
4. Option value (super-replication, Snell envelope)
5. No-early-exercise condition
6. Markovian setting
   (a) Continuation region, exercise region
   (b) Bellman equation, variational inequality
   (c) Exercise boundary, smooth-pasting condition
7. American puts
8. American calls

Readings

Domenico Cuoco’s lecture notes, part V.

Duffie, chapter 8, sections G-H.


Problems

1. Prove that if an American option’s discounted payoff process $\beta G$ is a $\mathcal{P}^*$-submartingale, then it is optimal not to exercise early.

2. Consider a complete, continuous-time financial market where $B^*_t$ is a standard 2-dimensional Brownian motion under the martingale measure $\mathcal{P}^*$. The interest rate $r_t$ is a nonnegative one-factor diffusion described by the equation

$$dr_t = \mu(r_t, t) dt + \sigma_r(r_t, t) dB^*_{1,t},$$

where $\mu$ and $\sigma_r$ are continuous and satisfy Lipschitz and linear growth conditions. The price of a stock, $S_t$, satisfies

$$\frac{dS_t}{S_t} = (r_t - \delta) dt + \sigma dB^*_{2,t},$$

where $\delta > 0$ and $\sigma \neq 0$ are constants. Let $c(S_t, r_t, t)$ be the time $t$ price of an American call on the stock with strike price $k$ and expiration date $T$.

(a) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$ and for all $s_2 > s_1 > 0$,

$$0 \leq \frac{c(s_2, r, t) - c(s_1, r, t)}{s_2 - s_1} \leq 1.$$

(b) Prove that for each $(r, t) \in \mathcal{R}^+ \times [0, T)$, if it is optimal to continue at $(s_2, r, t)$, then it is optimal to continue at $(s_1, r, t)$ for all $s_1 < s_2$. 