FRACTIONAL COINTEGRATION

A process is integrated of order \( d \), denoted by \( I(d) \), if its \( k \)-th difference has spectral density

\[
f(\lambda) \sim C|\lambda|^{-2(d-k)} \quad , \quad \lambda \to 0,
\]

where \( C > 0 \), and \( k \) is a nonnegative integer such that \( d - k < 1/2 \). Here, \( d \) is the memory parameter. An \( I(d) \) process without deterministic trends is weakly stationary if \( d < 1/2 \) and nonstationary otherwise. We say that \( \{X_t\} \) and \( \{Y_t\} \) are fractionally cointegrated if both processes are \( I(d) \) and there exists a linear combination \( U_t = Y_t - \beta X_t \) such that \( \{U_t\} \) is \( I(d_U) \), with \( d_U < d \). Fractional cointegration is a generalization of standard cointegration, where \( d = 1 \) and \( d_U = 0 \). Both fractional and standard cointegration were originally defined simultaneously in Engle and Granger (1987), but standard cointegration has been studied far more extensively. Standard cointegration allows only integer values for the memory parameter, and tests for the existence of cointegration rely on unit root theory. The fractional cointegration framework is more general since it allows the memory parameter to take fractional values and \( d - d_U \) to be any positive real number.

Fractional cointegration analysis often focuses on the reduction of the memory parameter from \( d \geq 1/2 \) to \( d_U < 1/2 \), since cointegration is commonly thought of as a stationary relationship between nonstationary variables. But cases where \( d < 1/2 \) are also of interest, particularly if one wishes to study fractional cointegration in volatility. A popular method for estimating the cointegration parameter \( \beta \) in standard cointegration analysis is the ordinary least squares (OLS) estimator. Robinson (1994) noted that for \( 0 < d < 1/2 \), the OLS estimator will in general be inconsistent in the presence of correlation between \( \{X_t\} \), \( \{U_t\} \), and he proposed a narrow-band least squares estimator (NBLSE) of \( \beta \) in the frequency domain, further studied in Robinson and Marinucci (2001).

Many economic series possess linear trends. In their original definition of cointegration, Engle and Granger (1987) excluded deterministic components. Furthermore, the narrowband estimator of Robinson and Marinucci (2001) does not allow for the possibility of deterministic trends in the levels.

Chen and Hurvich (2003) propose a tapered narrow-band least squares estimator of the cointegration parameter \( \beta \), based on a family of tapers introduced in Hurvich and Chen (2000). The new estimator is invariant with respect to deterministic polynomial trends in the series. In the proposed method, the data are differenced, \( p - 1 \) times in the presence of potential \( (p - 1)^{th} \) order polynomial trends. The differenced data are then tapered, i.e., multiplied by a sequence of constants, which depend on \( p \), where \( p \) is a positive integer.

The Tapered Narrow-Band Least Squares Estimator

Suppose that the observed series \( \{X_t, Y_t\} \) consists of two \( I(d) \) components with \( d \in (-0.5, p - 0.5) \), where \( p \geq 1 \) is a fixed integer, and the series may have additive deterministic polynomial trends of order less than or equal to \( (p - 1) \). The stochastic component of the process is nonstationary when \( d \geq 1/2 \). A widely used technique for detrending and inducing stationarity is differencing. The \( (p - 1)^{th} \) difference will convert the memory parameter to \( d - p + 1 \), and will completely remove a polynomial trend of the form described above, converting the trend into a constant. However, overdifferencing may arise as an unintended consequence of differencing, causing problems such as bias in parameter estimation. Tapering reduces this bias.

Suppose that observations on \( \{X_t\} \) and \( \{Y_t\} \) are available for \( t = -p + 2, \ldots , n \). Equivalently, the \( (p - 1)^{th} \) differences \( \{x_t\} \) and \( \{y_t\} \), \( t = 1, \ldots , n \), are generated from weakly stationary processes \( \{x_t\} = \)}
\( \{ \Delta^{p-1} X_t \} \) and \( \{ y_t \} = \{ \Delta^{p-1} Y_t \} \), with common memory parameter \( d_x = d_y = d-p+1 \in (-p+1/2, 1/2) \), where \( \Delta \) is the differencing operator.

Note that if \( \{ X_t \} \) and \( \{ Y_t \} \) are fractionally cointegrated with cointegration parameter \( \beta \), then \( \{ u_t \} = \{ \Delta^{p-1} U_t \} \). The memory parameter of \( \{ u_t \} \) is \( d_u \), and the degree of cointegration between \( \{ x_t \} \) and \( \{ y_t \} \) is \( d_x - d_u \).

The complex-valued taper in Hurvich and Chen (2000) is given by
\[
h_t = 0.5 \left( 1 - e^{i2\pi(t-0.5)/n} \right) , \quad t = 1, \ldots, n .
\]
Define the tapered discrete Fourier transform of a series \( \{ \xi_t \} \) by
\[
w^T_{\xi, j} = \frac{1}{\sqrt{2\pi \sum |h_t^{p-1}|^2}} \sum_{t=1}^{n} h_t^{p-1} \xi_t e^{i\lambda_j t} .
\]
and the tapered cross-periodogram of \( \{ \xi_t \} \) and \( \{ \zeta_t \} \) by
\[
I^{T}_{\xi, \zeta, j} = w^T_{\xi, j} w^T_{\zeta, j} .
\]

Since \( w^T_{\xi, j} \) can be written as a linear combination of \( w_{\xi, j}, \ldots, w_{\xi, j+p-1} \) (see Hurvich and Chen 2000), it follows that the tapered Fourier transform values at nonzero Fourier frequencies are invariant to the mean of the series. Thus, the tapered DFT, tapered periodogram and tapered cross-periodogram based on the \((p-1)\)th differences \( \{ x_t \}, \{ y_t \} \) are invariant to \((p-1)\)th degree polynomial trends in the levels \( \{ X_t \}, \{ Y_t \} \).

Next, define the averaged tapered periodogram
\[
\hat{F}^{T}_{\xi, \zeta}(m) = \frac{2\pi}{n} \sum_{j=1}^{m} \text{Re} \left\{ I^{T}_{\xi, \zeta, j} \right\} \quad 1 \leq m < n/2 .
\]
The Chen and Hurvich (2003) tapered estimator of the cointegration parameter \( \beta \) is
\[
\hat{\beta}^{T}_m = \hat{F}^{T}_{xy}(m)/\hat{F}^{T}_{xx}(m) \quad \text{, (1)}
\]
where \( m \geq 1 \) is fixed.

If instead of holding \( m \) fixed we take \( m = n/2 \) and avoid differencing and tapering, we obtain the ordinary least squares (OLS) estimator. The estimator of Robinson and Marinucci (2001) is also of form (1) but there is no tapering or differencing, and \( m \) tends to \( \infty \) with \( m/n \to 0 \).

We can interpret \( \hat{\beta}^{T}_m \) as estimating the unknown \( \beta \) in the regression model
\[
y_t = \beta x_t + u_t \quad , \quad t = 1, 2, \ldots . \quad \text{(2)}
\]
The mean-invariance of the tapered DFT allows us to ignore the intercept which would otherwise need to be included in (2). Equations (1) and (2) imply that
\[
\hat{\beta}^{T}_m - \beta = \hat{F}^{T}_{xu}(m)/\hat{F}^{T}_{xx}(m) \quad \text{. (3)}
\]
Chen and Hurvich (2003) showed that, under appropriate conditions, if \( m \) is fixed, \( n^{d_x - d_u} (\hat{\beta}^T_m - \beta) \) converges in distribution to a continuous, non-Gaussian limit. Note that if cointegration is present, \( d_x - d_u \) is positive, and thus \((\hat{\beta}^T_m - \beta)\) converges in probability to zero at rate \( n^{d_x - d_u} \). This rate becomes \( n^{-1} \) under classical cointegration, the same (fast) rate as attained by OLS when it works.

In general, OLS may be inconsistent, and \( \hat{\beta}^T_m \) with \( m \) fixed is superior in terms of rate of convergence to the estimator of Robinson and Marinucci (2001). Thus, the first \( m \) Fourier frequencies with \( m \) fixed contain all of the useful information for semiparametric estimation of fractional cointegration.

**Data Analysis: Interest Rates**

We consider the daily US Treasury rate, at constant maturities of 3 years, 5 years and 10 years, from 7/1/1969 to 10/8/2004, \( n = 8805 \). We worked with the returns (differences of the log rates). Log-periodogram regressions with \( m = n^{0.8} \) suggest that the returns all have memory parameter \( d_x = 0 \). The estimates (and \( t \)-statistics) were .017 (.97), .004 (.22) and .01 (.59) for maturities of 3 years, 7 years and 10 years, respectively.

Next, we computed the Chen-Hurvich (2003) estimates of \( \beta \) on the returns, without tapering since our final results indicate that tapering is not needed here. We tried \( m = 5 \) and \( m = 10 \), and the results were very similar. We therefore decided to use \( m = 5 \) since the theoretical results support the use of small, fixed \( m \). The estimates of \( \beta \) were .656, .544 and .872 for the choices of \( \{x_t\}, \{y_t\} \) corresponding to (3 Years, 7 years), (3 years, 10 years) and (7 years, 10 years), respectively. Note that these values are all quite different from 1, though it is not easy to perform a test.

Next, we computed the residuals based on the estimated values of \( \beta \). For example, the residual series for the (3 Years, 7 Years) pair is defined as: 7 year returns $- .656$ (3 Year returns).

Finally, we used a log-periodogram regression to estimate the memory parameter \( d_u \) for the three residual series. Since \( d_x \) is apparently zero, \( d_u \) represents the degree of cointegration for the given pair of series. A negative value of \( d_u \) would correspond to fractional cointegration. We used \( m = n^{0.7} \) for the estimates since our theory indicates that values of \( m \) that are too large in this context may bias the estimate of \( d_u \) towards zero. The resulting estimates of \( d_u \) (and \( t \)-statistics) are $- .058$ ($- .22$), $- .045$ ($- .18$) and $- .121$ ($- .45$) for the pairs (3 Years, 7 years), (3 years, 10 years) and (7 years, 10 years), respectively.

We find, therefore, that there is very strong evidence of fractional cointegration for the pair (7 years, 10 years), reasonably strong evidence for the pair (3 years, 7 years), and weak evidence for the pair (3 years, 10 years). Apparently, the shorter the time interval between the maturities, the stronger the evidence of fractional cointegration.

It is important to note that even when evidence for fractional cointegration was strong, the estimated degree of fractional cointegration was small. In no case did we find evidence of classical cointegration (\( d_u = -1 \)). Apparently, if there is any type of cointegration among interest rates, it is fractional. Thus, pairs of interest rate series revert quite slowly to an equilibrium, much more slowly than would be suggested by the exponential rate which would hold under classical cointegration.