COMPOUND INTEREST CALCULATIONS

Suppose that \$1,000 is invested for one year at simple interest of 5%. After one year, this $1,000 \text{ grows to } 1,000 \times (1 + 0.05) = 1,050$.

The simple interest calculation can be applied to periods other than one year through the formula I = P r t, where P is the principal, r is the interest rate (as a decimal such as 0.05), and t is the time in years. Thus, \$1,000 invested for ten years at simple interest of 5% earns interest of \$1,000 × 0.05 × 10 = \$500. The \$1,000 would grow to \$1,500 in ten years.

Interest can be compounded, of course. Consider the \$1,000 invested for one year at 5%, but now suppose that the interest is compounded quarterly. This means that the interest

rate of $\frac{5\%}{4} = 1.25\% = 0.0125$ is applied four times. The \$1,000 grows to \$1,000 × $(1.0125)^4 \approx $1,050.95$. You can see that for this one-year investment, the quarterly compounding has provided you with 95¢ over simple interest.

Compounding makes a serious difference over a long period, of course. Consider the \$1,000 invested at 5% for 10 years, compounded quarterly. Now the investment grows to $$1,000 \times (1.0125)^{40} \approx $1,643.62$. This is seriously larger than the \$1,500 obtained through simple interest.

Do not confuse simple interest with annual compounding. If you invest \$1,000 for 10 years at 5% with annual compounding. The value grows to $(1.05)^{10} \approx (1.05)^{10} \approx 1.628.89$

1

COMPOUND INTEREST CALCULATIONS

Here's a summary so far:

\$1,000 invested at 5%			
Time period	Interest calculation rule	Value at end	
1 year	Simple	\$1,050.00	
1 year	Annual compounding	\$1,050.00	
1 year	Quarterly compounding	\$1,050.95	
10 years	Simple	\$1,500.00	
10 years	Annual compounding	\$1,628.89	
10 years	Quarterly compounding	\$1,643.62	

What would happen if the interest were compounded monthly? Then the \$1,000 over one year would grow to $1,000 \times \left(1 + \frac{0.05}{12}\right)^{12} \approx 1,051.16$. It's interesting to look at the value after one year, as a function of the number of compoundings:

Principal of \$1,000 at 5%		
Number of	Value after one	
compoundings	year	
1	\$1,050.00	
2 (bi-annually)	\$1,050.62	
4 (quarterly)	\$1,050.95	
12 (monthly)	\$1,051.16	
52 (weekly)	\$1,051.25	
360 (daily)	\$1,051.27	
(continuous)	\$1,051.2711	

It seems that compounding many, many times becomes futile. Suppose that we consider n compoundings with the interest rate at r. The value of principal P after one year is

 $P\left(1+\frac{r}{n}\right)^n$. However, $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$. This is the basis of *continuous* compounding. The value after one year will be $P e^r$. Indeed, the value after t years is $P e^{rt}$.

We can help this out with subscripts. Let

 P_0 = value now (or at start)

 P_t = value *t* years later

Then $P_t = P_0 e^{rt}$.

s

The formula used in reverse can be described as the *present value*. If the interest rate is *r*, then the value today of P_t at time *t* from now is given by $P_0 = P_t e^{-rt}$.

We generally use continuous compounding because

- 1. The formula $P_t = P_0 e^{rt}$ is very easy to use.
- 2. There is no need to specify the compounding times. For instance, in quarterly compounding, you have to specify the four calendar dates on which the compounding occurs.
- 3. There is no need for rules about the principal amount on which the interest is applied.

For instance, suppose that compounding occurs on the last days of March, June, September, and December. You deposit \$1,000 on January 1, but you withdraw \$200 on March 18. How should the interest be computed on March 31?

4. Interest needs only to be computed on the dates of transactions. In the example in item 3, note that March 18 is the 77th day of the

year. The initial \$1,000 would be changed to $$1,000 \times e^{0.05 \times \frac{77}{360}} \approx $1,010.75$ on March 18, and then the \$200 withdrawal would reduce it to \$810.75. The arithmetic need not be done again until there is another deposit or withdrawal. By the way, it's customary to use 360-day years for this.

This should, of course, settle all the issues about interest calculations. People are not very good, however, at doing exponential calculations in their heads, so there are approximations.

These should be explored.

Let's say you get a 10% raise, then you get a 10% pay cut. Are you better off, worse off, or the same?

Suppose that your base salary is *B*. The actual value of *B* does not matter.

The 10% raise takes you to 1.1 *B*. The 10% reduction gets you to $(1 - 0.10) \times 1.1 \times B = 0.99 B$. That's worse by 1% than where you started. You may have observed that having the 10% cut first and then the 10% raise later is exactly the same.

The difference is trifling. Suppose however that it were a 60% raise and a 60% cut. That would leave you with $0.40 \times 1.60 \times B = 0.64 B$. Overall, that's a serious 36% pay cut.

These examples are all $(1-x)(1+x) = 1-x^2$. This is going to be less than 1. However, when x is small (like 10%) it's quite immaterial.



Suppose that you invest \$1,000 at 5% interest, compounded continuously, for six years. This will grow to $P_6 = \$1,000 \times e^{0.05 \times 6} = \$1,000 \times e^{0.30} \approx \$1,349.86$. The simple interest calculation can be done without paper. The \$1,000 will generate $0.05 \times 6 = 0.30$ in interest; thus the \$1,000 will earn \$300 in interest and grow to \$1,300. This is off target by \$49.86, but it's still a "ballpark" figure.

You may be aware of the mathematical form $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ This lets us examine $\$1,000 \times e^{0.30} = \$1,000 \left\{ 1 + 0.30 + \frac{0.30^2}{2} + \frac{0.30^3}{6} + \dots \right\}$

 $= \$1,000 \{1 + 0.30 + 0.045 + 0.0045 +\} = \$1,000 + \$300 + \$45 + \$4.50 + ...$

The calculation through the term with 0.30 gets the simple interest value \$1,300. Extending through the 0.30^2 term gets to \$1,345. Extending beyond to the 0.30^3 term gets the value as \$1,349.50. This is now 36¢ away from the continuous compounding value.

Let's suppose that you invest P_0 at rate r and you want to know how long it will take to double. That is, you want to solve for t in $P_t = 2 P_0 = P_0 e^{rt}$. The equation becomes just $2 = e^{rt}$. Logarithms (base e, of course) will give you $\log 2 = rt$. But $\log 2 \approx 0.6931 \approx 0.69$, so the time to double is then $t = \frac{0.69}{r}$. Let's agree, just for this formula, to think of interest rates without the % sign, meaning that a 3.8% interest rate will be written as 3.8 rather than 0.038. Then we'd have the "rule of 69" for the doubling time: $t = \frac{69}{r}$. As long as we're making approximations, let's change the 69 to 72; this gives the "rule of 72" as $t = \frac{72}{r}$.

An interest rate of 6% will lead to a doubling in 12 years. An interest rate of 4% will lead to a doubling in 18 years. An interest rate of 3% will lead to a doubling in 24 years.

The math gets more complicated for financial instruments that involve regular payments. Suppose that you needed a mortgage for \$400,000. You'd agree to make 360 monthly payments, each of amount M. If the interest rate is r, the value of the payment that you would make at 5 years, 3 months, is valued at $M e^{-5.25 r}$. The value of the mortgage is the sum over 360 such terms, and the value of M will be tuned to the target value \$400,000. This is, by the way, a calculation that you can do; it involves the sum of a finite geometric series.

