

SOME DRAWBACKS OF BLACK-SCHOLES

To provide one motivation for the development of ARCH models (next handout), we briefly discuss here some difficulties associated with the Black Scholes formula, which is widely used to calculate the price of an option. For example, consider a European call option for a stock. This is the right to buy a specific number of shares of a specific stock on a specific date in the future, at a specific price (the exercise price, also called the strike price). If all these quantities are fixed, the question becomes: what is a fair price to charge for the option?

The Black-Scholes formula gives the price of the option, in terms of other quantities, which are assumed known. These include the exercise price and the current price of the stock. The formula is derived under the assumption that the time interval between observations is very small, and that the log prices follow a random walk with normally distributed innovations. The formula is not affected by any linear drift in the random walk. The model for the stock prices themselves is called geometric Brownian motion.

A key input to the Black-Scholes formula is σ , the standard deviation of the stock's continuously compounded rate of return. (The continuously compounded rate of return is the change in the log price divided by the length of time elapsed). If the time interval between observations is sufficiently small, σ is just the standard deviation of the innovations in the random walk, so σ can be viewed as a measure of the volatility of the stock price. According to the random walk model, σ must remain constant over time. Its value will not be known, however, so it is usually estimated from the available data.

Several of the assumptions used in the Black-Scholes method may be unrealistic. First, the geometric Brownian motion model implies that the series of first differences of the log prices must be uncorrelated. But for the S&P 500 as a whole, observed over several decades, daily from 1 July 1962 to 29 Dec 1995, there are in fact small but statistically significant correlations in the differences of the logs at short time lags. (See the attached plot). (Of course, the detection of these correlations will require a large amount of data, as well as the assumption of stationarity, i.e. that the correlations have not changed over time).

The next question is whether the innovations (say, the returns) are normally distributed. The answer to this is almost surely NO! Experience has shown that returns are *leptokurtic*, i.e., have much more of a tendency to exhibit outliers than would be the case if they were normally distributed. An example is provided by the returns on the S&P 500 series. The attached Minitab Descriptive Statistics output shows a sample kurtosis of 43.69. By contrast, the true kurtosis of any normal random variable is just 3. The p -value on the Anderson-Darling Normality Test is 0.000, indicating that we can strongly reject the null hypothesis that the returns are normal. The graphical part of the output shows a histogram of the returns, with a superimposed normal curve matching the mean and variance observed in the data. The fit is not good, in that the histogram has much more mass in the center than the normal, and there is a large amount of data located several standard deviations from the mean, i.e., far in the tails of the normal distribution. This can be seen with the help of the boxplot of the returns located below the histogram. So there is overwhelming evidence that the returns are not normal, but instead have a leptokurtic (i.e., long-tailed) distribution.

Finally, there is the question of constancy of the variance of the innovations. In practice, it is often found that for financial time series, after taking logs (if needed) and first differences, the level of volatility (i.e., fluctuation) seems to change with time. Often, periods of high volatility follow immediately after a large change (often downward) in the level of the original series. It may take quite some time for this heightened volatility to subside. For example, the plot of differences of the logs of the S&P 500 shows very long periods of high volatility interspersed with periods of relative calm. This type of pattern is often referred to as volatility clustering. A way to measure this clustering is to look at the autocorrelations of the SQUARES of the differences of the logs (or of the squared returns). If there is volatility clustering, these autocorrelations should be significant for many lags, so that a shock to the volatility persists for many periods into the future. This is the case for the S&P 500.

Similar findings of leptokurtosis and volatility clustering have been obtained for the differences of the logs of individual stock prices, exchange rates, and interest rates.

Clearly, if any of its underlying assumptions is violated, the Black-Scholes formula may be invalid, so the computed options prices may not be fair. In particular, an estimate of variance based on

historical data may severely detract from the fairness of the computed options price if the stock has just entered a highly volatile phase. Leptokurtosis in the innovations can produce further pricing biases. Hull, in his book "Options, Futures, and Other Derivatives", 3rd Edition, pp. 492-495 (attached) discusses how long tails in the original stock price distribution relative to a log-normal distribution can cause Black-Scholes to systematically under-price or over-price an option.

In spite of these problems, and because of its simplicity, Black-Scholes is still very widely used, but with adjustments to account for the inadequacy of the model on which it is based. To some extent, the inadequacies can be gauged from looking at "implied volatility", i.e., the value of σ inferred from using the Black-Scholes formula in reverse, based on the observed option prices. If the model underlying Black-Scholes (i.e., that stock prices are a geometric Brownian motion) actually held, then implied volatility should be constant with respect to both strike price ("moneyness") and maturity. In fact, implied volatility changes with strike price (giving a "volatility smile"), and with maturity (giving a "volatility term structure"). These are discussed in Hull, pp. 502-504 (attached).

Although Black-Scholes will undoubtedly continue to be used for many years, it is very clear that the model underlying its use is strongly at odds with the observed data. In the following handouts, we will consider more appropriate models for innovations of financial time series (ARCH-GARCH models), and discuss ways of pricing options under such a model.