Lecture Notes 7

Optimal Risky Portfolios: Efficient Diversification

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II. Correlation Revisited: A Few Graphical Examples

III. Standard Deviation of Portfolio Return: Two Risky Assets
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V. Impact of Correlation: Two Risky Assets
VI. Portfolio Choice: Two Risky Assets
VII. Portfolio Choice: Combining the Two Risky Asset Portfolio with the Riskless Asset

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IX. Standard Deviation of Portfolio Return: \( n \) Risky Assets
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XII. Portfolio Choice: \( n \) Risky Assets and a Riskless Asset

XIII. Additional Readings

Buzz Words: Minimum Variance Portfolio, Mean Variance Efficient Frontier, Diversifiable (Nonsystematic) Risk, Nondiversifiable (Systematic) Risk, Mutual Funds.
I. Readings and Suggested Practice Problems

BKM, Chapter 8.1-8.6.
Suggested Problems, Chapter 8: 8-14

E-mail: Open the Portfolio Optimizer Programs (2 and 5 risky assets) and experiment with those.

II. Correlation Revisited: A Few Graphical Examples

A. Reminder: Don’t get confused by different notation used for the same quantity:

Notation for Covariance: Cov\[r_1,r_2\] or \(\sigma_{r_1,r_2}\) or \(\sigma_{1,2}\)
Notation for Correlation: Corr\[r_1,r_2\] or \(\rho_{r_1,r_2}\) or \(\rho_{1,2}\)

B. Recall that covariance and correlation between the random return on asset 1 and random return on asset 2 measure how the two random returns behave together.

C. Examples

In the following 5 figures, we Consider 5 different data samples for two stocks:
- For each sample, we plot the realized return on stock 1 against the realized return on stock 2.
- We treat each realization as equally likely, and calculate the correlation, \(\rho\), between the returns on stock 1 and stock 2, as well as the regression of the return on stock 2 (denoted y) on the return on stock 1 (x).
[Note: the regression \(R^2\) equals \(\rho^2\)]
1. A sample of data with $\rho = 0.630$:

![Graph showing linear relationship between Return on Stock 1 and Return on Stock 2 with $y = 0.9482x + 0.0506$, $R^2 = 0.3972$.]

2. A sample of data with $\rho = -0.714$:

![Graph showing linear relationship between Return on Stock 1 and Return on Stock 2 with $y = -0.8613x + 0.0726$, $R^2 = 0.51$.]
3. **Sample with \( \rho = +1 \):**

![Graph showing \( y = 0.02x + 0.05 \) with \( R^2 = 1 \)]

4. **Sample with \( \rho = -1 \):**

![Graph showing \( y = -0.8x + 0.05 \) with \( R^2 = 1 \)]

5. **Sample with \( \rho \approx 0 \):**

![Graph showing \( y = 0.009x + 0.0468 \) with \( R^2 = 0.0001 \)]
Foundations of Finance: Optimal Risky Portfolios: Efficient Diversification

D. Real-Data Example

Us Stocks vs. Bonds 1946-1995,
A sample of data with $\rho = 0.228$:

### Raw Data

<table>
<thead>
<tr>
<th>Year</th>
<th>STB Stocks (S&amp;P 500)</th>
<th>STB Bonds (Long Term US Gov't)</th>
<th>Inflation</th>
<th>Raw Data Excess over T-bill</th>
</tr>
</thead>
<tbody>
<tr>
<td>1946</td>
<td>-8.07%</td>
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<tr>
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<td>5.60%</td>
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<th>50</th>
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<td>8.31%</td>
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<td>3.92%</td>
<td>17.20%</td>
<td>10.13%</td>
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<td>Std.Error Mean</td>
<td>2.34%</td>
<td>1.49%</td>
<td>0.46%</td>
<td>0.54%</td>
<td>2.43%</td>
<td>1.43%</td>
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</table>

\[ y = 0.3592x + 0.1106 \]
\[ R^2 = 0.0522 \]
III. Standard Deviation of Portfolio Return: Two Risky Assets

A. Formula

\[
\sigma^2[r_p(t)] = w_{1,p} \sigma^2[r_1(t)] + w_{2,p} \sigma^2[r_2(t)] + 2 w_{1,p} w_{2,p} \sigma[r_1(t), r_2(t)]
\]

\[
\sigma[r_p(t)] = \sqrt{\sigma^2[r_p(t)]}
\]

where

- \(\sigma[r_1(t), r_2(t)]\) is the covariance of asset 1's return and asset 2's return in period \(t\),
- \(w_{i,p}\) is the weight of asset \(i\) in the portfolio \(p\),
- \(\sigma^2[r_p(t)]\) is the variance of return on portfolio \(p\) in period \(t\).

B. Example

Consider two risky assets. The first one is the stock of Microsoft. The second one itself is a portfolio of Small Firms. The following moments characterize the joint return distribution of these two assets.

- \(E[r_{\text{Small}}] = 1.912\), \(E[r_{\text{Msft}}] = 3.126\),
- \(\sigma[r_{\text{Small}}] = 3.711\), \(\sigma[r_{\text{Msft}}] = 8.203\), \(\sigma[r_{\text{Msft}}, r_{\text{Small}}] = 12.030\)

A portfolio formed with 60% invested in the small firm asset and 40% in Microsoft has standard deviation and expected return given by:

\[
\sigma^2[r_p] = w_{\text{Small},p}^2 \sigma^2[r_{\text{Small}}] + w_{\text{Msft},p}^2 \sigma^2[r_{\text{Msft}}] + 2 w_{\text{Small},p} w_{\text{Msft},p} \sigma[r_{\text{Small}}, r_{\text{Msft}}]
\]

\[
= 0.6^2 \times 3.711^2 + 0.4^2 \times 8.203^2 + 2 \times 0.6 \times 0.4 \times 12.030
\]

\[
= 4.958 + 10.766 + 5.774 = 21.498
\]

\[
\sigma[r_p] = \sqrt{\sigma^2[r_p]} = \sqrt{21.498} = 4.637
\]

\[
E[r_p] = w_{\text{Small},p} E[r_{\text{Small}}] + w_{\text{Msft},p} E[r_{\text{Msft}}] = 0.6 \times 1.912 + 0.4 \times 3.126 = 2.398
\]
IV. Graphical Depiction: Two Risky Assets

A. Representation in the “Mean-Variance Space”

The standard deviation, $\sigma_p$, of a return on a portfolio consisting of asset 1 and asset 2, and the portfolio’s expected return, $E_p$, can be expressed in terms of $w_1$, the weight of asset 1.

When plotting in the Mean-Variance plane $\sigma_p$ and $E_p$ for all possible values of $w_1$, we get a curve.

The curve is known as the portfolio possibility curve -, or as the portfolio frontier -, or as the set of feasible portfolios-, or as the opportunity set - with two risky assets.

An Algorithm to Plot the Portfolio Frontier:

1. Pick a value for $w_1$ (and then $w_2 = 1 - w_1$)

2. Compute expected return and standard deviation:

$$E[r_p] = w_1E[r_1] + w_2E[r_2] = w_1E[r_1] + (1 - w_1)E[r_2]$$

$$\sigma_p = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2\sigma_{1,2}}$$

$$= \sqrt{w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2w_1(1-w_1)\sigma_{1,2}}$$

3. Plot a single point \{ $\sigma_p$, $E[r_p]$ \}

4. Repeat 1-3 for various values of $w_1$
B. Example (cont.)

To get the portfolio possibility curve using the small-firm portfolio and Microsoft equity (i.e., to “get all possible p’s”), the standard deviation of return on a portfolio consisting of the small firm portfolio (asset 1) and Microsoft equity (asset 2) and its expected return can be indexed by the weight of the small firm portfolio within portfolio p: \( w_1 = w_{\text{Small},p} \).

<table>
<thead>
<tr>
<th>( w_{\text{Small},p} )</th>
<th>( w_{\text{Msft},p} )</th>
<th>( \sigma(r_p(t)) )</th>
<th>( E[r_p(t)] )</th>
</tr>
</thead>
<tbody>
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<td>1.2</td>
<td>9.574%</td>
<td>3.369%</td>
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<tr>
<td>0.0</td>
<td>1.0</td>
<td>8.203%</td>
<td>3.126%</td>
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<td>2.398%</td>
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<td>3.919%</td>
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<td>1.0</td>
<td>0.0</td>
<td>3.711%</td>
<td>1.912%</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.2</td>
<td>4.093%</td>
<td>1.670%</td>
</tr>
</tbody>
</table>

Figure here
Note: when one asset is risk-free the set of feasible portfolios is described by the CAL discussed in Lecture Notes 6 (in other words, the CAL together with its “mirror image” – obtained when shorting the risky asset – is the portfolio frontier of one risky asset and one riskless asset).

V. Impact of Correlation: Two Risky Asset Case

A. Standard Deviation Formula Revisited

The standard deviation formula can be rewritten in terms of correlation rather than covariance (using the definition of correlation):

\[ \sigma_p^2 = w_{1,p}^2 \sigma_1^2 + w_{2,p}^2 \sigma_2^2 + 2 w_{1,p} w_{2,p} \rho_{1,2} \sigma_1 \sigma_2 \]

where \( \rho_{1,2} \) is the correlation of asset 1's return and asset 2's return in period \( t \).

For a given portfolio with \( w_{1,p} > 0, w_{2,p} > 0, \) and \( \sigma_1, \sigma_2 \) fixed, \( \sigma_p \) decreases as \( \rho_{1,2} \) decreases.

B. Example (cont.)

Suppose the \( \mathbb{E}[r] \) and \( \sigma[r] \) for the small firm asset and for Microsoft remain the same but the correlation between the two assets is allowed to vary:
VI. Portfolio Choice: Two Risky Assets

A. A risk averse investor is not going to hold any combination of the two risky assets on the negative sloped portion of the portfolio frontier.

1. So the negative-sloped portion is known as the inefficient region of the curve.

2. And the positive-sloped portion is known as the efficient region of the curve, or as the efficient frontier, or as the minimum-variance frontier. A portfolio is efficient if it is on the efficient frontier (i.e., achieves the maximum expected return for a given level of standard deviation).
**B. The exact position on the efficient frontier that an individual holds depends on her tastes and preferences.**

**C. Example (cont.)**

The portfolio possibility curve for the small firm portfolio and Microsoft can be divided into its efficient and inefficient regions.

Any risk averse individual combining the small firm portfolio with Microsoft wants to lie in the efficient region: so wants to invest a positive fraction of her portfolio in Microsoft.

*Figure here*
VII. Portfolio Choice: Combining the Two Risky Asset Portfolio with the Riskless Asset

<table>
<thead>
<tr>
<th>Two-stage Decision Process.</th>
<th>Stage I: Asset Selection</th>
<th>Stage II: Asset Allocation</th>
</tr>
</thead>
</table>

Stage I: Asset Selection

What are the preferred weights of the two risky assets in the risky portfolio?

a. all risk averse individuals want access to the CAL with the largest slope; this involves combining the riskless asset with the same risky portfolio (\(\uparrow\) in the figure below).

b. this same risky portfolio is the one whose CAL is tangent to the efficient frontier; this is why \(\uparrow\) is known as the tangency portfolio, denoted \(T\).

We now know how to select the optimal portfolio of risky assets for asset allocation between risky and riskless assets:

The portfolio, denoted \(P\) in the previous lecture, should be chosen as simply the portfolio \(T\) on the efficient frontier (like the one labeled by \(\uparrow\) in the figure below), with a CAL tangent to the frontier.

Note: The optimal determination of \(P\) and that of the associated CAL is done simultaneously. The best \(P\) is the tangency portfolio \(T\).
c. Can calculate the weight of risky asset 1 in the tangency portfolio $T$ using the following formula:

$$w_{1T} = \frac{\sigma^2_{R_2} E[R_1] - \sigma_{R_1, R_2} E[R_2]}{\sigma^2_{R_2} E[R_1] - \sigma_{R_1, R_2} E[R_2] + \sigma^2_{R_2} E[R_2] - \sigma_{R_1, R_2} E[R_1]}$$

where $R_i = r_i - r_f$ is the excess return on asset $i$ (in excess of the riskless rate).

Stage II: Asset Allocation

What are the preferred weights of the risky portfolio $T$ and the riskless asset in the individual's portfolio?

As we discussed in the previous lecture, the weight of $T$ (★) in an individual's portfolio $w_{T,p}$ depends on the individual's tastes and preferences.

Figure here
VIII. Applications

A. Asset Allocation between Two Broad Classes of Assets

The two-risky-asset formulas can be used to determine how much to invest in each of two broad asset classes.

Example: The Wall Street Journal articles at the end of the previous Lecture Notes show recommendations for a composite portfolio $C$. The risky portfolio within $C$, can be thought of as the one which each strategist believes to be the tangent portfolio $T$. The weights within $T$ of the two broad asset classes “Stocks” and “Bonds” can be determined as above.

(The weights of “Stocks” relative to “Bonds” differ across strategists possibly because each one of them “sees” a different efficient frontier, and hence recommends to its clients a different $T$).

B. International Diversification

The two-risky-asset formulas can also be used when deciding how much to invest in an international equity fund and how much in a U.S. based fund.
IX. Standard Deviation of Portfolio Return: $n$ Risky Assets

A. Portfolios of many assets

There are $n$ risky assets, $i = 1, 2, \ldots, n$

Basic data $(2n + n(n-1)/2$ inputs):

$n$ Expected returns: $E[r_1], E[r_2], \ldots, E[r_n]$

$n$ Standard deviations: $\sigma_1, \sigma_2, \ldots, \sigma_n$

\[
\frac{n(n-1)}{2} \text{ coef. of corr.: } \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, \ldots
\]

Problems:

1. Given $p$ defined by $w_1, w_2, \ldots, w_n$, we know how to compute $E[r_p]$, but what about $\sigma_p$?

2. How do we form efficient portfolios (those which minimize $\sigma[r_p]$ given $E[r_p]$)?

B. Formula

\[
\sigma^2[r_p(t)] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{i,p} w_{j,p} \sigma[r_i(t), r_j(t)] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{i,p} w_{j,p} \sigma[r_i(t)]\sigma[r_j(t)]\rho[r_i(t), r_j(t)]
\]

where

- $\sigma[r_i(t)]$ is the standard deviation of asset $i$’s return in period $t$,
- $\sigma[r_i(t), r_j(t)]$ is the covariance of asset $i$’s return and asset $j$’s return in period $t$,
- $\rho[r_i(t), r_j(t)]$ is the correlation of asset $i$’s return and asset $j$’s return in period $t$;
- $w_{i,p}$ is the weight of asset $i$ in the portfolio $p$;
- $\sigma^2[r_p(t)]$ is the variance of return on portfolio $p$ in period $t$. 

C. Example: A 3-stock portfolio

\[ Er_p = w_1 Er_1 + w_2 Er_2 + w_3 Er_3 \]

\[ \sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2}\]
\[ + 2 w_1 w_3 \sigma_1 \sigma_3 \rho_{1,3} \]
\[ + 2 w_2 w_3 \sigma_2 \sigma_3 \rho_{2,3} \]

X. Effect of Diversification with \( n \) Risky Assets

To understand how to form efficient portfolios, we need to understand first the effect of diversification.

A. The Case of \( n \) Uncorrelated Risky Assets

Suppose all assets have the same expected return \( Er \) and same standard deviation \( \sigma[r] = \sigma \) and have returns which are uncorrelated:

\[ Er_1 = Er_2 = \ldots = Er_n = Er \]

\[ \sigma_1 = \sigma_2 = \ldots = \sigma_n = \sigma \]

\[ \rho_{1,2} = \rho_{2,3} = \rho_{1,3} = \ldots = 0 \]

Since stocks are identical, can a portfolio be better than each stock???

Since stocks are identical, there is nothing to be lost by putting an equal weight on each stock; so we consider an equally weighted portfolio, where \( w_{i,p} = 1/n \) for all \( i \).

Example: when \( n=2 \), an equally weighted portfolio has 50% in each asset.
Then:

With 2 stocks \((n=2)\):

\[
E[r_p(t)] = \frac{1}{2} E[r_1(t)] + \frac{1}{2} E[r_2(t)] = Er
\]
\[
\sigma^2[r_p(t)] = \left(\frac{1}{2}\right)^2 \sigma^2[r_1(t)] + \left(\frac{1}{2}\right)^2 \sigma^2[r_2(t)] = \frac{1}{2} \sigma^2
\]

With 3 stocks \((n=3)\):

\[
E[r_p(t)] = \frac{1}{3} E[r_1(t)] + \frac{1}{3} E[r_2(t)] + \frac{1}{3} E[r_3(t)] = Er
\]
\[
\sigma^2[r_p(t)] = \left(\frac{1}{3}\right)^2 \sigma^2[r_1(t)] + \left(\frac{1}{3}\right)^2 \sigma^2[r_2(t)] + \left(\frac{1}{3}\right)^2 \sigma^2[r_3(t)] = \frac{1}{3} \sigma^2
\]

Arbitrary \(n\):

\[
E[r_p(t)] = Er
\]
\[
\sigma^2[r_p(t)] = \sigma^2/n
\]

As \(n\) increases:

1. the variance of the portfolio declines to zero.
   (all the risk is diversifiable!)
2. the portfolio's expected return is unaffected.

\textit{This is known as the effect of diversification (can think of it as “risk reduction,” or as the “insurance” principle).}
B. The case of \( n \) identical positively correlated assets.

\[
\begin{align*}
Er_1 &= Er_2 = \cdots = Er_n = Er \\
\sigma_1 &= \sigma_2 = \cdots = \sigma_n = \sigma \\
\rho_{1,2} &= \rho_{2,3} = \rho_{1,3} = \cdots = \rho > 0
\end{align*}
\]

In this case the equally weighted \( p \) has

\[
E[r_p] = Er
\]

\[
\sigma_p^2 = \frac{\sigma^2}{n} + \frac{(n-1)\rho\sigma^2}{n}
\]

\[
= \frac{\sigma^2(1-\rho)}{n} + \rho\sigma^2 \xrightarrow{n \to \infty} \rho\sigma^2
\]

\( \sigma^{(1-\rho)/n} \) is the unique / idiosyncratic / firm specific / diversifiable / nonsystematic risk. It can be reduced by combining securities into portfolios. As we diversify into more assets, the risk reduction works for the specific-risk component.

\( \rho\sigma^2 \) is the market / nondiversifiable / systematic risk. This “portion” of risk we cannot diversify away. The lower is the correlation between assets, the lower is the nondiversifiable component.
XI. Opportunity Set: \( n \) Risky Assets

A. Set of Possible Portfolios

Because, in general, there is a limit to diversification, it follows that with \( n \) assets, although we have an infinite set of curves (each as in the two asset case), these are combined into the following general shape:

\[
\begin{align*}
E_r & \quad \text{Efficient set of risky assets} \\
\sigma & 
\end{align*}
\]

B. Minimum Variance (Standard Deviation) Frontier

Since individuals are assumed to have Mean-Variance (MV) preferences, can restrict attention to the set of portfolios with the lowest variance for a given expected return (as we did with 2 assets).

This set is a curve, and it is the minimum variance frontier (MVF) for the \( n \) risky assets.

Every other possible portfolio is dominated by a portfolio on the MVF (lower variance of return for the same expected return).
C. Adding risky assets

1. Adding risky assets to the opportunity set always causes the minimum variance frontier to shift to the left in \( \{\sigma[r], E[r]\} \) space.

   Why? -- For any given \( E[r] \), the portfolio on the MVF for the subset of risky assets is still feasible using the larger set of risky assets. Further, there may be another portfolio which can be formed from the larger set and which has same \( E[r] \) but a lower \( \sigma[r] \).

2. Example 2 (cont. ignoring DP)

   a. MVF for IBM, Apple, Microsoft, Nike and ADM is to the left of the MVF for IBM, Apple, Microsoft and Nike excluding ADM. This happens even though ADM has an \( \{\sigma[r], E[r]\} \) denoted by ✗ which lies to the right of the MVF for the 4 stocks excluding ADM.
XII. Portfolio Choice: \( n \) Risky Assets and a Riskless Asset

A. The analysis for the two risky asset and a riskless asset case applies here:

1. A Mean Variance investor combines the riskless asset with the risky portfolio whose Capital Allocation Line has the highest slope.

2. That risky portfolio is on the efficient frontier for the \( n \) risky assets and is in fact the tangency portfolio \( T \).

Calculating the weights of assets in the tangency portfolio can be performed via computer (see the Spreadsheet Model in BKM ch. 8, pp. 229-235).

3. Investors want to hold this tangency portfolio in combination with the riskless asset.

\[ \text{The associated Capital Allocation Line is the efficient frontier for the } n \text{ risky assets and the riskless asset.} \]

4. Only the weights of the tangency portfolio and the riskless asset in an individual’s portfolio depend on the individual’s tastes and preferences.
XIII. Additional Readings

- The articles about Gold as an investment, illustrate that even though it may be a “bad” investment in isolation, investing in gold makes sense as a hedge, i.e., as an insurance. This means that in some scenarios, perhaps very unlikely ones (like the Y2K computer problem discussed in one article), the gold fraction of the portfolio will help to maintain favorable returns at times of recession. Overall, adding gold improves the efficient frontier, analogously to how adding ADM improved the frontier of IBM, Apple, Microsoft, and Nike in our Example.

- The article about Mutual Funds explains, in layman terms, that it is the risk reduction through diversification, which is the major reason to hold mutual funds. Different clients of money managers may have different constraints, requirements, tax considerations, etc. Still, our class discussion suggests that a limited number of portfolios may be sufficient to serve many clients. This is the theoretical basis for the mutual fund industry. This is why funds were introduced in the first place, and this is why they are widely popular.

- There are more articles about funds: In particular Index Funds (the “Fast Trades...” article may be of interest to those who want to learn more about tax-issues related to mutual funds -- although we are not focusing on these in class); Total-Market Funds, Bond Funds, and Exchange Traded Funds (ETFs).

- Take a look at the article that illustrates that even Universities(“Emory...”) make investment mistakes, which could be easily avoided given what we learned in class!

- A Business Week article further elaborates on the Asset Selection and Asset Allocation problems.

- Another article illustrates that decision makers in Washington are paying attention to the benefits of diversification, and hence are considering investing Social Security funds in the market. The debate is regarding the “appropriately” diversified portfolio. …. And there are OTHER interesting articles… to READ!