3. PROBABILITY

"Anything that is measured contains some degree of randomness." --Cliff Hurvich

Do you agree?

**Probability** shows you the likelihood, or chances, for a future event, based on a set of assumptions about how the world works.

**Probability**
- Allows you to handle randomness (uncertainty)
- Is a central concept for risk management
- Is used for calculating expectations
- Is key for statistical inference (drawing conclusions from data).

If you toss a coin, what is the probability of getting a head?

Explain your answer in two different ways.
What did you mean by “probability”?

Consider a **random experiment**, such as tossing a coin many times.

If A is an **event** (something that may happen in the experiment) then P(A), the probability of A, is a number between 0 and 1.

- P(A) = 0 represents impossibility. P(A) = 1 represents certainty.

The **sample space** S is the set of all possible individual outcomes of the experiment.

- If all individual outcomes in S are equally likely then

  \[
  P(A) = \frac{\text{# Individual Outcomes in } A}{\text{# Individual Outcomes in } S}
  \]

**Eg:** Toss a coin twice. The sample space is \(S=\{HH,HT,TH,TT\}\).

Each of the four individual outcomes is equally likely. Can check this by actually tossing coins. Or can assume that successive tosses are independent.

(More on independence later.)

For the event \(A=\{\text{Exactly one Head}\}=\{HT,TH\}\), we have \(P(A)=2/4=1/2\).
If we can’t enumerate all the possibilities, estimate $P(A)$ by the observed relative frequency:

$P(A) \approx \frac{\text{Proportion of the time the event has occurred in the past}}{2} \approx 0.015$.

**Eg:** If there have been 135 launches of the Space Shuttle, and two of these resulted in a catastrophic failure, we can estimate the probability that the next launch will fail to be $2/135 = 0.015$.

**Union, Intersection, Complement**

Combining Events: The **union** $A \cup B$ is the event consisting of all outcomes in $A$ or in $B$ or in both.

The **intersection** $A \cap B$ is the event consisting of all outcomes in both $A$ and $B$.

If $A \cap B$ contains no outcomes (i.e., if $A$ and $B$ have no elements in common), then $A$, $B$ are said to be **mutually exclusive**.

The **complement** of the event $A$ consists of all outcomes in the sample space $S$ which are not in $A$.

- $P(A \cup B) = P(A \text{ or } B \text{ or both occur})$
- $P(A \cap B) = P(A \text{ and } B \text{ both occur})$
- $P(\overline{A}) = P(A \text{ does not occur}).$

**Eg 1:** In a taste test of soft drink preferences, a subject is given Coke, Pepsi and Sprite, and asked to state their preference, if any.

The sample space of possible outcomes is $S = \{\text{Coke, Pepsi, Sprite, No Preference}\}$.

Suppose $A = \{\text{Coke}\}$, $B = \{\text{Cola}\} = \{\text{Coke, Pepsi}\}$, $C = \{\text{Pepsi}\}$.

What are the events $A \cup C$, $A \cap B$, $A \cap C$, and $B$?

- Complement Rule: $P(\overline{A}) = 1 - P(A)$
- Addition Rule: $P( A \cup B ) = P(A) + P(B) - P(A \cap B)$.

If $A$, $B$ are mutually exclusive, then $P(A \cap B) = 0$, since $A \cap B$ contains no outcomes. This gives:

Addition Rule for mutually exclusive events:

$P(A \cup B) = P(A) + P(B)$,

if $A$, $B$ are mutually exclusive.
Eg 2: In the taste test, assume the following probabilities:

<table>
<thead>
<tr>
<th>Preference</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coke</td>
<td>0.3</td>
</tr>
<tr>
<td>Pepsi</td>
<td>0.4</td>
</tr>
<tr>
<td>Sprite</td>
<td>0.2</td>
</tr>
<tr>
<td>None</td>
<td>0.1</td>
</tr>
</tbody>
</table>

If $A = \{\text{Coke}\}$, $B = \{\text{Cola}\}$, $C = \{\text{Pepsi}\}$, compute $P(A \cup C)$, $P(A \cap B)$, $P(A \cap C)$, $P(A \cup B)$, $P(B)$.

Conditional Probability and Independence

Eg 3: Performance of mutual funds (Goetzmann and Ibbotson)

<table>
<thead>
<tr>
<th></th>
<th>Current Winner</th>
<th>Current Loser</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Past Winner</td>
<td>482</td>
<td>296</td>
<td>778</td>
</tr>
<tr>
<td>Past Loser</td>
<td>285</td>
<td>493</td>
<td>778</td>
</tr>
<tr>
<td>Total</td>
<td>767</td>
<td>789</td>
<td>1556</td>
</tr>
</tbody>
</table>

Compute the probability that a randomly selected mutual fund is a current winner.

If we are told that the fund was a past winner, does this partial knowledge change the probability that the fund is a current winner?

**Conditional Probability:** If $A$, $B$ are events with $P(B) > 0$, then the conditional probability that $A$ occurs, given that $B$ has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

**Independence:** $A$ and $B$ are independent if $P(A|B) = P(A)$. Otherwise they are dependent.

- A and B are independent if the knowledge that B has occurred doesn't change the probability that A will occur.

**Multiplication Rules:**

- If $A$ and $B$ are independent, then $P(A \cap B) = P(A)P(B)$.
- In General, $P(A \cap B) = P(A)P(B|A)$. 
Examples of conditional probability:

- Click-through rate (given search/visit history).
- Default probability on a loan (given credit score).
- The current value of a prediction market contract.

- "The probability of a Fed rate increase at its December meeting moved up to 67 percent, according to futures data compiled by Bloomberg."

Models that yield conditional probabilities:

- Linear Regression
- Logistic Regression

Eg 4: Classification of Managers (Past 3 Years)

<table>
<thead>
<tr>
<th></th>
<th>Promoted</th>
<th>Not Promoted</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>46</td>
<td>184</td>
<td>230</td>
</tr>
<tr>
<td>Female</td>
<td>8</td>
<td>32</td>
<td>40</td>
</tr>
<tr>
<td>Total</td>
<td>54</td>
<td>216</td>
<td>270</td>
</tr>
</tbody>
</table>

Note that only 8 out of 54 promotions went to women. Are the women being unfairly passed over for promotions?

Sol: No, since \( \Pr\{\text{Promoted}|\text{Female}\}=8/40=0.2=54/270=\Pr\{\text{Promoted}\}. \)

Alternatively:
\[
\Pr\{\text{Male}|\text{Promoted}\}=46/54=0.852=230/270=\Pr\{\text{Male}\}.
\]

Promotion status and gender are independent.

Eg: A coin is tossed twice. Suppose all outcomes (HH, HT, TH, TT) are equally likely.

If \( A = \{\text{Heads on First Toss}\} \) and \( B = \{\text{Heads on Second Toss}\} \), then \( A \) and \( B \) are independent, since

\[
P(A \cap B) = P(HH) = 1/4 = (1/2)(1/2) = P(A) \cdot P(B).
\]

Equivalently, \( P(B|A) = P(B) \), so the coin “doesn't remember” what the first toss was.

Eg: A “100-Year Flood” will occur in a given year with probability 1/100. The chance that there will be at least one “100-Year Flood” in the next 100 years (assuming results in each year are independent) is

\[
1 - \Pr\{\text{No 100-Year Flood in next 100 Years}\} = 1 - (99/100)^{100} = 0.63.
\]
Eg: HIV Screening Test (Wellcome Elisa).

Suppose that HIV has incidence of 25 per million in general population without known risk factors, and the screening test has a false positive rate of 0.01% and a false negative rate of 0.8%.

If someone tests positive for HIV, does this information change the probability that they are actually HIV-positive?

Does the probability increase? Is it over 99%?

“False Positive Rate” = Prob{Test Positive | HIV Negative}

“False Negative Rate” = Prob{Test Negative | HIV Positive}

<table>
<thead>
<tr>
<th></th>
<th>HIV Positive</th>
<th>HIV Negative</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Positive</td>
<td>248</td>
<td>1000</td>
<td>1248</td>
</tr>
<tr>
<td>Test Negative</td>
<td>2</td>
<td>9998750</td>
<td>9998752</td>
</tr>
<tr>
<td>TOTAL</td>
<td>250</td>
<td>9999750</td>
<td>10000000</td>
</tr>
</tbody>
</table>

Given a positive test result, 
Prob{HIV Positive | Test Positive} = 248/1248 = 0.20. Only 20% of those testing positive are actually HIV-Positive! The other 80% of the positives are false positives.

• Prob{Test Pos | HIV Pos} and Prob{HIV Pos | Test Pos} are two very different things (99.2% vs. 20%).

• “Independent” and “Mutually Exclusive” are not the same thing.

[Eg 5, Eg 6]

Making sound business decisions requires us to understand what types of patterns can be generated by randomness alone. To gauge our understanding of randomness, let’s do an experiment.

Coin-Tossing Demo

Toss an imaginary and real coin 25 times. Graph the running totals. I try to tell the difference.